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THE STRUCTURE OF SPLINE COLLOCATION MATRIX FOR SINGULARLY PERTURBATION PROBLEMS WITH TWO SMALL PARAMETERS

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Abstract. We consider a spline difference scheme on a piecewise uniform Shishkin mesh for a singularly perturbed boundary value problem with two parameters. We show that the discrete minimum principle holds for a suitably chosen collocation points. Furthermore, bounds on the discrete counterparts of the layer functions are given. Numerical results indicate uniform convergence.

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1. Introduction

We consider the two-parameter singularly perturbed boundary value problem

$$
\begin{aligned}
(1) \qquad Ly := \varepsilon y''(x) + \mu a(x) y'(x) - b(x) y(x) &= f(x), \quad x \in (0,1), \\
y(0) = p_0, \quad y(1) &= p_1,
\end{aligned}
$$

where a, b and f are sufficiently smooth functions, $0 < \varepsilon \ll 1$, $0 < \mu \ll 1$, and

$$
a(x) \geq a > 0, \quad b(x) \geq b > 0, \quad x \in I = [0,1].
$$

Under these assumptions the problem (1) has a unique solution which exhibits exponential boundary layers at $x = 0$ and $x = 1$. When the parameter $\mu = 1$ problem (1) becomes convection-diffusion problem with the boundary layer of width $\mathcal{O}(\varepsilon)$ in the neighbourhood of the point $x = 0$. In the case of $\mu = 0$ we have reaction-diffusion problem with boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ at $x = 0$ and $x = 1$. We consider problem (1) and offer a unified treatment for all possible classes of subproblems.

Numerical method constructed in this paper is based on a difference scheme obtained using quadratic spline function as an approximation function as in [5]. With the suitably chosen collocation points, we prove that our scheme has the inverse monotone matrix on the corresponding Shishkin mesh.

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One-dimensional two-parameter problem was recently treated numerically in [1], [2], [3], [4], [6]. Linss and Roos [2] and O'Riordan at al. [4] derived sharp estimates for the solution and its derivatives. Based on that information, in both papers, an error estimate for the simple upwind finite difference scheme on a properly chosen Shishkin mesh is given. In [1] Linss derived a general convergence theory for a first order inverse monotone scheme on arbitrary mesh and applied it on the Shishkin and Bakhvalov-type meshes. Roos and Uzelac [4] generated a uniformly convergent second order finite difference scheme using streamline diffusion as basic discretization. Vulanović [6] considered quasilinear two-parameter boundary value problem on Shishkin and Bakhvalov meshes and derived almost third-order scheme in the case of $\mu = \varepsilon^{1+p}$, $p > 0$.

2. Properties of the exact solution and its derivatives

In order to describe the layer structure of the problem (1) we present the following lemmas from [3].

Throughout the paper M will represent a constant independent of μ, ε and of the mesh.

The problem (1) satisfies the continuous minimum principle.

Lemma 1. *If $g \in C^2[0,1]$ such that $Lg \le 0$, $x \in (0,1)$ and $g(0) \ge 0$, $g(1) \ge 0$, then $g(x) \ge 0$, $x \in [0,1]$.*

Lemma 2. *The solution y of the problem (1) satisfies the following bounds*

$$\|y\| \le \max(|y(0)|, |y(1)|) + \frac{1}{b}\|f\|.$$

Lemma 3. *The derivatives $y^{(k)}$ of the solution of (1) satisfy the following bounds*

$$\|y^{(k)}\| \le \frac{M}{(\sqrt{\varepsilon})^k}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right)\max(\|y\|, \|f\|), \quad \text{for } k = 1, 2,$$

$$\|y^{(3)}\| \le \frac{M}{(\sqrt{\varepsilon})^3}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right)\max(\|y\|, \|f\|, \|f'\|).$$

The solution y has the representation

$$y = v + w_L + w_R, \tag{2}$$

where

$$\begin{aligned} Lv &= f, && v(0), v(1) \quad \text{chosen} \\ Lw_L &= 0, && w_L(0) = y(0) - v(0), \quad w_L(1) = 0 \\ Lw_R &= 0, && w_R(0) = 0, \quad w_R(1) = y(1) - v(1). \end{aligned}$$

Here v is the regular component of the solution and satisfy the following bounds

$$\|v^{(k)}\| \le M, \quad k = 0, 1, 2, \quad \|v^{(3)}\| \le M\varepsilon^{-1}.$$

The singular components w_L and w_R satisfy the bounds of Lemma 3 and the sharper bounds of the following lemma.

Lemma 4. *When the solution of (1) is decomposed as in (2) we have*

$$|w_L(x)| \leq Me^{-\theta_1 x}, \qquad |w_R(x)| \leq Me^{-\theta_2(1-x)}$$

where

$$\theta_1 = \frac{\mu a + \sqrt{\mu^2 a^2 + 4\varepsilon b}}{2\varepsilon}, \quad \theta_2 = \frac{-\mu A + \sqrt{\mu^2 A^2 + 4\varepsilon b}}{2\varepsilon}, \quad A = \max_{0\leq x\leq 1} |a(x)|.$$

θ_1 *and* θ_2 *are respective positive roots of the equations*

$$\varepsilon\theta_1^2 - \mu a\theta_1 - b = 0 \quad and \quad \varepsilon\theta_2^2 + \mu A\theta_2 - b = 0.$$

The following estimates for θ_1 and θ_2 are given in [3]:

$$\theta_1 \geq \max\left\{\frac{\sqrt{\beta}}{\sqrt{\varepsilon}}, \frac{\alpha\mu}{\varepsilon}\right\}, \quad \theta_2 \geq \left\{\begin{array}{lll} \frac{M}{\sqrt{\varepsilon}} & if & \mu \leq M\sqrt{\varepsilon} \\ \frac{M}{\mu} & if & \mu \geq M\sqrt{\varepsilon} \end{array}\right. .$$

3. Discrete problem

In order to obtain discrete counterpart of the problem (1), we first discretize the domain I as $\Delta_n = \{x_0 = 0 < x_1 < x_2 < \ldots < x_n = 1\}$.

The piecewise uniform mesh is defined as follows. Let our discretization parameter n be a positive integer divisible with 4. We define two mesh transition points

$$\sigma_1 = \min\{\frac{1}{4}, \frac{2}{\theta_1}\ln n\}, \qquad \sigma_2 = \min\{\frac{1}{4}, \frac{2}{\theta_2}\ln n\}.$$

Then divide $[0, \sigma_1]$ and $[1-\sigma_2, 1]$ into $n/4$ subintervals each, while $[\sigma_1, 1-\sigma_2]$ divide into $n/2$ subintervals. The mesh points are given by

$$x_i = \left\{\begin{array}{ll} \frac{4\sigma_1 i}{n}, & i \leq \frac{n}{4} \\ \sigma_1 + \frac{2}{n}(i - \frac{n}{4})(1-\sigma_1-\sigma_2), & \frac{n}{4} \leq i \leq \frac{3n}{4} \\ 1 - \sigma_2 + (i - \frac{3n}{4})\frac{4\sigma_2}{n}, & \frac{3n}{4} \leq i \leq n, \end{array}\right.$$

The solution $y(x)$ of the problem (1) is approximated with the quadratic spline $u(x) \in C^2(I)$ on each subinterval $I_i = [x_i, x_{i+1}]$, $i = 0, 1, ..., N-1$:

$$u(x) = u_i + (x - x_i)u_i' + \frac{(x-x_i)^2}{2}u_i''. \tag{3}$$

Then we have

$$u_{i+1} = h_i + h_{i+1}u_i' + \frac{h_{i+1}^2}{2}u_i'', \tag{4}$$

$$u_{i+1}' = u_i' + h_{i+1}u_i'' \tag{5}$$

where $h_i = x_i - x_{i-1}, u_i = u(x_i)$. For the collocation points we use the points

$$\xi_i = \alpha x_i + (1-\alpha)x_{i+1}, \quad 0 \leq \alpha \leq 1, \quad i = 0, ..., n-1.$$

The collocation equations have the form

$$\varepsilon u''(\xi_i) + \mu a(\xi_i)u'(\xi_i) - b(\xi_i)u(\xi_i) = f(\xi_i), \tag{6}$$

$$u(\xi_i) = u_i + (1-\alpha)h_{i+1}u_i' + (1-\alpha)^2\frac{h_{i+1}^2}{2}u_i'' \tag{7}$$

$$u'(\xi_i) = u_i' + (1-\alpha)h_{i+1}u_i'' \tag{8}$$

for $i = 0, ...n-1$, where $u''(\xi_i) = u_i''$. From (4)-(8) we obtain the system of equations

$$\begin{aligned} L_n u_i \equiv \quad & R_i^- u_{i-1} + R_i^c u_i + R_i^+ u_{i+1} = q^- f_{\xi_{i-1}} + q^+ f_{\xi_i}, \\ u_0 = \quad & p_0, \quad u_n = p_1, \end{aligned} \tag{9}$$

where $f_{\xi_i} = f(\xi_i)$ and

$$R_i^- = 1 + \frac{h_i}{2C_{i-1}}b_{\xi_{i-1}} + \frac{A_{i-1}}{C_{i-1}h_i}, \quad R_i^+ = -\frac{h_i A_i}{h_{i-1}^2 C_i},$$

$$R_i^c = -1 + \frac{h_i}{2C_i}b_{\xi_i} + \frac{A_i h_i}{h_{i+1}^2 C_i} - \frac{A_{i-1}}{h_i C_{i-1}},$$

$$q^- = -\frac{h_i}{2C_{i-1}}, \qquad q^+ = -\frac{h_i}{2C_i},$$

$$C_i = \frac{1}{h_{i+1}}(2\varepsilon + h_{i+1}\mu a_{\xi_i}(-1+2\alpha) - b_{\xi_i}\alpha(1-\alpha)h_{i+1}^2)$$

$$A_i = \varepsilon + \mu a_{\xi_i}(1-\alpha)h_{i+1} - b_{\xi_i}\frac{(1-\alpha)^2}{2}h_{i+1}^2.$$

For the standard collocation method we have $\alpha = \frac{1}{2}$. But the discrete analogue does not satisfy the discrete minimum principle, i.e. the matrix of the system (9) is not an inverse monotone matrix.

In order to obtain the inverse monotone matrix we introduce the parameter α which moves the collocation points so that

$$R_i^+ \geq 0, \quad R_i^- \geq 0, \quad R_i^c < 0.$$

Since $R_i^- = \frac{S_{i-1}}{2C_{i-1}h_i}$ and $C_{i-1} < 0$ for $\alpha \leq \frac{1}{2}$ we determine α from the condition

$$S_{i-1} = -2\varepsilon + 2\mu a_{\xi_{i-1}}\alpha h_i + b_{\xi_{i-1}}h_i^2\alpha^2 \leq 0.$$

From $R_i^+ = \frac{h_i}{2h_{i+1}^2}\frac{Q_i}{C_i} \geq 0$ we obtain

$$Q_i = S_i - 2\mu a_{\xi_i}h_{i+1} + b_{\xi_i}h_{i+1}^2(1-2\alpha) \leq 0,$$

and finally, α is determined to satisfy the inequality

$$S_i \leq \min(0, 2\mu a_{\xi_i}h_{i+1} - \|b\|h_{i+1}^2), \quad \frac{n}{4} \leq i \leq n. \tag{10}$$

The following theorem holds.

Theorem 1. *(Discrete Minimum Principle) Let us determine the parameter α in the system (9) to satisfy (10) for $i \geq \frac{n}{4}$ and $\alpha = \frac{1}{2}$ otherwise. If W is any mesh function with properties $L_n W \leq 0$, $W_0 \geq 0$, $W_n \geq 0$, then $W \geq 0$.*

Proof. Using the inverse monotonicity of the matrix the proof follows immediately. □

We use the discrete decomposition of the solution U,

$$U = V + W_L + W_R,$$

where

$$L_n V = f(x_i), \quad V(0) = v(0), \quad V(1) = v(1),$$
$$L_n W_L = 0, \quad W_L(0) = w_L(0), \quad W_L(1) = 0,$$
$$L_n W_R = 0, \quad W_R(0) = 0, \quad W_R(1) = w_R(1).$$

We will prove the following bounds on the discrete counterparts of the singular components w_L and w_R.

Theorem 2. *(The Estimates) Let α be determened as in Theorem 1 and let $a(x) = a = const, b(x) = b = const$. Let $\theta_L h_{j+1} \geq 1$, then we have*

$$|W_L(x_j)| \leq M \prod_{i=1}^{j} (1 + \theta_L h_i)^{-1} = \psi_{L,j}, \quad \psi_{L,0} = M,$$

$$|W_R(x_j)| \leq M \prod_{i=j+1}^{n} (1 + \theta_R h_i)^{-1} = \psi_{R,j}, \; \psi_{R,0} = M,$$

where parameters θ_L and θ_R are defined to be the positive roots of the equations

$$2\varepsilon\theta_L^2 - \mu a \theta_L - b = 0, \quad 2\varepsilon\theta_R^2 + \mu A \theta_R - b = 0. \tag{11}$$

Proof. We start with W_L. Consider the function

$$\Phi_{L,j} = \psi_{L,j} \pm W_L(x_j).$$

Then we have

$$L_n \Phi_{L,j} = M\psi_{L,j+1}(\delta + \gamma + R^- \theta_L^2 h_j h_{j+1})$$

where

$$\delta = R^- + R^+ + R^C = \frac{h_i b_{\xi_{i-1}}}{2C_{i-1}} + \frac{h_i b_{\xi_i}}{2C_i} < 0,$$

$$\gamma = \theta_L (R^- h_j + R^- h_{j+1} + R^C h_{j+1}).$$

From

$$\gamma = \theta_L \left(\frac{h_j h_{j+1}}{2C_{j-1}} b_{\xi_{j-1}} + \frac{h_j h_{j+1}}{2C_j} b_{\xi_j} - \frac{h_j h_{j+1}(1-\alpha)^2}{2C_j} b_{\xi_j} + \frac{\alpha^2 h_j^2}{2C_{j-1}} b_{\xi_{j-1}} \right.$$

$$-\frac{\varepsilon}{C_{j-1}} - \frac{\mu\alpha h_j}{C_{j-1}} a_{\xi_{j-1}} + \frac{\mu(1-\alpha)h_j}{C_j} a_{\xi_j}\Bigg)$$

and (11) we have

$$\gamma \le \theta_L \left(\frac{h_j b_{\xi_{j-1}}}{2C_{j-1}}(h_{j+1} + \alpha^2 h_j) - \frac{\varepsilon}{C_{j-1}} - \frac{\mu a_{\xi_{j-1}} \alpha h_j}{C_{j-1}} \right.$$
$$\left. + \frac{\mu a_{\xi_j}(1-\alpha)h_j^2}{C_j} - \frac{h_{j+1}\mu a}{2C_{j-1}} \right) + \frac{h_{j+1}\theta_L^2 \varepsilon}{C_{j-1}} - \frac{b h_{j+1}}{2C_{j-1}}.$$

Further

$$L_n \Phi_{L,j} \le M\psi_{L,j+1} \left[\frac{h_j b_{\xi_{j-1}}}{2C_{j-1}} + \frac{h_j b_{\xi_j}}{2C_j} - \frac{b h_{j+1}}{2C_{j-1}} \right.$$
$$+\theta_L \left(\frac{h_j \varepsilon}{h_{j+1} C_j} - \frac{\varepsilon}{C_{j-1}} - \frac{\mu\alpha h_j a_{\xi_{j-1}}}{C_{j-1}} - \frac{\mu\alpha h_j a_{\xi_j}}{C_j} - \frac{\mu a h_{j+1}}{2C_{j-1}} + \frac{\mu h_j a_{\xi_j}}{C_j} \right)$$
$$\left. + \frac{\theta_L^2 h_{j+1}}{2C_{j-1}} (2\mu a_{\xi_{j-1}} \alpha h_j + b_{\xi_{j-1}} \alpha^2 h_j^2) \right].$$

Since $\theta_L h_{j+1} \ge 1$ and $\alpha \le \frac{1}{2}$ and we obtain

$$\text{(12)} \qquad L_n \Phi_{L,j} \le M\psi_{L,j+1} \left[\frac{h_j b_{\xi_{j-1}}}{2C_{j-1}} + \frac{h_j b_{\xi_j}}{2C_j} - \frac{b h_{j+1}}{2C_{j-1}} \right.$$
$$\left. +\theta_L \left(\frac{h_j \varepsilon}{h_{j+1} C_j} - \frac{\varepsilon}{C_{j-1}} + \frac{\mu h_j a_{\xi_j}}{2C_j} - \frac{\mu a h_{j+1}}{2C_{j-1}} \right) + \frac{\theta_L^2 h_{j+1}}{2C_{j-1}} b_{\xi_{j-1}} \alpha^2 h_j^2 \right].$$

Now for $a(x) = const$ and $b(x) = const$, we have

$$\text{(13)} \qquad L_n \Phi_{L,j} \le M\psi_{L,j+1} \left(\frac{h_j b_{\xi_{j-1}}}{2C_{j-1}} + \frac{\theta_L^2 h_{j+1}}{2C_{j-1}} b_{\xi_{j-1}} \alpha^2 h_j^2 \right) < 0.$$

Using the discrete minimum principle we obtain the results. The same idea applies to W_R. □

Remark 1. Note that the condition $\theta_L h_{j+1} \ge 1$ is not restrictive. In the case when this condition is not satisfied the standard scheme can be applied.

4. Numerical results

We test the performance of our method applied to the boundary value problem

$$\varepsilon y''(x) + \mu y'(x) - y(x) = 1, \quad x \in (0,1)$$
$$y(0) = y(1) = 0.$$

The exact solution is given with

$$y(x) = \frac{e^{\frac{-\mu+\sqrt{\mu^2+4\varepsilon}}{2\mu}} - 1}{e^{\frac{\sqrt{\mu^2+4\varepsilon}}{\mu}} - 1} \, e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\mu}(1-x)} + \frac{e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\mu}} - 1}{e^{\frac{\sqrt{\mu^2+4\varepsilon}}{\mu}} - 1} \, e^{\frac{-\mu+\sqrt{\mu^2+4\varepsilon}}{2\mu} x} - 1.$$

The error is measured in the discrete maximum norm

$$E^n_{\varepsilon,\mu} = \max_{x_i \in \Delta_n} |y(x_i) - u_i|,$$

and order of convergence is calculated using

$$p^n_{\varepsilon,\mu} = \log_2 \frac{E^n_{\varepsilon,\mu}}{E^{2n}_{\varepsilon,\mu}}.$$

ε	N					
	32	64	128	256	512	1024
2^{-2}	4.15(-5)	1.04(-5)	2.59(-6)	6.49(-7)	1.62(-7)	4.05(-8)
2^{-4}	2.26(-4)	5.65(-5)	1.41(-5)	3.53(-6)	8.82(-7)	2.20(-7)
2^{-6}	1.37(-3)	3.40(-4)	8.49(-5)	2.12(-5)	5.31(-6)	1.33(-6)
2^{-8}	1.22(-2)	2.92(-3)	7.37(-4)	1.84(-4)	4.59(-5)	1.15(-5)
2^{-10}	7.89(-2)	2.89(-2)	8.84(-3)	2.19(-3)	5.50(-4)	1.36(-4)
2^{-12}	9.55(-2)	3.44(-2)	1.05(-2)	3.43(-3)	1.08(-3)	3.31(-4)
2^{-14}	1.02(-1)	3.67(-2)	1.12(-2)	3.64(-3)	1.14(-3)	3.52(-4)
2^{-16}	1.04(-1)	3.74(-2)	1.14(-2)	3.71(-3)	1.16(-3)	3.58(-4)

Table 1. Errors for $\mu = 2^{-4}$

ε	N				
	32	64	128	256	512
2^{-2}	2.00	1.99	2.00	2.00	2.
2^{-4}	2.00	1.99	2.00	2.00	2.
2^{-6}	2.00	2.00	1.99	2.00	2.00
2^{-8}	2.07	1.98	2.00	2.00	2.00
2^{-10}	1.45	1.71	2.01	2.00	2.00
2^{-12}	1.47	1.71	1.62	1.67	1.70
2^{-14}	1.47	1.71	1.62	1.67	1.70
2^{-16}	1.48	1.71	1.62	1.67	1.70

Table 2. Estimated convergence orders for $\mu = 2^{-4}$

	N				
ε	128	256	512	1024	2048
2^{-2}	2.51(-6)	6.27(-7)	1.57(-7)	3.92(-8)	9.80(-9)
2^{-4}	1.04(-5)	2.61(-6)	6.52(-7)	1.63(-7)	4.07(-8)
2^{-6}	3.10(-5)	7.74(-6)	1.93(-6)	4.84(-7)	1.21(-7)
2^{-8}	1.25(-4)	3.11(-4)	7.78(-6)	1.94(-6)	4.86(-7)
2^{-10}	5.21(-4)	1.29(-4)	3.23(-5)	8.08(-6)	2.02(-6)
2^{-12}	2.30(-3)	5.61(-4)	1.40(-4)	3.48(-5)	8.71(-6)
2^{-14}	2.30(-2)	1.10(-3)	3.47(-4)	1.07(-4)	3.23(-5)

Table 3. Errors for $\mu = 2^{-10}$

	N			
ε	128	256	512	1024
2^{-2}	2.00	2.00	2.	2.00
2^{-4}	2.00	2.00	1.99	2.00
2^{-6}	2.00	2.00	2.00	1.99
2^{-8}	2.00	2.00	2.00	1.99
2^{-10}	2.00	2.00	2.00	1.99
2^{-12}	2.03	2.00	2.00	1.99
2^{-14}	1.67	1.67	1.70	1.72

Table 4. Estimated convergence orders for $\mu = 2^{-10}$

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