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CONVERGENCE OF THE MRV METHOD AT SINGULAR POINTS [1]

Zorana Lužanin[2], Tibor Lukić[3]

Abstract. In this paper we give sufficient conditions for convergence of the Newton-like method with modification of the right-hand-side vector (MRV) for a class of singular problems. The rate of convergence is sublinear. Numerical results are included witch agree well with the theoretically proven results.

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1. Introduction

Consider the system of nonlinear equations

$$F(x) = 0, \tag{1}$$

where $F \in C^2$ is a nonlinear mapping, $F : D \subset R^n \longrightarrow R^n$. The main purpose of this paper is to implement MRV Method [7] to determine solution x^* of $F(x) = 0$ when the derivative, $F'(x^*)$ is singular i.e., $F'(x^*) = 0$. In this case, we will say the point x^* is singular. The Newton iterates, $x^{n+1} = x^n - F'(x^n)^{-1}F(x^n)$, in singular case converge local linear, see [9, 4]. While, the rate of convergence for chord method $x^{n+1} = x^n - F'(x^0)^{-1}F(x^n)$, is only sublinear, that is $\lim_{k\to\infty} \|x^n - x^*\|/\|x^{n+1} - x^*\| = 1$.

The method with modification of the right-hand-side vector (MRV) for regular case is introduced in [7]. The essential idea was acceleration of the fixed Newton method by relaxation parameter and modification of the right-hand-side vector leading to low linear algebra cost. The MRV method is given by the following algorithm.

Algorithm. MRV:
Let $x^0 \in R^n$ and $F'(x^0)$ is a nonsingular matrix be given. For $k = 0, 1, 2, \ldots$

- Step 1. Solve
$$F'(x^0)s^n = (\alpha_k H(x^n) - I)F(x^n),$$

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[1]Supported by Grant No. 1840 of the Ministry of Science, Technologies and Development of the Republic of Serbia.

[2]Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia and Montenegro, zorana@im.ns.ac.yu

[3]Faculty of Engineering, University of Novi Sad, Trg Dositeja Obradovića 6, 21000 Novi Sad, Serbia and Montenegro, e-mail: tibor@uns.ns.ac.yu

where $H(x^n) = F'(x^n) - F'(x^0)$, I is the identity matrix and α_k is a real parameter

- Step 2. Define $x^{n+1} = x^n + s^n$,

The algorithm uses the relaxation parameter α_n. One possibility is to take $\alpha_n = \alpha$ during the whole process. In a special case, $\alpha_n = \alpha = 0$ would lead to the chord method. Obviously, the easiest way to choose the parameter is to assume $\alpha_n = \alpha$. Other possibility is to determine the parameter α_n such that the new iteration becomes as close as possible in $\|\cdot\|_2$ to the Newton iteration. This choice of α_k is called optimal parameter, because it is a solution of the optimization problem

$$\alpha_k^{opt} := \arg\min_{\alpha\in\mathcal{R}} \|L(x^n + s^n)\|_2^2,$$

with the MRV correction s^n and linear model $L(x^n + d) = F(x^n) + F'(x^n)d$.

We give some notation, which is fairly standard. We denote by N the null space of $F'(x^*)$, and by X the range space of $F'(x^*)$. Let P_N be a projector onto N parallel to X, and let $P_X = I - P_N$.

We assume throughout that $F'(x^*)$ has a one dimensional null space N and closed range X such that $R^n = N \oplus X$. For $x \in R^n$, we define $\widetilde{x} = x - x^*$ and define θ_n, ρ_n and ζ_n for the n^{th} iterate x^n by

$$\begin{aligned} \theta_n \|P_N \widetilde{x}^n\| &= \|P_X \widetilde{x}^n\|, \\ \zeta_n P_N \widetilde{x}^0 &= P_N \widetilde{x}^n, \\ \rho_n &= \|\widetilde{x}^n\| \end{aligned} \tag{2}$$

We define operators $D(x)$ and $\bar{D}(x)$ on N by

$$D(x) = P_N F''(x^*)(\widetilde{x}, P_N\cdot), \tag{3}$$

$$\bar{D}(x) = P_N F''(x^*)(P_N\widetilde{x}, P_N\cdot)\,. \tag{4}$$

The satisfies guess x^0, must be chosen so that $F'(x^0)$ be invertible. A set which satisfies these requirements can be defined as follows: for ρ and θ positive define [1, 9] $W_{\rho,\theta}$ by

$$W_{\rho,\theta} = \{x \in R^n \,|0 < \|x - x^*\| \le \rho,\ \|P_X(x - x^*)\| \le \theta\|P_N(x - x^*)\|\}. \tag{5}$$

We let $\beta_m(x)$ denote any term of order $\|\widetilde{x}\|^m$ and $\beta_m^X(x)$ (resp $\beta_m^N(x)$) any term of order $\|P_X\widetilde{x}\|^m$ (resp $\|P_N\widetilde{x}\|^m$). Let $\gamma_p^q(x)$ denote any term of order $\|\widetilde{x}\|^p$ such that $P_X\gamma_p^q(x) = \beta_{p+q}(x)$.

The following theorem contains some results that will be needed in what follows.

Theorem 1.1. *[1] Let $x^0 \in W_{\rho,\theta}$, dim(N)=1. Assume that there is $\alpha > 0$ so that for all $\phi \in N$*

$$\|F''(x^*)(\phi,\phi)\| \ge \alpha\|\phi\|^2 \tag{6}$$

Then for ρ and θ sufficiently small, $F'(x)^{-1}$, $D(x)^{-1}$ exist for all $x \in W_{\rho,\theta}$ and

$$\begin{aligned} F'(x)^{-1} &= P_N D(x)^{-1} P_N + \beta_0(x) \\ &= P_N \bar{D}(x)^{-1} P_N + \theta\beta_{-1}(x) \\ &= \beta_{-1}(x)\,. \end{aligned} \tag{7}$$

Moreover, the Newton iterates, $x^n = x^{n-1} - F'(x^{n-1})^{-1}F(x^{n-1})$, $n \geq 1$ remain in $W_{\rho,\theta}$, converge to x^*, and

$$\lim_{n\to\infty} \frac{\|P_N(x^{n+1} - x^*)\|}{\|P_N(x^n - x^*)\|} = \frac{1}{2}, \tag{8}$$

$$\|P_X(x^{n+1} - x^*)\| \leq K\|x^n - x^*\|^2, \quad \textit{for some } K > 0,\ n = 0, 1, \ldots \tag{9}$$

2 Main Result

The following lemmas are used for proving the convergence result.

Lemma 2.1 *If $x^0 \in W_{\rho,\theta}$ and $x^{n+1} = x^n - F'(x^0)^{-1}(I - \alpha_n H(x^n))F(x^n)$, then*

$$\widetilde{x}^{n+1} = P_N\widetilde{x}^n - \frac{1}{2}F'(x^0)^{-1}F''(x^*)(\widetilde{x}^n, \widetilde{x}^n) + E_n, \tag{10}$$

where

$$\begin{aligned} E_n &= \gamma_0^1(x^0)\beta_1^X(x^n) + \gamma_{-1}^1(x^0)\beta_3(x^n) + \tau_n\,, \\ \tau_n &= -\alpha_n[\gamma_{-1}^1(x^0)H(x^n)\beta_1^X(x^n) + \tfrac{1}{2}\gamma_{-1}^1(x^0)H(x^n)F''(x^*)(\widetilde{x}^n, \widetilde{x}^n) \\ &\quad + \gamma_{-1}^1(x^0)H(x^n)\beta_1(x^0)\beta_1^X(x^n) + \gamma_{-1}^1(x^0)H(x^n)\beta_3(x^n)]. \end{aligned} \tag{11}$$

Proof. Let $\hat{F} = P_X F'(x^*) P_X$. From the Taylor expansions

$$\begin{aligned} F(x^n) &= F'(x^*)\widetilde{x}^n + \tfrac{1}{2}F''(x^*)(\widetilde{x}^n, \widetilde{x}^n) + \beta_3(x^n) \\ &= \hat{F}P_X\widetilde{x}^n + \tfrac{1}{2}F''(x^*)(\widetilde{x}^n, \widetilde{x}^n) + \beta_3(x^n) \end{aligned}$$

and

$$\begin{aligned} F'(x^0)P_X\widetilde{x}^n &= \hat{F}P_X\widetilde{x}^n + F''(x^*)(\widetilde{x}^0, P_X\widetilde{x}^n) + \beta_2(x^0)\beta_1^X(x^n) \\ &= \hat{F}P_X\widetilde{x}^n + \beta_1(x^0)\beta_1^X(x^n). \end{aligned}$$

We obtain

$$F(x^n) = F'(x^0)P_X\widetilde{x}^n + \frac{1}{2}F''(x^*)(\widetilde{x}^n, \widetilde{x}^n) + \beta_1(x^0)\beta_1^X(x^n) + \beta_3(x^n).$$

As $P_X F'(x^0)^{-1} = \beta_0(x^0)$ and $F'(x^0)^{-1} = \gamma^1_{-1}(x^0)$ we obtain

$$\begin{aligned} F'(x^0)^{-1}(I-\alpha_n H(x^n))F(x^n) &= P_X\widetilde{x}^n + \tfrac{1}{2}F'(x^0)^{-1}F''(x^*)(\widetilde{x}^n,\widetilde{x}^n) \\ &\quad -\alpha_n F'(x^0)^{-1}H(x^n)P_X\widetilde{x}^n + F'(x^0)^{-1}\beta_3(x^n) \\ &\quad +F'(x^0)^{-1}\beta_1(x^0)\beta_1^X(x^n) \\ &\quad -\tfrac{1}{2}\alpha_n F'(x^0)^{-1}H(x^n)F''(x^*)(\widetilde{x}^n,\widetilde{x}^n) \\ &\quad -\alpha_n F'(x_0)^{-1}H(x^n)\beta_3(x^n) \\ &\quad -\alpha_n F'(x^0)^{-1}H(x^n)\beta_1(x^0)\beta_1^X(x^n) \\ &= P_X\widetilde{x}^n + \tfrac{1}{2}F'(x^0)^{-1}F''(x^*)(\widetilde{x}^n,\widetilde{x}^n) + E_n. \end{aligned}$$

As $x^{n+1} = x^n - F'(x^0)^{-1}(I-\alpha_n H(x^n))F(x^n)$ we have

$$\widetilde{x}^{n+1} = (I-P_X)\widetilde{x}^n - \frac{1}{2}F'(x^0)^{-1}F''(x^*)(\widetilde{x}^n,\widetilde{x}^n) + E_n.$$

This completes the proof. □

Let $\kappa^* = \|P_X F'(x^0)^{-1}F''(x^*)(\cdot,\cdot)\|$ and $\kappa = 2(\kappa^*+1)$. In [1, 3, 9] it was shown that if $x^0 \in W_{\rho,\theta}$ and $x^1 = x^0 - F'(x^0)^{-1}F(x^0)$, then

(12)
(i) there is $K_0 > 0$ so that
$\|P_X\widetilde{x}^1\| \le K_0\|P_X\widetilde{x}^0\|\rho_0$,
(ii) $\|P_X\widetilde{x}^1\| \le \kappa\|P_N\widetilde{x}^0\|^2$,
(iii) $\rho_1 \le \rho_0$,
(iv) $\|P_X\widetilde{x}^1\| \le 2\kappa\|P_N\widetilde{x}^1\|^2$,
(v) there is $c > 0$, so that
$(\frac{1}{2} - c\theta_0)\|P_N\widetilde{x}^0\| \le \|P_N\widetilde{x}^1\| \le (\frac{1}{2} + c\theta_0)\|P_N\widetilde{x}^0\|$.

A consequence of (i) and (v) is

$$\theta_1 \le \left(\frac{1}{2} - c\theta_0\right)^{-1} K_0\rho_0\theta_0 < \theta_0 \tag{13}$$

for ρ_0 and θ_0 sufficiently small.

The following sequence of lemmas give the estimates for the higher MRV iterates.

Lemma 2.2 *Assume $x^0 \in W_{\rho,\theta}$, $n \ge 0$ and $\alpha_n| < \alpha'$. Assume that for $1 \le k \le n$*

$$\|P_X\widetilde{x}^k\| \le 2\kappa\|P_N\widetilde{x}^k\|^2 \tag{14}$$

and

$$\|P_N\widetilde{x}^k\| \le \|P_N\widetilde{x}^0\| \tag{15}$$

then for sufficiently small ρ and θ

$$\|P_X\widetilde{x}^{n+1}\| \le \kappa\|P_N\widetilde{x}^n\|^2\,. \tag{16}$$

Proof. If $n = 1$ (14) and (15) are results of Newton's method [1, 3]. For $n > 1$ we apply P_X to both sides of (10),

$$\begin{aligned} P_X \widetilde{x}^{n+1} &= P_X P_N \widetilde{x}^n - \tfrac{1}{2} P_X F'(x^0)^{-1} F''(x^*)(\widetilde{x}^n, \widetilde{x}^n) + P_X E_n \\ &= -\tfrac{1}{2} P_X F'(x^0)^{-1} F''(x^*)(\widetilde{x}^n, \widetilde{x}^n) + P_X E_n. \end{aligned}$$

From (14), for $k = n$ we have,

$$\begin{aligned} \|\widetilde{x}^n\| &\le \|P_N \widetilde{x}^n + P_X \widetilde{x}^n\| \le \|P_N \widetilde{x}^n\| + \|P_X \widetilde{x}^n\| \\ &\le \|P_N \widetilde{x}^n\|(1 + 2\kappa \|P_N \widetilde{x}^n\|) \le \|P_N \widetilde{x}^n\|(1 + 2\kappa \|P_N \widetilde{x}^0\|), \end{aligned} \tag{17}$$

and there is $c_1' > 0$ so that

$$\begin{aligned} \|P_X \gamma_0^1(x^0) \beta_1^X(x^n)\| &= \beta_1(x^0) \cdot \beta_1^X(x^n) = \|\widetilde{x}^0\| \|P_X \widetilde{x}^n\| \\ &\le \|\widetilde{x}^0\| 2c_1' \kappa \|P_N \widetilde{x}^n\|^2 = 2c_1' \kappa \rho_0 \cdot \|P_N \widetilde{x}^n\|^2. \end{aligned} \tag{18}$$

By (11) we have

$$\begin{aligned} \|P_X \tau_n\| \le |\alpha_n| &[\|P_X \gamma_{-1}^1(x^0) H(x^n) \beta_1^X(x^n)\| \\ &+ \tfrac{1}{2} \|P_X \gamma_{-1}^1(x^0) H(x^n) F''(x^*)(\widetilde{x}^n, \widetilde{x}^n)\| \\ &+ \|P_X \gamma_{-1}^1(x^0) H(x^n) \beta_1(x^0) \beta_1^X(x^n)\| + \|P_X \gamma_{-1}^1(x^0) H(x^n) \beta_3(x^n)\|] \\ \le \|\alpha_n H(x^n)\| &[\, \|P_X \gamma_{-1}^1(x^0)\| \|\beta_1^X(x^n)\| \\ &+ \tfrac{1}{2} \|P_X \gamma_{-1}^1(x^0)\| \|F''(x^*)(\widetilde{x}^n, \widetilde{x}^n)\| \\ &+ \|P_X \gamma_{-1}^1(x^0)\| \|\beta_1(x^0) \beta_1^X(x^n)\| + \|P_X \gamma_{-1}^1(x^0) \beta_3(x^n)\|\,]. \end{aligned}$$

Since H is continuous function and sequence $\{\alpha_n\}$ is bounded , there exists $\delta > 0$ such that that $\|\alpha_n H(x^n)\| < \delta$. By definition of κ^* and by assumption (14) we have

$$\begin{aligned} \|P_X \tau_n\| &\le \delta\,[\, \|P_X \widetilde{x}^n\| + \kappa^* \|\widetilde{x}^n\|^2 + \|P_X \gamma_0^1(x_0)\| \|P_X \widetilde{x}^n\| + \|\beta_3(x^n)\|\,] \\ &\le \delta \|P_N \widetilde{x}^n\|^2\,[\, 2\kappa + \kappa^*(1 + 2\kappa \|P_X \widetilde{x}^0\|)^2 + 2\kappa\rho_0 \\ &\quad + \|P_N \widetilde{x}^n\|(1 + 2\kappa \|P_N \widetilde{x}^0\|)^3\,], \end{aligned}$$

and there exist constants $c_2 > 0$ and $c_2' > 0$ so that

$$\|P_X \tau_n\| \le \|P_N \widetilde{x}^n\|^2 \cdot [c_2' + \|P_N \widetilde{x}^n\| c_2\,] \tag{19}$$

By inequalities (19), (18) and (17) we obtain

$$\begin{aligned} \|P_X E_n\| &\le 2c_1' \kappa \rho_0 \cdot \|P_N \widetilde{x}^n\|^2 + \|\widetilde{x}^n\|^3 + \|P_N \widetilde{x}^n\|^2 \cdot [\, c_2' + c_2 \|P_N \widetilde{x}^n\|\,] \\ &\le \|P_N \widetilde{x}^n\|^2 \cdot [\, 2c_1' \kappa \rho_0 + \|P_N \widetilde{x}^n\|(1 + 2\kappa \|P_N \widetilde{x}^0\|)^3 + c_2' + c_2 \|P_N \widetilde{x}^n\|\,] \\ &\le \|P_N \widetilde{x}^n\|^2 \cdot [\, 2\kappa c_1 \rho_0 + c_3 \|P_N \widetilde{x}^n\|\,], \end{aligned}$$

for some constants $c_1 > 0$ and $c_3 > 0$. By $\|P_N\widetilde{x}^n\| \leq (1-\theta_0)^{-1}\rho_0$ and by (15), we have

$$\begin{aligned}\|P_X E_n\| &\leq \|P_N\widetilde{x}^n\|^2(2\kappa c_1\rho_0 + c_3(1-\theta_0)^{-1}\rho_0)\\ &= \|P_N\widetilde{x}^n\|^2\rho_0(2\kappa c_1 + (1-\theta_0)^{-1}c_3).\end{aligned} \tag{20}$$

By definition of κ^*,

$$\begin{aligned}\|P_X F'(x^0)^{-1}F''(x^*)(\widetilde{x}^n,\widetilde{x}^n)\| &\leq \kappa^*\|\widetilde{x}^n\|^2\\ &\leq \kappa^*\|P_N\widetilde{x}^n\|^2(1+2\kappa\|P_N\widetilde{x}^0\|)^2\\ &\leq \kappa^*\|P_N\widetilde{x}^n\|^2(1+2\kappa\rho_0(1-\theta_0)^{-1})^2.\end{aligned} \tag{21}$$

By (20) and (21) we have

$$\|P_X\widetilde{x}^{n+1}\| \leq \|P_N\widetilde{x}^n\|^2\left[\frac{1}{2}\kappa^*(1+2\kappa\rho_0(1-\theta_0)^{-1})^2 + 2\kappa c_1\rho_0 + c_3(1-\theta_0)^{-1}\rho_0\right]$$

if ρ and θ are sufficiently small, we obtain

$$\|P_X\widetilde{x}^{n+1}\| \leq \|P_N\widetilde{x}^n\|^2[\kappa^*+2] \leq \kappa\|P_N\widetilde{x}^n\|^2$$

which completes the proof. □

Lemma 2.3 *[4] Let $x^0 \in W_{\rho,\theta}$. Assume that, for $1 \leq k \leq n$ that (14) and (15) holds for $x^n \in W_{\rho,\theta}$ and*

$$0 \leq \zeta_{k-1}(1-\frac{3}{4}\zeta_{k-1}) \leq \zeta_k \leq \zeta_{k-1}(1-\frac{1}{4}\zeta_{k-1}), \tag{22}$$

then

$$0 \leq \zeta_n(1-\frac{3}{4}\zeta_n) \leq \zeta_{n+1} \leq \zeta_n(1-\frac{1}{4}\zeta_n). \tag{23}$$

The next Lemma unites the previous two.

Lemma 2.4 *[4] Let $x^0 \in W_{\rho,\theta}$ and ρ and θ sufficiently small, then for $k \geq 1$ hold*

$$\begin{aligned}&a)\quad \rho_k = \|\widetilde{x}^k\| \leq \rho_0,\\ &b)\quad \theta_k \leq 2\kappa(1-\theta_0)^{-1}\rho_0\zeta_k < \theta,\\ &c)\quad 0 \leq \zeta_{k-1}(1-\tfrac{3}{4}\zeta_{k-1}) \leq \zeta_k \leq \zeta_{k-1}(1-\tfrac{1}{4}\zeta_{k-1}),\\ &d)\quad \|P_X\widetilde{x}^k\| \leq 2\kappa\|P_N\widetilde{x}^k\|^2,\\ &e)\quad \|P_X\widetilde{x}^k\| \leq \kappa\|P_N\widetilde{x}^{k-1}\|^2.\end{aligned} \tag{24}$$

Now, we are able to give a Theorem about sublinear convergence of the MRV Method.

Theorem 2.5. *Let F be twice Lipschitz continuously differentiable in a neighborhood of x^* and let $x^0 \in W_{\rho,\theta}$ and dim(N)=1. If exist $\alpha > 0$ so that $\phi \in N$*

$$\|F''(x^*)(\phi,\phi)\| \geq \alpha\|\phi\|^2,$$

then for ρ and θ sufficiently small and $|\alpha_n| < \alpha'$, the MRV iteration

$$x^{n+1} = x^n - F'(x^0)^{-1}(I - \alpha_n H(x^n))F(x^n),$$

where $|\alpha_n| < \overline{\alpha}$ for some $\overline{\alpha} > 0$, remain in $W_{\rho,\theta}$, converge to x^* and for $n \geq 1$ holds

$$0 \leq \zeta_n(1 - \frac{3}{4}\zeta_n) \leq \zeta_{n+1} \leq \zeta_n(1 - \frac{1}{4}\zeta_n), \tag{25}$$

$$\left(1 + \frac{3n}{4}\right)^{-1} \|P_N(x^0 - x^*)\| \leq \|P_N(x^n - x^*)\| \leq \|P_N(x^0 - x^*)\| \left(1 + \frac{n}{4}\right)^{-1}, \tag{26}$$

$$\|P_X(x^n - x^*)\| \leq K\|x^{n-1} - x^*\|^2 \quad \textit{for some } K > 0. \tag{27}$$

Proof. (24)(a) and (24)(b) imply that $x^n \in W_{\rho,\theta}$ for all $n \geq 0$. Estimate (24)(c) gives (25) and inequality (24)(e) gives (27). The second part of inequality (24)(c) by [5, 8] guarantees $\zeta_n \leq \left(1 - \frac{n}{4}\right)^{-1}$. By the same way the first part of inequality (24)(c) guarantees $\zeta_n \geq \left(1 + \frac{3n}{4}\right)^{-1}$, which together proof the convergence estimate (26). □

3. Numerical Results

Consider the nonlinear mapping

$$F(x) = \begin{bmatrix} x_1 + x_1x_2 + x_2^2 \\ x_1^2 - 2x_1 + x_2^2 \end{bmatrix} \tag{28}$$

on R^2. F has a root at $x^* = [0,0]^T$ and $F'(x^*)$ has one-dimensional null space $N = \text{span}(\phi)$, where $\phi = [0,1]^T$. It is easy to see $F''(x^*)(\phi,\phi) = \phi^T F''(x^*)\phi = [2,2]^T$. The last equality implies that the assumption (6) for $\alpha = 1 > 0$ holds.

The numerical results given by MRV method are shown in Table 1. The convergence is sublinear.

n	x_1	x_2	$\frac{\|P_X(x^n - x^*)\|}{\|x^{n-1} - x^*\|^2}$
0	0.05	0.1	
1	0.00017985611510790	0.060791366906474	0.0143885
2	0.000473269935359905	0.033471103604271	0.128062
3	0.00028703685232103	0.0232246812657045	0.25616
4	0.00015720724249627	0.0180412794200752	0.291412
5	0.000099789131768687	0.0141246527661112	0.30656
6	0.000062906915799622	0.0115597478405662	0.315298
7	0.000043033228138149	0.0095823314193569	0.322029
8	0.000029737613468795	0.0081419789994001	0.323858

Table 1.

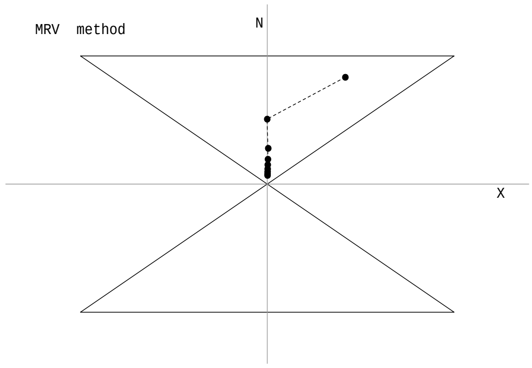

Figure 1: Behavior of the MRV iterates in the set $W_{\rho,\theta}$.

Figure 1 shows that the convergence acceleration in X-direction is faster than the acceleration in N-direction. It agrees with the result of Theorem 2.5.

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