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ON THE IMPROVED FAMILY OF SIMULTANEOUS METHODS FOR THE INCLUSION OF MULTIPLE POLYNOMIAL ZEROS

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Abstract. Starting from a family of iterative methods for the simultaneous inclusion of multiple complex zeros, we construct efficient iterative methods with accelerated convergence rate by the use of Gauss-Seidel procedure and the suitable corrections. The proposed methods are realized in the circular complex interval arithmetic and produce disks that contain the wanted zeros. The suggested algorithms possess a high computational efficiency since the increase of the convergence rate is attained without additional calculations. Using the concept of the R-order of convergence of mutually dependent sequences, the convergence analysis of the proposed methods is presented. Numerical results are given to demonstrate the convergence properties of the considered methods.

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1. Introduction

The aim of this paper is the construction of inclusion methods with very high computational efficiency for the simultaneous inclusion of multiple polynomial zeros. The improvement is obtained using the Gauss-Seidel approach and suitable corrections of Schröder's and Halley's type. The proposed acceleration is attained without additional calculations, which means that the obtained algorithms possess a great computational efficiency.

The proposed iterative methods are realized in circular complex interval arithmetic. These methods produce approximations in the form of complex intervals (disks or rectangles) containing the sought zeros. The main advantage of circular arithmetic methods is the self-validation of the obtained results. In this manner the information about the upper error bounds of approximations to the zeros is provided. For more details about the circular complex arithmetic and inclusion root-finding methods see the books [1], [8] and [13].

Throughout this paper a circular closed region (disk) $Z := \{z : |z - c| \le r\}$, with the center $c :=$ mid Z and the radius $r :=$ rad Z, will be denoted briefly by the parametric notation $Z = \{c; r\}$. The basic operations of circular arithmetic

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and its properties may be found in the books [1] and [13]. Here we stress only that the inversion of a non-zero disk Z can be defined by the Möbius transformation (exact inversion)

$$(1)\quad Z^{-1} = \{c; r\}^{-1} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} = \left\{ \frac{1}{z} : z \in Z \right\} \ (|c| > r, \text{ i.e., } 0 \notin Z),$$

or by Taylor's form (centered inversion)

$$(2)\qquad Z^{I_c} = \left\{ \frac{1}{c}; \frac{r}{|c|(|c| - r)} \right\} \supseteq Z^{-1} \quad (|c| > r).$$

We will use the symbol INV to denote one of the introduced inversions. The square root of a disk $\{c; r\}$ in the centered form, where $c = |c|e^{i\theta}$ and $|c| > r$ (that is, the disk $\{c; r\}$ does not contain the origin) is defined as the union of two disjoint disks (see [3]):

$$(3)\qquad \begin{aligned} \{c; r\}^{1/2} \ &:= \ \left\{ \sqrt{|c|}e^{i\theta/2}; \frac{r}{\sqrt{|c|} + \sqrt{|c| - r}} \right\} \\ &\bigcup \left\{ -\sqrt{|c|}e^{i\theta/2}; \frac{r}{\sqrt{|c|} + \sqrt{|c| - r}} \right\}. \end{aligned}$$

In what follows, the disks in the complex plane will be denoted by capital letters. In addition, let us note that all operations posses the so-called *inclusion isotonicity* property, that is, they preserve the subset relation.

2. Family of total step methods

Let the zeros $\zeta_1, \ldots, \zeta_\nu$ $(2 \le \nu \le n)$ of a monic polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ have the multiplicities $\mu_1, \mu_2, \ldots, \mu_\nu (\mu_1 + \ldots + \mu_\nu = n)$, respectively. Efficient procedures for the determination of the order of multiplicity can be found, for instance, in [5] and [6]. Further, let $z_1, \ldots, z_\nu$ be mutually distinct approximations to the polynomial zeros. By ε_i we denote the error terms, that is, $\varepsilon_i = z_i - \zeta_i$. For the point $z = z_i$ $(i \in I_\nu := \{1, \ldots, \nu\})$ and a complex parameter α $(\ne -1)$, let us introduce the abbreviations

$$\Sigma_{k,i} = \sum_{\substack{j=1 \\ j \ne i}}^{\nu} \frac{\mu_j}{(z_i - \zeta_j)^k} \quad \text{and} \quad q_i^* = \mu_i(\alpha + 1)\Sigma_{2,i} - \alpha(\alpha + 1)\Sigma_{1,i}^2.$$

Using identities

$$\delta_{1,i} = \frac{P'(z_i)}{P(z_i)} = \sum_{j=1}^{\nu} \frac{\mu_j}{z_i - \zeta_j} = \frac{\mu_i}{z_i - \zeta_i} + \Sigma_{1,i}$$

and

$$\begin{aligned} \delta_{2,i} \ &= \ \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2} = -\frac{d}{dz}\left(\frac{P'(z)}{P(z)} \right) \Bigg|_{z=z_i} \\ &= \ \sum_{j=1}^{\nu} \frac{\mu_j}{(z_i - \zeta_j)^2} = \frac{\mu_i}{(z_i - \zeta_i)^2} + \Sigma_{2,i}, \end{aligned}$$

after elementary manipulations we obtain

$$\mu_i(\alpha+1)\delta_{2,i}-\alpha\delta_{1,i}^2-q_i^*=\Big(\frac{\mu_i(\alpha+1)}{\varepsilon_i}-\alpha\delta_{1,i}\Big)^2. \tag{4}$$

From the identity (4) we single out the term ε_i $(=z_i-\zeta_i)$ and obtain the following fixed point relation

$$\zeta_i=z_i-\frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i}\pm\sqrt{\mu_i(\alpha+1)\delta_{2,i}-\alpha\delta_{1,i}^2-q_i^*}}\quad (i\in I_\nu). \tag{5}$$

Let us introduce the disks

$$S_{k,i}(\boldsymbol{X},\boldsymbol{W}):=\sum_{j=1}^{i-1}\mu_j\Big(\mathrm{INV}_1(z-X_j)\Big)^k+\sum_{j=i+1}^{n}\mu_j\Big(\mathrm{INV}_1(z-W_j)\Big)^k, \tag{6}$$

where $k=1,2$, $\boldsymbol{X}=(X_1,...,X_n)$ and $\boldsymbol{W}=(W_1,...,W_n)$ are vectors whose components are disks and $\mathrm{INV}_1\in\{()^{-1},()^{I_c}\}$. Using (6) let us define the disk

$$Q_i(\boldsymbol{X},\boldsymbol{W})=\mu_i(\alpha+1)S_{2,i}(\boldsymbol{X},\boldsymbol{W})-\alpha(\alpha+1)S_{1,i}^2(\boldsymbol{X},\boldsymbol{W}).$$

Then, by (6) and the definition of q_i^*, according to the inclusion isotonicity it follows $q_i^*\in Q_i(\boldsymbol{X},\boldsymbol{W})$.

In order to consider the method without correction and the methods with Schröder's and Halley's correction simultaneously, we introduce the additional indices $p=0,1$ and 2, respectively, and denote the corresponding vectors of disks as follows:

$$\begin{aligned}
\boldsymbol{Z}^{(0)} &= \boldsymbol{Z}=\big(Z_1,\dots,Z_\nu\big)\ \text{(current disk approximations)},\\
\widehat{\boldsymbol{Z}}^{(0)} &= \widehat{\boldsymbol{Z}}=\big(\widehat{Z}_1,\dots,\widehat{Z}_\nu\big)\ \text{(new disk approximations)},\\
\boldsymbol{Z}^{(1)} &= \boldsymbol{Z}_N=\big(Z_1-N(z_1),\dots,Z_\nu-N(z_\nu)\big)\ \text{(Schröder's disks)},\\
\boldsymbol{Z}^{(2)} &= \boldsymbol{Z}_H=\big(Z_1-H(z_1),\dots,Z_\nu-H(z_\nu)\big)\ \text{(Halley's disks)},
\end{aligned}$$

where

$$N(z_i)=\mu_i\frac{P(z_i)}{P'(z_i)}\quad\text{and}\quad H(z_i)=\frac{P(z_i)}{\Big(\dfrac{1+1/\mu_i}{2}\Big)P'(z_i)-\dfrac{P(z_i)P''(z_i)}{2P'(z_i)}}$$

are corrections that appear in the well known iterative formulas $\hat z=z-N(z)$ and $\hat z=z-H(z)$ (for multiple zero) with the order of convergence two and three, respectively.

Using the inclusion isotonicity property, from the fixed point relation (5) we obtain the new family of total step inclusion methods

$$\widehat{Z}_i=z_i-\mu_i(\alpha+1)\mathrm{INV}_2(A_i)\quad (i\in\mathcal{I}_\nu,\ p=0,1,2), \tag{7}$$

where $\mathrm{INV}_2 \in \{()^{-1}, ()^{I_c}\}$ and

$$A_i = \alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\big(\delta_{1,i}\big)^2 - Q_i\big(\boldsymbol{Z}^{(p)}, \boldsymbol{Z}^{(p)}\big)\Big]_*^{1/2}.$$

The subscripts "1" and "2" of INV in (6), (7) and subsequent formulas point to the order of application of the inversion; namely, in the realization of the iterative formula (7) we first apply the inversion INV_1 to the sums (6), and then the inversion INV_2 in the final step. In our consideration of the new family we will always assume that $\alpha \neq -1$. However, the particular case $\alpha = -1$ reduces (by applying a limiting process) to the already known Halley-like interval method studied in [9] and [15].

By virtue of (3), the square root of a disk in (7) produces two disks; the symbol $*$ indicates that one of the two disks should be selected. That disk will be called a "*proper*" disk. Considering (4) and the inclusion $q_i^* \in Q_i$ we conclude that proper disk is the one that contains $\mu_i(\alpha+1)/\varepsilon_i - \alpha\delta_{1,i}$. According to (3), we have

$$\big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - Q_i\big]^{1/2} = G_{1,i} \cup G_{2,i}, \quad \mathrm{mid}\ G_{k,i} = g_{k,i}, \quad g_{1,i} = -g_{2,i}$$

for $i \in \mathcal{I}_\nu,\ k = 1, 2$. An efficient and reliable criterion for the choice of the proper disk is considered in [3] (see also [8]) and reads:

If disks $Z_1, \dots, Z_\nu$ are reasonably small, then we have to choose the disk (between $G_{1,i}$ and $G_{2,i}$), whose center minimizes $|P'(z_i)/(\mu_i P(z_i)) - g_{k,i}|$ $(k = 1, 2)$.

In order to estimate the order of convergence of the interval methods with corrections, we usually deal with the sequences $\epsilon_i^{(m)} = |\mathrm{mid}\ Z_i^{(m)} - \zeta_i|$, $r_i^{(m)} = \mathrm{rad}\ Z_i^{(m)}$ $(i \in \mathcal{I}_\nu)$, where $Z_i^{(m)}$ is an outer approximation to the zero ζ_i produced at the mth iterative step. In the convergence analysis of an iterative inclusion method (IM) we use the concept of the R-order of convergence, introduced in [7] (denoted by $O_R(IM)$), and operate with mutually dependent sequences of centers and radii of disks. We will use the following assertion given in [4] (see, also [14]) :

Theorem 1. *Given the error-recursion*

$$u_i^{(m+1)} \le c_i \prod_{j=1}^{k} \big(u_j^{(m)}\big)^{t_{ij}}, \quad (i \in \mathcal{I}_k;\ m \ge 0), \tag{8}$$

where $t_{ij} \ge 0,\ c_i > 0,\ 1 \le i, j \le k$ and $u_j^{(m)} = \epsilon_j^{(m)}$ or $u_j^{(m)} = r_j^{(m)}$. Denote the matrix of exponents appearing in (8) with T_k, that is $T_k = [t_{ij}]_{k\times k}$. If the non-negative matrix T_k has the spectral radius $\rho(T_k) > 1$ and a corresponding eigenvector $\boldsymbol{x}_\rho > 0$, then the R-order of convergence of all sequences $\{u_i^{(m)}\}$ $(i \in \mathcal{I}_\nu)$ is at least $\rho(T_k)$.

In the sequel the matrix $T_k = [t_{ij}]$ will be called the *R-matrix* because it is related to the R-order of convergence. Further, for two real or complex numbers w_1 and w_2 satisfying $|w_1| = O(|w_2|)$ we write $w_1 \sim w_2$ (the same order of magnitude). In the convergence analysis of inclusion methods it is adopted that $0 < |\varepsilon^{(0)}| = r^{(0)} < 1$ (the "worst case" model). This assumption has no influence to the final result of the limit process in finding the lower bound of the R-order of convergence.

According to Theorem 1, using the approach proposed in [2] and [11], and later in [10] and [12], the following assertion can be proved:

Theorem 2. *The total-step methods* (7) *are convergent with the convergence order*

$$O_R(7) \geq \begin{cases} \begin{cases} 4, & p = 0, \\ 2+\sqrt{7} \cong 4.646, & p = 1,2, \end{cases} & \textit{if} \ \ \mathrm{INV}_1 = ()^{-1}, \\ \ p+4, \qquad\qquad\qquad\quad p = 0,1,2, & \textit{if} \ \ \mathrm{INV}_1 = ()^{I_c}. \end{cases}$$

3. Family of single step methods

Further acceleration of the convergence of the family of inclusion methods (7) can be attained using the new inclusion disks serially, that is, employing the already calculated disks as soon as they are available (the so-called Gauss-Seidel approach or single step mode). In this manner we obtain the corresponding family of single-step methods:

$$\widehat{Z}_i = z_i - \mu_i(\alpha+1)\cdot \mathrm{INV}_2(B_i) \quad (i \in \mathcal{I}_\nu, \ p = 0,1,2), \tag{9}$$

where $\mathrm{INV}_2 \in \{()^{-1}, ()^{I_c}\}$ and $B_i = \alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - Q_i(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}^{(p)})\Big]_*^{1/2}$.

In order to find the R-order of convergence of the single step methods (9), one has to handle 2ν mutually dependent sequences of centers and radii of produced disks, which is a difficult task. However, we can estimate easily the bounds of the R-order regarding the limit cases $\nu = 2$ and very large ν.

Since the convergence rate of a single-step method becomes almost the same to the one of the corresponding total-step method when the polynomial degree is very large, according to Theorem 2 we have

$$O_R((9,\nu)) \cong O_R(7) \geq \begin{cases} \begin{cases} 4, & p = 0, \\ 2+\sqrt{7} \cong 4.646, & p = 1,2, \end{cases} & \text{if} \ \ \mathrm{INV}_1 = ()^{-1}, \\ \ p+4, \qquad\qquad\quad\ \ p = 0,1,2, & \text{if} \ \ \mathrm{INV}_1 = ()^{I_c}. \end{cases}$$

Consider now the single-step methods (9) for $\nu = 2$ and assume that $|\varepsilon_1^{(0)}| = |\varepsilon_2^{(0)}| = r_1^{(0)} = r_2^{(0)} < 1$ (the "worst case" model). After an extensive calculation we derive the following estimates:

(i) Case $\mathrm{INV}_1 = ()^{-1}$:

$$|\hat{\varepsilon}_1| \sim \begin{cases} |\varepsilon_1|^3|\varepsilon_2|, & p = 0, \\ |\varepsilon_1|^3 r_2^2, & p = 1,2, \end{cases} \qquad |\hat{\varepsilon}_2| \sim \begin{cases} |\varepsilon_1|^3|\varepsilon_2|^4, & p = 0, \\ |\varepsilon_1|^3|\varepsilon_2|^3 r_2^2, & p = 1,2, \end{cases}$$

$$\hat{r}_1 \sim |\varepsilon_1|^3 r_2, \qquad \hat{r}_2 \sim |\varepsilon_1|^3|\varepsilon_2|^3 r_2, \qquad p = 0,1,2.$$

(ii) Case $\mathrm{INV}_1 = ()^{I_c}$:

$$\begin{aligned} |\hat{\varepsilon}_1| &\sim |\varepsilon_1|^3|\varepsilon_2|^{p+1}, \quad |\hat{\varepsilon}_2| \sim |\varepsilon_1|^3|\varepsilon_2|^{p+4}, \quad p = 0, 1, 2, \\ \hat{r}_1 &\sim |\varepsilon_1|^3 r_2, \qquad\qquad \hat{r}_2 \sim |\varepsilon_1|^3|\varepsilon_2|^3 r_2. \end{aligned}$$

The corresponding R-matrices and their spectral radii and eigenvectors are:

(i) Case $\mathrm{INV}_1 = ()^{-1}$:

$$p = 0, \quad T_4^{(e)} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}, \quad \begin{array}{l} \rho\big(T_4^{(e)}\big) = 5.30278, \\ \boldsymbol{x}_\rho^{(e)} = (1, 2.30277, 1, 2.30277) > 0, \end{array}$$

$$p = 1, 2, \ T_4^{(e)} = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 3 & 3 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}, \quad \begin{array}{l} \rho\big(T_4^{(e)}\big) = 6.29654, \\ \boldsymbol{x}_\rho^{(e)} = (1, 1.91, 0.7382, 1.6483) > 0. \end{array}$$

(ii) Case $\mathrm{INV}_1 = ()^{I_c}$:

$$p = 0, 1, 2, \ T_4^{(c)} = \begin{bmatrix} 3 & p+1 & 0 & 0 \\ 3 & p+4 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}, \ \rho\big(T_4^{(c)}\big) = \begin{cases} 5.30278, & p = 0, \\ 6.64575, & p = 1, \\ 7.8541, & p = 2, \end{cases}$$

$$\boldsymbol{x}_\rho^{(c)} = \begin{cases} (1, 2.30277, 1, 2.30277) > 0, & p = 0, \\ (1, 1.8229, 0.6771, 1.5) > 0, & p = 1, \\ (1, 1.6180, 0.5279, 1.1459) > 0, & p = 2. \end{cases}$$

The superscripts "e" and "c" are used to indicate the type of the inversion used in (6).

According to the previous results we can formulate the following assertion:

Theorem 3. *The ranges of the lower bounds of the R-order of convergence of the single-step methods* (9) *are*

$$\begin{aligned} O_R(9) &\in \begin{cases} (4, 5.303), & p = 0, \\ (4.646, 6.297), & p = 1, 2, \end{cases} \qquad \textit{if} \quad \mathrm{INV}_1 = ()^{-1}, \\ O_R(9) &\in \begin{cases} (4, 5.303), & p = 0, \\ (5, 6.646), & p = 1, \\ (6, 7.855), & p = 2, \end{cases} \qquad \textit{if} \quad \mathrm{INV}_1 = ()^{I_c}. \end{aligned}$$

Remark 1. Similarly as in [10], it can be proved that any correction of the order higher than *two* (for instance, Halley's correction) cannot increase the convergence speed of the inclusion algorithm if the exact inversion is applied to (6) (that is, $\mathrm{INV}_1 = ()^{-1}$).

Remark 2. From Theorems 2 and 3 we see that (when corrections are used) centered inversion gives a better convergence although it produces an enlarged disk compared to the exact inversion (see (2)). The explanation lies in the fact that the centered inversion enables the better convergence of the midpoints of the disks produced by (6). In this way, the faster convergence of the midpoints of disks forces a better convergence of the radii (see [12]).

4. Numerical examples

The presented families of inclusion methods have been tested in solving many polynomial equations. To provide the enclosure of the zeros in the second and third iteration, which produce very small disks, we used the programming package *Mathematica 5* with the multi-precision arithmetic. For demonstration, we give two numerical examples.

Example 1. We applied the interval methods (7) and (9) (with $p = 0, 1, 2$) in order to find the circular inclusion approximations to the zeros of the polynomial

$$\begin{aligned} P(z) &= z^{12} + z^{11} - 20z^{10} + 2z^9 + 153z^8 - 179z^7 - 418z^6 + 1052z^5 \\ &\quad -204z^4 - 1436z^3 + 1224z^2 + 432z - 864. \end{aligned}$$

The zeros of P are $\zeta_1 = -3,\ \zeta_2 = -1,\ \zeta_3 = 1+i,\ \zeta_4 = 1-i,\ \zeta_5 = 2$, with the respective multiplicities $\mu_1 = 3,\ \mu_2 = \mu_3 = \mu_4 = 2,\ \mu_5 = 3$. The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.5\}$ with the centers:

$$\begin{aligned} z_1^{(0)} &= -2.8 - 0.1i, \quad z_2^{(0)} = -1.2 + 0.2i, \quad z_3^{(0)} = 1.1 + 0.8i, \\ z_4^{(0)} &= 0.8 - 1.1i, \quad z_5^{(0)} = 1.8 - 0.2i. \end{aligned}$$

The maximal radii of the inclusion disks, produced in the third iterative steps, are given in Table 1. The terms “exact inversions” and “centered inversions” point out that the same type of inversion is applied in the implemented iterative formulas, that is, $\mathrm{INV}_1, \mathrm{INV}_2 = ()^{-1}$ and $\mathrm{INV}_1, \mathrm{INV}_2 = ()^{I_c}$, respectively.

exact inversions						
α	(7), $p=0$	(9), $p=0$	(7), $p=1$	(9), $p=1$	(7), $p=2$	(9), $p=2$
1	1.03(−32)	2.11(−39)	7.67(−41)	3.69(−51)	3.78(−40)	6.20(−52)
0.5	7.09(−40)	3.92(−45)	3.09(−44)	3.87(−53)	5.95(−45)	2.74(−54)
$\mu_i/(n-\mu_i)$	1.36(−41)	1.40(−44)	1.47(−48)	1.68(−53)	1.43(−47)	7.76(−55)
0	3.60(−40)	1.34(−44)	7.72(−44)	2.01(−54)	2.03(−46)	1.32(−55)
−1	7.24(−29)	7.89(−32)	4.82(−28)	2.11(−39)	2.00(−30)	1.23(−41)
centered inversions						
α	(7), $p=0$	(9), $p=0$	(7), $p=1$	(9), $p=1$	(7), $p=2$	(9), $p=2$
1	1.89(−39)	4.48(−43)	1.49(−58)	3.42(−59)	2.23(−80)	7.75(−86)
0.5	7.77(−45)	8.95(−50)	1.64(−59)	4.41(−66)	6.10(−88)	6.87(−94)
$\mu_i/(n-\mu_i)$	1.04(−46)	4.61(−49)	3.25(−65)	1.57(−67)	4.82(−92)	6.14(−95)
0	1.17(−42)	6.97(−49)	1.49(−57)	5.07(−65)	3.63(−89)	6.67(−92)
−1	1.88(−31)	2.34(−36)	1.17(−47)	8.75(−52)	6.10(−71)	1.77(−77)

Table 1 The radii of inclusion disks in the third iteration. $A(-h)$ means $A \times 10^{-h}$.

Example 2 To find the circular inclusion approximations to the zeros of the polynomial

$$P(z) = z^{12} - (2-3i)z^{11} + (16-6i)z^{10} - (26-38i)z^9$$

$$+(101-58i)z^8-(120-131i)z^7+(250-76i)z^6$$
$$-(72+20i)z^5-(84-432i)z^4+(864-292i)z^3$$
$$-504z^2+432iz+864,$$

we implemented the same interval methods as in Example 1.

The zeros of P are $\zeta_1=-1,\ \zeta_2=2i,\ \zeta_3=1+i,\ \zeta_4=1-i,\ \zeta_5=-3i$ of the multiplicities $\mu_1=2,\ \mu_2=3,\ \mu_3=2,\ \mu_4=2,\ \mu_5=3$, respectively. The initial disks were selected to be $Z_i^{(0)}=\{z_i^{(0)};0.6\}$, with the centers:

$$\begin{aligned} z_1^{(0)} &= -1.2+0.2i, \quad z_2^{(0)}=-0.1+2.3i, \quad z_3^{(0)}=1.2+0.8i,\\ z_4^{(0)} &= 0.8-1.2i, \quad z_5^{(0)}=0.2-2.8i. \end{aligned}$$

The maximal radii of the inclusion disks produced in the first three iterative steps, are given in Table 2.

	exact inversions			**centered inversions**		
α	$m=1$	$m=2$	$m=3$	$m=1$	$m=2$	$m=3$
			(7), $p=0$			
1	3.18(−2)	1.33(−9)	2.96(−43)	5.20(−2)	7.77(−10)	6.19(−45)
0.5	1.82(−2)	3.91(−10)	1.67(−46)	2.62(−2)	6.82(−11)	6.13(−51)
$\mu_i/(n-\mu_i)$	1.33(−2)	1.57(−10)	3.53(−46)	1.81(−2)	1.54(−11)	1.91(−50)
0	9.86(−3)	5.91(−11)	6.44(−46)	1.29(−2)	6.31(−12)	5.95(−50)
−1	2.92(−2)	9.21(−9)	1.01(−36)	4.33(−2)	1.50(−9)	2.18(−41)
			(7), $p=1$			
1	2.58(−2)	1.12(−9)	3.07(−45)	3.79(−2)	9.23(−12)	1.45(−64)
0.5	1.68(−2)	7.90(−11)	1.56(−51)	2.31(−2)	9.25(−13)	2.39(−71)
$\mu_i/(n-\mu_i)$	1.15(−2)	8.99(−12)	9.55(−55)	1.51(−2)	1.45(−13)	6.10(−72)
0	7.84(−3)	1.35(−12)	4.27(−57)	1.01(−2)	2.60(−14)	6.07(−71)
−1	2.60(−2)	3.35(−10)	4.25(−46)	3.74(−2)	3.80(−12)	9.32(−62)
			(7), $p=2$			
1	2.53(−2)	5.27(−10)	1.79(−45)	3.74(−2)	5.83(−14)	1.90(−89)
0.5	1.67(−2)	4.75(−11)	3.31(−51)	2.30(−2)	8.62(−15)	8.03(−95)
$\mu_i/(n-\mu_i)$	1.15(−2)	8.03(−12)	2.06(−53)	1.52(−2)	2.09(−15)	1.29(−98)
0	7.98(−3)	2.38(−12)	3.10(−55)	1.03(−2)	5.39(−16)	7.69(−99)
−1	2.61(−2)	4.41(−10)	1.24(−45)	3.76(−2)	3.18(−14)	4.45(−89)
			(9), $p=0$			
1	1.38(−2)	4.66(−11)	6.08(−47)	3.60(−2)	8.81(−12)	1.15(−50)
0.5	1.29(−2)	3.76(−12)	8.05(−53)	1.77(−2)	2.58(−13)	5.35(−58)
$\mu_i/(n-\mu_i)$	1.04(−2)	2.27(−12)	3.58(−52)	1.39(−2)	5.12(−13)	3.88(−56)
0	6.45(−3)	2.64(−12)	2.08(−51)	8.42(−3)	5.85(−13)	3.36(−54)
−1	1.55(−2)	2.51(−10)	1.15(−42)	2.24(−2)	7.16(−11)	1.21(−45)
			(9), $p=1$			
1	1.81(−2)	1.49(−10)	9.84(−50)	2.59(−2)	2.02(−13)	7.04(−68)
0.5	9.75(−3)	3.46(−12)	1.40(−56)	1.30(−2)	3.93(−15)	7.42(−76)
$\mu_i/(n-\mu_i)$	7.73(−3)	1.76(−13)	2.54(−62)	1.01(−2)	2.78(−15)	5.36(−77)
0	4.39(−3)	2.13(−13)	4.00(−64)	5.60(−3)	3.57(−15)	7.46(−75)
−1	1.15(−2)	2.60(−11)	1.08(−53)	1.53(−2)	7.13(−13)	3.41(−64)
			(9), $p=2$			
1	1.83(−2)	1.98(−10)	9.65(−50)	2.64(−2)	9.66(−15)	4.00(−92)
0.5	9.90(−3)	4.33(−12)	2.64(−56)	1.32(−2)	6.29(−16)	2.13(−98)
$\mu_i/(n-\mu_i)$	7.87(−3)	1.98(−13)	5.20(−62)	1.03(−2)	6.82(−17)	1.85(−102)
0	4.49(−3)	2.64(−13)	3.53(−64)	5.75(−3)	8.72(−18)	4.59(−104)
−1	1.15(−2)	2.91(−11)	5.56(−53)	1.52(−2)	1.74(−15)	8.94(−93)

Table 2 The radii of inclusion disks in the first three iterations. $A(-h)$ means $A\times 10^{-h}$.

Tables 1 and 2 show very fast convergence of the proposed methods. The third iteration is given to demonstrate very high accuracy of approximations, rarely required in practice. Furthermore, the corresponding columns in Tables 1 and 2, concerned with exact inversions, show that the application of Halley's correction in (7) and (9) does not produce, in general, smaller disks if $\mathrm{INV}_1 = ()^{-1}$ (compared to the methods with Newton's corrections), which coincides with the theoretical results given by Theorems 2 and 3.

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