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ON TRACES OF MAXIMAL CLONES

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Abstract.

In this paper we study the partially ordered set of endomorphism monoids of Rosenberg relations, which are used to characterize maximal clones on a finite set. The problem naturally splits into 49 cases of interrelationships of endomorphism monoids of Rosenberg relations. In order to make the paper self-contained, in addition to new results, we survey some previously published partial results providing therefore the complete solution to all but five cases.

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1. Introduction

In this paper we study the structure of the poset of traces of maximal clones on a finite set. This research was motivated by the papers [1] and [4], where the completeness for some special structures (concrete near-rings) was investigated using techniques from clone theory. It appeared that unary parts (traces) of the maximal clones that contain the operation + correspond to the maximal near-rings containing the identity map. Moreover, if for every two distinct unary parts $M_i^{(1)}$ and $M_j^{(1)}$ of such maximal clones we have $M_i^{(1)} \not\subseteq M_j^{(1)}$, then every unary part is a maximal near-ring. It is natural to ask what goes on in the general case, i.e. what the relationship between any two traces of maximal clones on a finite set is. As was expected, the width of this poset is doubly exponential, but it was rather surprising to find out that its height is equal to the size of the underlying set. Moreover, it turned out that the structure of this poset is quite rich.

Some of the results related to endomorphism monoids of central and regular relations can be found in [6], [8] and [5], and we quote them here without proof.

In Section 2 we fix the notions and notation used in the paper. Some properties of endomorphisms of regular relations are given in Section 3. The overview of the relationships between the traces of maximal clones is given in Section 4. The paper concludes with Section 5 in which we give some properties of the poset of traces of maximal clones.

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2. Preliminaries

Throughout the paper we assume that A is a finite set and $|A| \geqslant 3$. Let $O_A^{(n)}$ denote the set of all n-ary operations on A (so that $O_A^{(1)} = A^A$) and let $O_A := \bigcup_{n \geqslant 1} O_A^{(n)}$ denote the set of all finitary operations on A. For $F \subseteq O_A$ let $F^{(n)} := F \cap O_A^{(n)}$ be the set of all n-ary operations in F. A set $C \subseteq O_A$ of finitary operations is a *clone of operations on* A if it contains all projection maps $\pi_i^n : A^n \to A : (x_1, \dots, x_n) \mapsto x_i$ and is closed with respect to composition of functions in the following sense: whenever $g \in C^{(n)}$ and $f_1, \dots, f_n \in C^{(m)}$ for some positive integers m and n then $g(f_1, \dots, f_n) \in C^{(m)}$, where the composition $h := g(f_1, \dots, f_n)$ is defined by $h(x_1, \dots, x_m) := g(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m))$.

For a clone C, the unary part $C^{(1)}$ of C, will be referred to as *the trace of* C.

We say that an n-ary operation f preserves an h-ary relation ϱ if the following holds:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{h1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{h2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{hn} \end{bmatrix} \in \varrho \text{ implies } \begin{bmatrix} f(a_{11}, a_{12}, \dots, a_{1n}) \\ f(a_{21}, a_{22}, \dots, a_{2n}) \\ \vdots \\ f(a_{h1}, a_{h2}, \dots, a_{hn}) \end{bmatrix} \in \varrho.$$

For a set Q of relations and for a set F of operations let

$$\mathrm{Pol}Q \quad := \quad \{f \in O_A \mid f \text{ preserves every } \varrho \in Q\}.$$

Let $\mathrm{Pol}_n Q := (\mathrm{Pol}Q) \cap O_A^{(n)}$. For an h-ary relation $\theta \subseteq A^h$ and a unary operation $f \in A^A$ it is convenient to write

$$f(\theta) := \{(f(x_1), \dots, f(x_h)) \mid (x_1, \dots, x_h) \in \theta\}.$$

Then clearly f preserves θ if and only if $f(\theta) \subseteq \theta$. It follows that $\mathrm{Pol}_1 Q$ is the endomorphism monoid of the relational structure $\langle A, Q \rangle$. Therefore instead of $\mathrm{Pol}_1 Q$ we simply write $\mathrm{End}Q$. Also, we denote by $\mathrm{Aut}Q$ the automorphism group of the relational structure $\langle A, Q \rangle$, i.e. $\mathrm{Aut}Q := S_A \cap \mathrm{Pol}Q$, where S_A is the full symmetric group on A.

If the underlying set is finite and has at least three elements, then the lattice of clones has cardinality $2^{\aleph_0}$. However, one can show that the lattice of clones on a finite set has a finite number of coatoms, called *maximal clones*, and that every clone distinct from O_A is contained in one of the maximal clones. One of the most influential results in clone theory is the explicite characterization of the maximal clones, obtained by I. G. Rosenberg as the culmination of the work of many mathematicians. It is usually stated in terms of the following six classes of finitary relations on A (the so-called *Rosenberg relations*).

(R1) *Bounded partial orders.* These are partial orders on A with a least and a greatest element.

(R2) *Nontrivial equivalence relations.* These are equivalence relations on A distinct from $\Delta_A := \{(x,x) \mid x \in A\}$ and A^2.

(R3) *Permutational relations.* These are relations of the form $\{(x,\pi(x)) \mid x \in A\}$ where π is a fixpoint-free permutation of A with all cycles of the same length p, where p is a prime.

(R4) *Affine relations.* For a binary operation $\oplus$ on A let

$$\lambda_\oplus := \{\langle x,y,u,v\rangle \in A^4 \mid u \oplus v = x \oplus y\}.$$

A relation ϱ is called *affine* if there is an elementary Abelian p-group $\langle A, \oplus, \ominus, 0\rangle$ on A such that $\varrho = \lambda_\oplus$.

Suppose now that A is an elementary Abelian p-group. Then it is well-known that $f \in \mathrm{Pol}\{\lambda_\oplus\}$ if and only if

$$f(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = f(x_1,\ldots,x_n) \oplus f(y_1,\ldots,y_n) \ominus f(0,\ldots,0)$$

for all $x_i, y_i \in A$. In case f is unary, this condition becomes

$$f(x \oplus y) = f(x) \oplus f(y) \ominus f(0).$$

(R5) *Central relations.* All unary relations are central relations. For central relations ϱ of arity $h \geqslant 2$ the definition is as follows: ϱ is said to be *totally symmetric* if $(x_1,\ldots,x_h) \in \varrho$ implies $(x_{\pi(1)},\ldots,x_{\pi(h)}) \in \varrho$ for all permutations π, and it is said to be *totally reflexive* if $(x_1,\ldots,x_h) \in \varrho$ whenever there are $i \neq j$ such that $x_i = x_j$. An element $c \in A$ is *central* if $(c, x_2, \ldots, x_h) \in \varrho$ for all $x_2,\ldots,x_h \in A$. Finally, $\varrho \neq A^h$ is called *central* if it is totally reflexive, totally symmetric and has a central element. According to this, every central relation can be written as $C \cup R \cup T$, where C consists of all the tuples of distinct elements containing at least one central element (the central part), R consists of all the tuples $(x_1,\ldots,x_h)$ such that there are $i \neq j$ with $x_i = x_j$ (the reflexive part) and T consists of all the tuples $(x_1,\ldots,x_h)$ such that $x_1,\ldots,x_h$ are distinct non-central elements. We will call T *the tail* of ϱ.

(R6) *h-regular relations.* Let $\Theta = \{\theta_1,\ldots,\theta_m\}$ be a family of equivalence relations. We say that Θ is an *h-regular family* if every θ_i has precisely h blocks, and additionally, if B_i is an arbitrary block of θ_i for $i \in \{1,\ldots,m\}$, then $\bigcap_{i=1}^m B_i \neq \emptyset$.

An h-ary relation $\varrho \neq A^h$ is h-regular if $h \geqslant 3$ and there is an h-regular family Θ such that $(x_1,\ldots,x_h) \in \varrho$ if and only if for all $\theta \in \Theta$ there are distinct i, j with $(x_i, x_j) \in \theta$.

Note that regular relations are totally reflexive and totally symmetric.

Theorem 2.1. *[Rosenberg [10]] A clone M of operations on a finite set is maximal if and only if there is a relation ϱ from one of the classes (R1)–(R6) such that $M = \mathrm{Pol}\{\varrho\}$.*

Let $f \in O_A^{(n)}$ for some $n \in N$. We say that the ith argument of f is *essential* if there exist $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b, c \in A$ with $b \neq c$ such that

$$f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n),$$

i.e. the value of f is not independent of the ith argument. We shall call an operation f a *Słupecki operation* if it is surjective and has at least two essential arguments. It is easy to check that the set U_A of all non–Słupecki operations (that is, operations that are essentially unary or not surjective) is a clone. Moreover this clone is maximal, and it is the only maximal clone that contains all unary maps. From this it follows that the partially ordered set of traces of maximal clones has the greatest element.

Let us recall the notion of the irreducible element in a partially ordered set. Let $(A, \leqslant)$ be a partially ordered set. We say that an element b *covers* an element a ($a \prec b$) if $a < b$ and there is no $c \in A$ such that $a < c < b$. An $a \in A$ is $\wedge$*–irreducible* if there exists exactly one element $b \in A$ such that $a \prec b$. Analogously, an $a \in A$ is $\vee$*–irreducible* if there exists exactly one element $b \in A$ such that $b \prec a$. Finally, an element a is irreducible if it is $\wedge$–irreducible or $\vee$–irreducible and in that case we say that b is the *primary neighbor* of a. Note that an arbitrary element may have 0, 1 or 2 primary neighbors.

3. Some properties of endomorphisms of regular relations

Note first that there is another way to define regular relations. Given a finite set A, $|A| \geqslant 3$, and an h-regular family $\Theta = \{\theta_1, \dots, \theta_m\}$ on the set A, let

$$R_\Theta = \{(x_1, \dots, x_h) \mid (\forall \theta \in \Theta)(\exists i \neq j) x_i \theta x_j\}$$

denote the corresponding h-regular relation. Now, take the set $\{1, \dots, h\}^m$. We define the *elementary (h, m)-relation* $\Psi_{h,m}$ on this set in the following way:

$$\Psi_{h,m} = \left\{ \left(\begin{bmatrix} a_1^1 \\ \vdots \\ a_m^1 \end{bmatrix}, \dots, \begin{bmatrix} a_1^h \\ \vdots \\ a_m^h \end{bmatrix} \right) \mid (\forall i \in \{1, \dots, m\})(\exists j \neq k) a_i^j = a_i^k \right\}.$$

Note that the elementary (h, m)-relation is the h-regular relation on the set $\{1, \dots, h\}^m$ defined by the h-regular family $\Theta^* = \{\theta_1^*, \dots, \theta_m^*\}$, where

$$\theta_i^* = \left\{ \left(\begin{bmatrix} b_1^1 \\ \vdots \\ b_m^1 \end{bmatrix}, \begin{bmatrix} b_1^2 \\ \vdots \\ b_m^2 \end{bmatrix} \right) \mid b_i^1 = b_i^2 \right\}.$$

Then, there exists a surjective mapping $\lambda : A \to \{1, \dots, h\}^m$ such that

$$R_\Theta = \{(x_1, \dots, x_h) \mid (\lambda(x_1), \dots, \lambda(x_h)) \in \Psi_{h,m}\}.$$

The complete characterization of all mappings that preserve regular relations can be found in [3]. We present this result without proof.

Denote by $\overrightarrow{x}^{(i)}$ the i-th coordinate of the vector $\overrightarrow{x}$, which is an element of the set $\{1, \ldots, h\}^m$. Let f be an n-ary function on the set A. We define the function $f_i' : A^n \to \{1, \ldots, h\}$ in the following way:

$$f_i'(x_1, \ldots, x_n) := (\lambda(f(x_1, \ldots, x_n)))^{(i)}.$$

Proposition 3.1. *[3] An n-ary function f on a set A preserves an h-regular relation R_Θ if and only if for each function f_i' either f_i' has at most $h-1$ distinct values or there exist a permutation s on $\{1, \ldots, h\}$, a $j \in \{1, \ldots, n\}$ and a $v \in \{1, \ldots, m\}$ such that*

$$f_i'(x_1, \ldots, x_n) := s((\lambda(x_j))^{(v)}).$$

We continue with some properties of h-regular and elementary (h, m)-relations.

Lemma 3.2. *[5] Let $A = \{1, \ldots, h\}^m$, $m \geqslant 2$ and let $\Psi_{h,m}$ be an elementary h-regular relation on A. Then for every pair of distinct elements $\overrightarrow{c}, \overrightarrow{d} \in A$ there exist elements $\overrightarrow{e^2}, \ldots, \overrightarrow{e^h}$ such that $\overrightarrow{d} \notin \{\overrightarrow{e^2}, \ldots, \overrightarrow{e^h}\}$, $(\overrightarrow{d}, \overrightarrow{e^2}, \ldots, \overrightarrow{e^h}) \in \Psi_{h,m}$ and $(\overrightarrow{c}, \overrightarrow{e^2}, \ldots, \overrightarrow{e^h}) \notin \Psi_{h,m}$.*

Lemma 3.3. *[5] Let $\Theta = \{\theta\}$ be an h-regular family and let R_Θ be the h-regular relation defined by Θ. Then* $\mathrm{End}\{\theta\} \subseteq \mathrm{End}\{R_\Theta\}$.

Lemma 3.4. *Let $\Theta = \{\theta_1, \ldots, \theta_m\}$ be an h–regular family and let R_Θ be the h–regular relation defined by Θ. Then*

$$\cap_{i=1}^m \theta_i = \{(a, b) \mid (\forall c_3, \ldots, c_h \in A)(a, b, c_3, \ldots, c_h) \in R_\Theta\}.$$

Proof. $(\subseteq)$ Let $(a, b) \in \cap_{i=1}^m \theta_i$ and take arbitrary $c_3, \ldots, c_h \in A$. Then for every i, $1 \leqslant i \leqslant m$, we have $(a, b) \in \theta_i$, so $(a, b, c_3, \ldots, c_h) \in R_\Theta$ by definition.

$(\supseteq)$ Suppose that $(a, b) \notin \cap_{i=1}^m \theta_i$. Then there exists an i, $1 \leqslant i \leqslant m$, such that $(a, b) \notin \theta_i$. Let $T_1^i, \ldots, T_h^i$ be the equivalence classes of θ_i and let $a \in T_1^i$, $b \in T_2^i$. For arbitrary choice of $c_j \in T_j^i$, $3 \leqslant j \leqslant h$, we have that $(a, b, c_3, \ldots, c_h) \notin R_\Theta$–contradiction. □

Lemma 3.5. *Let $\Theta = \{\theta_1, \ldots, \theta_m\}$ be an h–regular family, let R_Θ be the h–regular relation defined by Θ and let $\Phi = \cap_{i=1}^m \theta_i$. Then* $\mathrm{Aut}\{R_\Theta\} \subseteq \mathrm{Aut}\{\Phi\}$.

Proof. Let $f \in \mathrm{Aut}\{R_\Theta\}$. To see that $f \in \mathrm{Aut}\{\Phi\}$, it suffices to prove that $(f(a), f(b)) \in \Phi$ for every $(a, b) \in \Phi$. Take arbitrary elements $c_3, \ldots, c_h \in A$. Then $(a, b, f^{-1}(c_3), \ldots, f^{-1}(c_h)) \in R_\Theta$ (by Lemma 3.4) and

$$(f(a), f(b), f(f^{-1}(c_3)), \ldots, f(f^{-1}(c_h))) = (f(a), f(b), c_3, \ldots, c_h) \in R_\Theta,$$

so $(f(a), f(b)) \in \Phi$ by definition. $\square$

Lemma 3.6. *Let $\Theta = \{\theta_1, \dots, \theta_m\}$ be an h–regular family, let R_Θ be the h–regular relation defined by Θ and let $\Phi = \cap_{i=1}^{m} \theta_i$. If $\Phi = \Delta_A$, then $A \cong A/\theta_1 \times \cdots \times A/\theta_m$.*

Proof. Let $\varphi : A \to A/\theta_1 \times \cdots \times A/\theta_m$ be the mapping defined by $\varphi(a) = ([a]_{\theta_1}, \dots, [a]_{\theta_m})$. We are going to show that φ is bijective. Let $\varphi(a) = \varphi(b)$. Then $([a]_{\theta_1}, \dots, [a]_{\theta_m}) = ([b]_{\theta_1}, \dots, [b]_{\theta_m})$. It follows that $[a]_{\theta_i} = [b]_{\theta_i}$, for every i, $1 \leqslant i \leqslant m$, so $(a, b) \in \cap_{i=1}^{m} \theta_i = \Phi$. Since $\Phi = \Delta_A$, we obtain that $a = b$. Hence, φ is injective. Let $(T_1, \dots, T_m) \in A/\theta_1 \times \cdots \times A/\theta_m$. Since Θ is h–regular relation, we have $T := \cap_{i=1}^{m} T_i \neq \emptyset$. Moreover, for all $a, b \in T$ we have $(a, b) \in \Phi$. Hence, $a = b$ and $T = \{a\}$. Now, $\varphi(a) = (T_1, \dots, T_m)$ by construction. Hence, φ is surjective. Altogether φ is bijective. Hence $A \cong A/\theta_1 \times \cdots \times A/\theta_m$. $\square$

4. Characterizations

The results are summarized in the following table:

$\varrho \backslash \sigma$	Bounded partial order	Equivalence relation	Permutational relation	Affine relation	Unary central relation	k–ary central relation, $k \geqslant 2$	h–regular relation
Bounded partial order	$-$ 4.1	$-$ 4.6	$-$ 4.13	$-$ 4.14	$-$ 4.19	$\pm?$ [8, 4.10]	$\pm?$ 4.27
Equivalence relation	$-$ 4.2	$-$ 4.7	$-$ 4.13	$-$ 4.15	$-$ 4.19	$-$ [8, 4.11]	$\pm$ [5, 4.1]
Permutational relation	$-$ 4.3	$\pm$ 4.9	$-$ 4.13	$\pm$ 4.16	$-$ 4.19	$-$ [8, 4.12]	$\pm$ 4.30
Affine relation	$-$ 4.4	$-$ 4.10	$-$ 4.13	$-$ 4.17	$-$ 4.19	$-$ [8, 4.13]	$\pm$ 4.31
Unary central relation	$-$ [8, 4.2]	$\pm$ [8, 4.3]	$-$ [8, 4.4]	$-$ [8, 4.5]	$-$ [8, 4.1]	$\pm$ [8, 4.1]	$\pm$ [8, 4.6,4.8]
k–ary central relation, $k \geqslant 2$	$-$ [8, 4.2]	$\pm$ [8, 4.3]	$-$ [8, 4.4]	$-$ [8, 4.5]	$-$ [8, 3.2]	$\pm$ [6, 3.4]	$\pm?$ [8, 4.6,4.8]
h–regular relation	$-$ 4.5	$-$ 4.12	$-$ 4.13	$-$ 4.18	$-$ 4.19	$-?$ 4.24	$\pm?$ [5, 5.1]

Table 1.

The entries in this table are to be interpreted in the following way:

- we write $-$, if $\mathrm{End}\{\varrho\} \not\subseteq \mathrm{End}\{\sigma\}$ for every pair (ϱ, σ) of relations of indicated type;

- we write $\pm$, if $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$ in general, but there are ϱ and σ such that $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{\sigma\}$ and in that case we give necessary and sufficient condition;
- we write $\pm?$, if the work is still in progress, but there is a partial result;
- we write $-?$, if $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$ in general, but it is still unknown if there exist ϱ and σ such that $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{\sigma\}$.

Each of the subsections that follow presents results from one column of the table.

4.1. Rosenberg Relations vs. Bounded Partial Orders

In this subsection we are going to show that no trace of a maximal clone is subset of the trace of a maximal clone defined by a bounded partial order.

We assume that σ is a partial order on A with the least element 0 and the greatest element 1 and ϱ will range through the list of Rosenberg relations.

Proposition 4.1. *Let ϱ be a bounded partial order distinct from σ and σ^{-1}. Then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. The result follows immediately from Theorem 1 in [2]. □

Proposition 4.2. *Let ϱ be a nontrivial equivalence relation. Then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. Consider the following two functions:

$$f_{a,b}(x) = \begin{cases} a, & \text{if } x = b, \\ b, & \text{if } x = a, \\ x, & \text{otherwise,} \end{cases} \qquad g_{a,b}(x) = \begin{cases} a, & \text{if } x \in [b]_\varrho, \\ b, & \text{if } x \in [a]_\varrho, \\ x, & \text{otherwise.} \end{cases}$$

If there exist a and b such that $a <_\sigma b$ and $[a]_\varrho = [b]_\varrho$, then $f_{a,b}$ preserves ϱ but does not preserve σ. If for all $a <_\sigma b$ we have $[a]_\varrho \neq [b]_\varrho$, then for any such pair a, b the mapping $g_{a,b}$ preserves ϱ, but does not preserve σ. Hence, $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$. □

Proposition 4.3. *Let ϱ be a permutational relation arising from a p–regular permutation α. Then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. If 0 and 1 are contained in different cycles of the permutation α, say $\alpha = (0a_1 \dots a_{p-1})(1a_{p+1} \dots a_{2p-1}) \dots$ (so, $a_0 = 0$ and $a_p = 1$), then we take the mapping

$$f(x) = \begin{cases} a_{p+i}, & \text{if } x = a_i, 0 \leqslant i \leqslant p-1 \\ a_{i-p}, & \text{if } x = a_i, p \leqslant i \leqslant 2p-1 \\ x, & \text{otherwise.} \end{cases}$$

Since f exchanges the cycles $(0a_1 \dots a_{p-1})$ and $(1a_{p+1} \dots a_{2p-1})$, it follows that it preserves ϱ. On the other hand, we have $0 \leqslant_\sigma 1$, but $f(0) = 1 \not\leqslant_\sigma 0 = f(1)$, so σ is not preserved by f.

Suppose now that 0 and 1 are contained in the same cycle of α, say $\alpha = (0 \dots a_{j-1} 1 a_{j+1} \dots a_{p-1}) \dots$ Then the mapping

$$f(x) = \begin{cases} a_{i+j-1}, & \text{if } x = a_i, 0 \leqslant i \leqslant p-1 \\ x, & \text{otherwise,} \end{cases}$$

preserves ϱ, but does not preserve σ, since $0 \leqslant_\sigma 1$, but $f(0) = 1 \not\leqslant_\sigma a_{2j-1} = f(1)$ (indices are, of course, mod p). To conclude, $\operatorname{End}\{\varrho\} \nsubseteq \operatorname{End}\{\sigma\}$. □

Proposition 4.4. *Let ϱ be an affine relation. Then* $\operatorname{End}\{\varrho\} \nsubseteq \operatorname{End}\{\sigma\}$.

Proof. For the proof, we take the affine mapping $f(x) = 0 + 1 - x$. Now, $0 \leqslant_\sigma 1$, but $f(0) = 1 \not\leqslant_\sigma 0 = f(1)$ and σ is not preserved. □

Proposition 4.5. *Let ϱ be an h–regular relation. Then* $\operatorname{End}\{\varrho\} \nsubseteq \operatorname{End}\{\sigma\}$.

Proof. It is sufficient to take the mapping

$$f(x) = \begin{cases} 0, & \text{if } x = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Since $0 \leqslant_\sigma 1$ and $f(0) = 1 \not\leqslant_\sigma 0 = f(1)$, it follows that σ is not preserved. On the other hand, f preserves every totally reflexive at least ternary relation and, therefore, preserves h–regular relations. □

4.2. Rosenberg Relations vs. Equivalence Relations

Next we show that the traces of maximal clones defined by bounded partial orders, equivalence, affine and h–regular relations are not subsets of the trace of a maximal clone defined by any equivalence relation. However, it can happen that the trace of a maximal clone defined by a permutational or a central relation is a subset of the trace of a maximal clone defined by some equivalence relation. The complete characterizations of all such relations are given in Propositions 4.9 and 4.11.

In this subsection we assume that σ is a nontrivial equivalence relation and that ϱ ranges through the Rosenberg relations.

Proposition 4.6. *Let ϱ be a partial order with the least element* 0 *and the greatest element* 1. *Then* $\operatorname{End}\{\varrho\} \nsubseteq \operatorname{End}\{\sigma\}$.

Proof. Consider the following ϱ preserving mappings:

$$f_{a,b}(x) = \begin{cases} a, & \text{if } x = a, \\ b, & \text{otherwise,} \end{cases} \qquad g_a(x) = \begin{cases} 0, & \text{if } x \leqslant_\varrho a, \\ x, & \text{otherwise.} \end{cases}$$

If $[0]_\sigma \neq [1]_\sigma$ and $|[1]_\sigma| \geqslant 2$, then there is an $a \in [1]_\sigma$ such that $a \neq 1$. Clearly, $f_{1,0}$ does not preserve σ. Dually, if $[0]_\sigma \neq [1]_\sigma$ and $|[0]_\sigma| \geqslant 2$, then there is an $a \in [0]_\sigma$ such that $a \neq 0$. Analogously, $f_{0,1}$ does not preserve the equivalence relation σ.

Further, if $[0]_\sigma \neq [1]_\sigma$ and $|[0]_\sigma| = |[1]_\sigma| = 1$, then there is at least one equivalence class of σ with at least two elements, say a and b, since σ is a nontrivial equivalence relation. Then there are two possibilities: either $b \not\leqslant_\varrho a$ or $b \leqslant_\varrho a$. In the first case, we observe that g_a is not an endomorphism of σ. Analogously, the counterexample for the latter case is g_b.

Finally, if $[0]_\sigma = [1]_\sigma$, then there exists an $a \in A$ such that $a \notin [0]_\sigma$. However, in this case, $f_{1,a}$ does not preserve σ. To conclude, $\mathrm{End}\{\varrho\} \not\subseteq \mathrm{End}\{\sigma\}$. $\square$

Proposition 4.7. *Let ϱ be a nontrivial equivalence relation. Then* $\mathrm{End}\{\varrho\} \not\subseteq \mathrm{End}\{\sigma\}$.

Proof. Denote by $R_1, \dots, R_n$ the equivalence classes of ϱ and by $S_1, \dots, S_m$ the equivalence classes of σ. Now, we consider the following cases:

Case 1: There exists a class of ϱ which intersects at least two nontrivial classes of σ.

Let R_i, S_j, S_k be such that $|S_j|, |S_k| \geqslant 2$, $j \neq k$, and R_i has a nonempty intersection with both S_j and S_k. Let $a \in R_i \cap S_j$ and $b \in R_i \cap S_k$. Then

$$f(x) = \begin{cases} b, & \text{if } x = a, \\ x, & \text{otherwise} \end{cases}$$

clearly preserves ϱ. On the other hand, there exists a $c \in S_j \setminus \{a\}$, so $(a, c) \in \sigma$, but $(f(a), f(c)) = (b, c) \notin \sigma$. Therefore, $f(\sigma) \not\subseteq \sigma$.

Case 2: Every class of ϱ intersects at most one nontrivial class of σ. Now, we have two possibilities:

(1) $R_i \cap S_j = \emptyset$ whenever R_i and S_j are nontrivial. Let R_1 be a nontrivial class of ϱ and let S_1 be a nontrivial class of σ. Then for every element $c \in S_1$ we have $[c]_\varrho = \{c\}$. Now, take arbitrary $a \in S_1$, $b \in R_1$ and the mapping from the previous case. It clearly preserves ϱ and does not preserve σ.

(2) $R_i \cap S_j \neq \emptyset$ for some nontrivial R_i, S_j. Let $a \in R_i \cap S_j$.

If there is a $b \in R_i \setminus S_j$, then we take the same mapping as in (1) with the same conclusion. Otherwise, it follows that $R_i \subseteq S_j$. Now, if $R_i \subset S_j$, then there is some $c \in S_j \setminus R_i$. On the other hand, σ is not trivial, so, there is at least one more equivalence class, say S_k. Let $d \in S_k$. Now, we take the mapping

$$f(x) = \begin{cases} d, & \text{if } x \in R_i, \\ x, & \text{otherwise} \end{cases}$$

which preserves ϱ and does not preserve σ, since $(a, c) \in \sigma$ and $(f(a), f(c)) = (d, c) \notin \sigma$. Finally, if $R_i = S_j$, then at least one of the relations ϱ and σ has at least one more nontrivial class which is not a nontrivial class of both relations. If it has a nonempty intersection with some nontrivial class of the other relation, then we act as above. Otherwise, it is the union of some trivial classes of the other relation. Now, if that class is a class of σ, then we act as in (1). Otherwise, it is a class of ϱ, say R_l. Then take $p, q \in R_l$, $r \in R_i$ and define the mapping

$$f(x) = \begin{cases} p, & \text{if } x = r, \\ q, & \text{otherwise} \end{cases}$$

which preserves ϱ and does not preserve σ, since for $s \in S_j$, $s \neq r$, we have $(r, s) \in \sigma$ and $(f(r), f(s)) = (p, q) \notin \sigma$. Therefore, $f(\sigma) \nsubseteq \sigma$. □

In order to give the answer in the case of a permutational relation ϱ we need the following lemma.

Lemma 4.8. *Let ϱ be a permutational relation arising from a p–regular permutation α on A. If* $\mathrm{Aut}\{\varrho\} \subseteq \mathrm{Aut}\{\sigma\}$*, then all equivalence classes of σ are nontrivial.*

Proof. Let $\mathrm{Aut}\{\varrho\} \subseteq \mathrm{Aut}\{\sigma\}$. Suppose that there exists a trivial class $[a]_\sigma = \{a\}$ and let C_1 be a cycle of α such that $a \in C_1$. If C_1 contains an element b from a nontrivial class of σ, say $C_1 = (b \dots a \dots a_{p-1})$, where $a_0 = b$ and $a_m = a$, then just take the mapping

$$f(x) = \begin{cases} \alpha^m(x), & \text{if } x \in C_1, \\ x, & \text{otherwise.} \end{cases}$$

It is clear that $f \in \mathrm{Aut}\{\varrho\} \setminus \mathrm{Aut}\{\sigma\}$.

If C_1 is the union of trivial classes of σ, then there exists a cycle C_2 that contains an element b from some nontrivial class. Then $C_1 = (aa_1 \dots a_{p-1})$ and $C_2 = (ba_{p+1} \dots a_{2p-1})$, where $a_0 = a$ and $a_p = b$. Now,take the mapping

$$f(x) = \begin{cases} a_{p+i}, & \text{if } x = a_i, 0 \leqslant i \leqslant p-1 \\ a_{i-p}, & \text{if } x = a_i, p \leqslant i \leqslant 2p-1 \\ x, & \text{otherwise.} \end{cases}$$

It is clear that f preserves ϱ, but does not preserve σ, since the elements of $[b]_\sigma$ are mapped by f to the elements of at least two equivalence classes of σ. Hence, all equivalence classes of σ are nontrivial. □

Proposition 4.9. *Let ϱ be a permutational relation arising from a p–regular permutation α on A. Then* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{\sigma\}$ *if and only if every cycle of α is an equivalence class of σ.*

Proof. ($\Leftarrow$) Suppose that every cycle of α is an equivalence class of σ and let $f \in \operatorname{End}\{\varrho\}$. Since f preserves ϱ, f maps each cycle of α onto a cycle of α. But, then f maps every equivalence class of σ onto an equivalence class of σ, so σ is also preserved by f. Hence, $\operatorname{End}\{\varrho\} \subseteq \operatorname{End}\{\sigma\}$.

($\Rightarrow$) Suppose that $\operatorname{End}\{\varrho\} \subseteq \operatorname{End}\{\sigma\}$. Then $\operatorname{Aut}\{\varrho\} \subseteq \operatorname{Aut}\{\sigma\}$, so all equivalence classes of σ are nontrivial, by Lemma 4.8.

First, we are going to show that for every cycle C there exists an equivalence class S such that either C (considered as a set of elements) is a subset of S or S is a subset of C. Suppose that this is not the case. Then there exists a cycle C_j such that for every equivalence class S we have that C_j is not contained in S and S is not contained in C_j. Then C_j contains elements of at least two equivalence classes, say S_1 and S_2. Since S_1 is not contained in C_j, it follows that there exists an element $a \in S_1 \setminus C_j$, so there exists a cycle C_i such that $a \in C_i$. Let $b \in S_1 \cap C_j$ and $c \in S_2 \cap C_j$. Since $b, c \in C_j$ it follows that $c = \alpha^m(b)$ for some m. Now, take the mapping

$$f(x) = \begin{cases} \alpha^m(x), & \text{if } x \in C_j, \\ x, & \text{otherwise.} \end{cases}$$

It obviously preserves ϱ and does not preserve σ, since $(a, b) \in \sigma$ and $(f(a), f(b)) = (a, c) \notin \sigma$.

Further, if there exists a cycle C which consists of $k \geqslant 2$ equivalence classes of σ, since the permutation α preserves σ, it follows that all equivalence classes of C contain the same number of elements, say $n \geqslant 2$; then C contains $k \cdot n = p$ elements, where $k, n \geqslant 2$ and p is a prime number – contradiction.

Similarly, assume that there exists an equivalence class of σ, say S, which contains at least two cycles of α, say $C_1, C_2, \ldots$. Then there is a cycle $C_0 \not\subseteq S$, so a mapping that takes C_1 to C_0 in an appropriate way and leaves everything else fixed obviously preserves ϱ and does not preserve σ. Therefore, every cycle of α is an equivalence class of σ. □

Proposition 4.10. *Let ϱ be an affine relation. Then* $\operatorname{End}\{\varrho\} \not\subseteq \operatorname{End}\{\sigma\}$.

Proof. Let $[a]_\sigma$ be a nontrivial equivalence class of σ, $b \in [a]_\sigma$, $b \neq a$ and let $c \in A \setminus [a]_\sigma$. Then there is an affine mapping f which takes b to c and leaves a fixed. Clearly, σ is not preserved by f, so $\operatorname{End}\{\varrho\} \not\subseteq \operatorname{End}\{\sigma\}$. □

The following result was obtained in [8]. We repeat it here without proof.

Proposition 4.11. *Let ϱ be a central relation on A whose center is C_ϱ.*

Then $\operatorname{End}\{\varrho\} \subseteq \operatorname{End}\{\sigma\}$ *if and only if ϱ is a unary or a binary central relation with no tail such that C_ϱ is a nontrivial equivalence class of σ and all other equivalence classes of σ are trivial. (Note that for unary central relations $C_\varrho = \varrho$ and that they have no tail by definition.)*

Proposition 4.12. *Let ϱ be an h–regular relation. Then* $\operatorname{End}\{\varrho\} \not\subseteq \operatorname{End}\{\sigma\}$.

Proof. Let $[a]_\sigma$ be a nontrivial equivalence class, so there is $b \in [a]_\sigma$, $b \neq a$, and let $c \notin [a]_\sigma$ (such a c exists because σ is nontrivial). Then the mapping

$$f(x) = \begin{cases} a, & \text{if } x = a, \\ c, & \text{otherwise} \end{cases}$$

preserves every totally reflexive at least ternary relation. On the other hand, $(a, b) \in \sigma$, but $(f(a), f(b)) = (a, c) \notin \sigma$, so f does not preserve σ. Hence, $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$. □

4.3. Rosenberg Relations vs. Permutational Relations

In this subsection we show that the trace of the maximal clone defined by a permutational relation cannot contain the trace of any other maximal clone. Let σ be a permutational relation arising from a p–regular permutation α.

Proposition 4.13. *Let ϱ be any Rosenberg relation. Then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. **Case 1:** Suppose that ϱ is not a permutational relation. Since α is fixpoint free (by definition), it follows that σ is not preserved by constant mappings.

On the other hand, bounded partial orders, equivalence, affine, central and h–regular relations are always preserved by a suitably chosen constant mapping, so if ϱ is one of those, it is clear that $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Case 2: Let ϱ now be a permutational relation arising from a q–regular permutation β, then we proceed as follows:

If $p \neq q$, we can prove that $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$ in the following way:

Since p and q are distinct prime numbers, there is a cycle of α, say $S = (a_0 \dots a_{p-1})$, which has a nonempty intersection with at least two cycles of β, say $R_1 = (b_0 \dots b_{q-1})$ and $R_2 = (b_q \dots b_{2q-1})$ and at least one of these cycles is not contained in S, say R_1. Without loss of generality, assume that $a_0 = b_0 \in R_1 \cap S$, $a_j = b_q \in R_2 \cap S$ and $b_m \in R_1 \setminus S$, for some $1 \leqslant m \leqslant q - 1$. Now, just take the mapping

$$f(x) = \begin{cases} \beta^m(x), & \text{if } x \in R_1, \\ x, & \text{otherwise.} \end{cases}$$

It preserves ϱ, but it does not preserve σ, since $a_0, a_j \in S$, but $f(a_0) = f(b_0) = \beta^m(b_0) = b_m \notin S$ and $f(a_j) = f(b_q) = b_q \in S$, so S is not mapped onto a cycle by f.

Now, suppose that $p = q$. We also assume that α is not a power of β. Let $R_1, R_2, \dots, R_l$ be the cycles of ϱ and $S_1, S_2, \dots, S_l$ the cycles of σ. Since ϱ and σ are distinct, we have two possibilities: either there exists a cycle of ϱ contained in at least two cycles of σ or $R_1 = S_1, R_2 = S_2, \dots, R_l = S_l$ as sets, but they differ in the arrangement of elements in cycles.

Suppose first that there is a cycle, say R_1, which intersects at least two cycles of σ, say S_1 and S_2, so R_1 is not a cycle of α. Then β has at least two cycles, so $\beta = (r_0^1 \dots r_{p-1}^1)(r_0^2 \dots r_{p-1}^2) \dots (r_0^l \dots r_{p-1}^l)$, where $R_1 = (r_0^1 \dots r_{p-1}^1)$. Now, take the mapping $f(r_j^i) = r_j^1$, $0 \leqslant j \leqslant p-1$, $1 \leqslant i \leqslant l$. Clearly, it preserves ϱ, since it maps all cycles of β onto R_1 and does not preserve σ, since the image of every cycle of α either contains less then p elements as a set or it is R_1, which is not a cycle of α.

Now, if $R_1 = S_1, R_2 = S_2, \dots R_l = S_l$ as sets and there exists a pair of cycles, say R_1 and S_1, such that $R_1 = S_1$, but $\alpha|S_1$ is not a power of $\beta|S_1$, then there exists r_0 such that $\alpha(r_0) = \beta^k(r_0)$ and $\alpha(r_1) = \beta^j(r_1)$, $k \not\equiv j \pmod p$, where $r_1 = \beta(r_0)$. For $f = \beta$ we have $f(\alpha(r_0)) = f(\beta^k(r_0)) = \beta^{k+1}(r_0)$ and $\alpha(f(r_0)) = \alpha(r_1) = \beta^j(\beta(r_0)) = \beta^{j+1}(r_0)$, so $f(\alpha(r_0)) \neq \alpha(f(r_0))$ and σ is not preserved.

If every pair of cycles with the same sets of elements is such that $\alpha|S_i$ is a power of $\beta|R_i$, for all i, then there are two pairs, say $R_1 = (r_0 \dots r_{p-1}), S_1 = (s_0 \dots s_{p-1})$ and $R_2 = (r_p \dots r_{2p-1}), S_2 = (s_p \dots s_{2p-1})$, such that $\alpha|S_1 = (\beta|R_1)^m$ and $\alpha|S_2 = (\beta|R_2)^n$, $m \neq n$, $m, n < p$. Then we take the mapping

$$f(r_k) = \begin{cases} r_{k+p}, & \text{if } r_k \in R_1, \\ r_k, & \text{otherwise.} \end{cases}$$

Since f maps R_1 onto R_2 and every other cycle of ϱ onto itself, it preserves ϱ. We will show that it does not preserve σ.

Without loss of generality we can assume that $r_0 = s_0$ and $r_p = s_p$. Then we have the following: $s_1 = \alpha(s_0) = \alpha(r_0) = \beta^m(r_0) = r_m$, so it follows that $f(\alpha(s_0)) = f(s_1) = f(r_m) = r_{m+p}$. On the other hand, $\alpha(f(s_0)) = \alpha(r_p) = \beta^n(r_p) = r_{n+p} \neq r_{m+p}$, since $n \neq m$. So, $f(\alpha(s_0)) \neq \alpha(f(s_0))$ and it follows that $f \in \mathrm{End}\{\varrho\} \setminus \mathrm{End}\{\sigma\}$. □

4.4. Rosenberg Relations vs. Affine Relations

We now show that only the trace of a maximal clone defined by a permutational relation can be in certain cases contained in the trace of a maximal clone defined by an affine relation. The complete characterization is given in Proposition 4.16. We assume that $\sigma = \{\langle x, y, u, v\rangle \in A^4 \mid x + y = u + v\}$ is an affine relation, where $\langle A, +, -, 0\rangle$ is an elementary Abelian p–group and that ϱ is one of the Rosenberg relations.

Proposition 4.14. *Let ϱ be a partial order with the least element 0_ϱ and the greatest element 1_ϱ. Then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. We define the mapping

$$f_{a,b,c}(x) = \begin{cases} a, & \text{if } x = b, \\ c, & \text{otherwise.} \end{cases}$$

If $0 = 0_\varrho$, then we take the order–preserving mapping $f_{0_\varrho,0,1_\varrho}$ that is not affine.

Further, if $0 \neq 0_\varrho$ and $p \geqslant 3$, then we take the order–preserving mapping $f_{0_\varrho,0_\varrho,1_\varrho}$ which is not affine. If, however, $p = 2$, then we have two possibilities:

either $0 = 1_\varrho$ or $0 \neq 1_\varrho$. In the first case just take the nonaffine order-preserving mapping $f_{1_\varrho,1_\varrho,0_\varrho}$. In the latter case, the mapping

$$f(x) = \begin{cases} 0_\varrho, & \text{if } x <_\varrho 0, \\ 0, & \text{if } x = 0, \\ 1_\varrho, & \text{otherwise,} \end{cases}$$

preserves ϱ, but does not preserve σ. Hence, $\mathrm{End}\{\varrho\} \not\subseteq \mathrm{End}\{\sigma\}$. □

Proposition 4.15. *Let ϱ be a nontrivial equivalence relation. Then* $\mathrm{End}\{\varrho\} \not\subseteq \mathrm{End}\{\sigma\}$*.*

Proof. For the proof we take the mapping

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ a, & \text{otherwise} \end{cases}$$

where a is any element of $A \setminus \{0\}$ if $[0]_\varrho = \{0\}$, and $a \in [0]_\varrho \setminus \{0\}$, if $[0]_\varrho$ is a nontrivial equivalence class. Then $f \in \mathrm{End}\{\varrho\} \setminus \mathrm{End}\{\sigma\}$. □

Proposition 4.16. *Let ϱ be a permutational relation arising from a p–regular permutation α on A. Then* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{\sigma\}$ *if and only if α is a cyclic permutation of the form $\alpha(x) = ax + b$, $a \neq 0$ (where A is considered as vector space over the p–element field* $\mathrm{GF}(p)$*), or $|A| = 4$ and α has two cycles.*

Proof. ($\Rightarrow$) Suppose that $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{\sigma\}$ and that α has at least three cycles, so that $\alpha = (a_0^0 \dots a_{p-1}^0)(a_0^1 \dots a_{p-1}^1) \dots (a_0^l \dots a_{p-1}^l)$, where $a_0^0 = 0$. Then the mapping

$$f(x) = \begin{cases} a_j^1, & \text{if } x = a_j^k, k \in \{2, \dots, l\} \\ x, & \text{otherwise,} \end{cases}$$

obviously preserves ϱ, so it also preserves σ. Hence, f is affine. Now, notice that $\mathrm{Ker} f = \{0\}$, whence follows that f is an injective affine mapping – a contradiction. Hence, α has at most two cycles.

If α has precisely two cycles, then $|A| = 2p$, where p is prime. Since elementary Abelian q–group has q^k elements, it follows that $p = 2$, so $|A| = 4$ and $\alpha = (a_0a_1)(a_2a_3)$.

Now, suppose that α is a cyclic permutation and take any nontrivial $f \in \mathrm{End}\{\varrho\}$. Then $f \circ \alpha = \alpha \circ f$ whence follows that f is a power of α. On the other hand, since $f \in \mathrm{End}\{\sigma\}$, we have $f(x) = kx + n$ for some $k \in \mathrm{GF}(f)$, $n \in A$ (see [9]). Since f is nontrivial, we also have that α is a power of f. Hence, $\alpha \in \mathrm{End}\{\sigma\}$ and $\alpha(x) = ax + b$ for some $a \neq 0$.

($\Leftarrow$) Conversely, if $\alpha(x) = ax + b$ is a cyclic permutation, then $f \in \mathrm{End}\{\varrho\}$ if and only if $f = \alpha^m$ for some m. Therefore, $f(x) = kx + n$, so it follows that $f \in \mathrm{End}\{\sigma\}$.

If α has precisely two cycles, then $|A| = 4$ and $\alpha = (a_0 a_1)(a_2 a_3)$. Now, every mapping f which preserves ϱ maps a cycle onto a cycle, so $f(A)$ has either two or four elements. To prove that f is always affine, it suffices to show that $f(x+y) = f(x) + f(y) - f(0)$, for every $x, y \in A$. If at least one of x, y is 0 then it is clear that the above equation is satisfied. Further, if $x = y$, then, since A is a 2–group, $x + x = 0$ and $f(0) + f(0) = 0 = f(x) + f(x)$. Finally, if $x \neq y$ and $x, y \neq 0$ then $x + y = z$, where z is the third nonzero element. We have $f(x+y) - f(x) - f(y) + f(0) = f(x+y) + f(x) + f(y) + f(0) = f(z) + f(x) + f(y) + f(0) = 0$, since $f(0), f(x), f(y), f(z)$ are either all distinct or two values appear twice. We conclude, $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{\sigma\}$. □

Proposition 4.17. *Let ϱ be an affine relation. Then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. It is known that every two elementary Abelian p–groups with the same carrier are isomorphic, so it follows that $|\mathrm{End}\{\varrho\}| = |\mathrm{End}\{\sigma\}|$. Now, if $\mathrm{End}\{\varrho\} \neq \mathrm{End}\{\sigma\}$, then $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$. □

Proposition 4.18. *Let ϱ be an h–regular relation or a central relation with the center C_ϱ. Then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. Take the mapping

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ a, & \text{otherwise,} \end{cases}$$

where $a \in A \setminus \{0\}$. For central relations we only require that $a \in C_\varrho \setminus \{0\}$ if $C_\varrho \neq \{0\}$. □

4.5. Rosenberg Relations vs. Central Relations

First we will show that the trace of a maximal clone is never contained in the trace of the maximal clone defined by a unary central relation. Then we will give a partial answer to the question for central relations of arity at least 1.

Proposition 4.19. *Let σ be a unary central relation and let ϱ be any Rosenberg relation. Then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. Note that a unary central relation σ is a proper nonempty subset of A. Hence there exists an element $a \in A \setminus \sigma$ and the constant mapping $f(x) = a$ does not preserve σ.

On the other hand, bounded partial orders, equivalence, affine, k–ary central ($k \geqslant 2$) and h–regular relations are preserved by any constant mapping, so if ϱ is one of these, it is clear that $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{\sigma\}$.

It is left to be seen what is the relationship between $\mathrm{End}\{\varrho\}$ and $\mathrm{End}\{\sigma\}$, if ϱ is a permutational or another unary central relation.

Let ϱ be a permutational relation arising from a p–regular permutation α. We consider the following cases:
Case 1: α has at least two cycles. Then there is a cycle that contains an element $a \in A \setminus \sigma$ and there is another cycle that contains $b \in \sigma$, say

$$\alpha = (aa_1 \dots a_{p-1})(ba_{p+1} \dots a_{2p-1}) \dots$$

(so, $a_0 = a$ and $a_p = b$.) Then we define

$$f(x) = \begin{cases} a_{p+i}, & \text{if } x = a_i, 0 \leqslant i \leqslant p \\ a_{i-p}, & \text{if } x = a_i, p \leqslant i \leqslant 2p-1 \\ x, & \text{otherwise.} \end{cases}$$

Obviously, $f \in \mathrm{End}\{\varrho\}$. On the other hand, since $f(b) = a \notin \sigma$, it follows that σ is not preserved by f.
Case 2: α is a one-cycle permutation. We take $a \in A \setminus \sigma$ and $b \in \sigma$ and assume that $\alpha = (ba_1 \dots a_{j-1}aa_{j+1} \dots a_{p-1})$, $a_0 = b$, $a_j = a$. Then $\alpha^j(b) = a$ so that $\alpha^j \in \mathrm{End}\{\varrho\} \setminus \mathrm{End}\{\sigma\}$.

Now, let ϱ be a unary central relation. Then we have two possibilities: either $\varrho \subset \sigma$ or $\varrho \not\subset \sigma$.

In the first case, there exists an element $b \in \sigma \setminus \varrho$. We define the mapping

$$f(x) = \begin{cases} a, & \text{if } x \in \sigma \setminus \varrho, \\ x, & \text{otherwise,} \end{cases}$$

where $a \in A \setminus \sigma$. It is obvious that f preserves ϱ and does not preserve σ. On the other hand, if $\varrho \not\subset \sigma$, then there is an element $b \in \varrho \setminus \sigma$ and the constant mapping $f(x) = b$ does not preserve σ. So, $\mathrm{End}\{\varrho\} \not\subseteq \mathrm{End}\{\sigma\}$. $\square$

In the sequel let σ be a central relation of arity $\geqslant 2$. The following result was obtained in [8]:

Proposition 4.20. *[8] Let ϱ be a bounded partial order on A that is not a chain, with the least element 0 and the greatest element 1 and let σ be a binary central relation on the same set with the center C_σ. Then* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{\sigma\}$ *if and only if $\sigma = \varrho \cup \varrho^{-1}$.*

Proposition 4.23 was obtained in [6] and we present it here without proof.

Definition 4.21. Let ϱ be a central relation of arity k and let $m \geqslant k$. Then for $\overrightarrow{a} = (a_1, \dots, a_m) \in A^m$ we define

$$\mathrm{type}_\varrho(\overrightarrow{a}) = \{\{a_{i_1}, \dots, a_{i_k}\} \mid i_1, \dots, i_k \in \{1, \dots, m\} \text{ and } (a_{i_1}, \dots, a_{i_k}) \in \varrho^{irr}\}.$$

Definition 4.22. Let $\overrightarrow{a} = (a_1, \dots, a_m)$, $\overrightarrow{b} = (b_1, \dots, b_m)$ and $m \geqslant k = \mathrm{ar}(\varrho)$. We say that $\mathrm{type}_\varrho(\overrightarrow{a})$ *embeds* into $\mathrm{type}_\varrho(\overrightarrow{b})$ if there is a bijective mapping

$$f : \{a_1, \dots, a_m\} \to \{b_1, \dots, b_m\}$$

such that for all $\{a_{i_1}, \ldots, a_{i_k}\} \subseteq \{a_1, \ldots, a_m\}$ we have

$$\{a_{i_1}, \ldots, a_{i_k}\} \in \mathrm{type}_\varrho(\overrightarrow{a}) \Rightarrow \{f(a_{i_1}), \ldots, f(a_{i_k})\} \in \mathrm{type}_\varrho(\overrightarrow{b}).$$

In this case we write $f : \mathrm{type}_\varrho(\overrightarrow{a}) \hookrightarrow \mathrm{type}_\varrho(\overrightarrow{b})$.

Proposition 4.23. *Let ϱ_1 and ϱ_2 be central relations such that* $\mathrm{ar}(\varrho_1) = k$ *and* $\mathrm{ar}(\varrho_2) = m$. *Then* $\mathrm{End}\{\varrho_1\} \subseteq \mathrm{End}\{\varrho_2\}$ *if and only if*

- *$k \leqslant m$ and*
- *for all $\overrightarrow{a} \in \varrho_2^{irr}$ and $\overrightarrow{b} \in A^m$ we have*

$$\mathrm{type}_{\varrho_1}(\overrightarrow{a}) \hookrightarrow \mathrm{type}_{\varrho_1}(\overrightarrow{b}) \Rightarrow \overrightarrow{b} \in \varrho_2.$$

Let R_Θ be the h–regular relation defined by an h–regular family $\Theta = \{\theta_1, \ldots, \theta_m\}$ and let $\lambda : A \to \{1, \ldots, h\}^m$ be the surjective mapping such that $R_\Theta = \{(x_1, \ldots, x_h) \mid (\lambda(x_1), \ldots, \lambda(x_h)) \in \Psi_{h,m}\}$.

Proposition 4.24. *If $|A| = h^m$, then* $\mathrm{End}\{R_\Theta\} \nsubseteq \mathrm{End}\{\sigma\}$.

Proof. First, note that if $|A| = h^m$, then λ is a bijective mapping.

Now, since σ is a nontrivial central relation, there exist noncentral elements $b_1, b_2, \ldots, b_k$ such that $(b_1, b_2, \ldots, b_k) \notin \sigma$. Further, take an arbitrary central element c. Since λ is bijective, there exist $\overrightarrow{c_\lambda}, \overrightarrow{y_1}, \ldots, \overrightarrow{y_k} \in \{1, \ldots, h\}^m$ such that $\overrightarrow{y_i} = \lambda(b_i)$, $1 \leqslant i \leqslant k$, and $\overrightarrow{c_\lambda} = \lambda(c)$, where

$$\overrightarrow{y_i} = \begin{bmatrix} y_1^i \\ y_2^i \\ \vdots \\ y_m^i \end{bmatrix} \text{ and } \overrightarrow{c_\lambda} = \begin{bmatrix} c_1^\lambda \\ c_2^\lambda \\ \vdots \\ c_m^\lambda. \end{bmatrix}.$$

We define the family $\Phi = \{\varphi_1, \varphi_2 \ldots, \varphi_m\}$ of bijective mappings on the set $\{1, \ldots, h\}$ in the following way:

$$\varphi_j(x) = \begin{cases} y_j^1, & \text{if } x = c_j^\lambda, \\ c_j^\lambda, & \text{if } x = y_j^1, \\ x, & \text{otherwise,} \end{cases}$$

where $1 \leqslant j \leqslant m$.

Now, define the mapping

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\right) = \begin{bmatrix} \varphi_1(x_1) \\ \varphi_2(x_2) \\ \vdots \\ \varphi_m(x_m) \end{bmatrix}.$$

It is easy to see that $f \in \mathrm{Aut}\{\Psi_{h,m}\}$. Also, note that $f(\overrightarrow{c_\lambda}) = \overrightarrow{y_1}$. Now, take $\overrightarrow{x_2}, \dots, \overrightarrow{x_k} \in \{1, \dots, h\}^m$ such that $\overrightarrow{x_i} = f^{-1}(\overrightarrow{y_i})$ and let $a_2, \dots, a_k \in A$ be such that $\overrightarrow{x_i} = \lambda(a_i)$, $2 \leqslant i \leqslant k$. Since $c \in C_\sigma$, it follows that $(c, a_2, \dots, a_k) \in \sigma$. But, then $(\lambda^{-1} \circ f \circ \lambda(c), \lambda^{-1} \circ f \circ \lambda(a_2), \dots, \lambda^{-1} \circ f \circ \lambda(a_k)) = (\lambda^{-1} \circ f(\overrightarrow{c_\lambda}), \lambda^{-1} \circ f(\overrightarrow{x_2}), \dots, \lambda^{-1} \circ f(\overrightarrow{x_k})) = (\lambda^{-1}(\overrightarrow{y_1}), \lambda^{-1}(\overrightarrow{y_2}), \dots, \lambda^{-1}(\overrightarrow{y_k})) = (b_1, b_2, \dots, b_k) \notin \sigma$ and we have that $\lambda^{-1} \circ f \circ \lambda \notin \mathrm{End}\{\sigma\}$.

On the other hand, $\lambda^{-1} \circ f \circ \lambda \in \mathrm{Aut}\{R_\Theta\} \subset \mathrm{End}\{R_\Theta\}$. To conclude, $\mathrm{End}\{R_\Theta\} \nsubseteq \mathrm{End}\{\sigma\}$. □

4.6. Rosenberg Relations vs. h–regular Relations

We will show that the trace of a maximal clone defined by an affine relation is never included in the trace of a maximal clone defined by an h–regular relation. For all other Rosenberg relations we can obtain inclusion under some conditions. We give a complete characterization for permutational relations (Proposition 4.30) and for equivalence relations the complete characterization can be found in [5]. However, the complete answer concerning bounded partial orders, central and h–regular relations is still unknown. Some partial characterizations are given in Propositions 4.27, 4.33 and 4.34.

In this subsection we assume that σ is an h–regular relation. Since every h–regular relation is defined by an h–regular family of equivalence relations Θ, we will also denote this relation by R_Θ. The relation ϱ ranges through Rosenberg relations.

Proposition 4.25. *Let ϱ be a bounded partial order. If R_Θ is an h–regular relation defined by $\Theta = \{\theta_1, \dots, \theta_m\}$, where $m \geqslant 2$, then $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{R_\Theta\}$.*

Proof. Let 0 be the least element of ϱ and let $\lambda : A \to \{1, \dots, h\}^m$ be a surjective mapping such that $R_\Theta = \{(x_1, \dots, x_h) \mid (\lambda(x_1), \dots, \lambda(x_h)) \in \Psi_{h,m}\}$. Further, let $B = A \setminus \{y \mid \lambda(0) = \lambda(y)\}$. Note that B is a nonempty set, since $|\lambda(A)| \geqslant 3$, so $\min B \neq \emptyset$. Now, take any $a \in \min B$ and the mapping

$$f(x) = \begin{cases} 0, & \text{if } x \leqslant a, \\ x, & \text{otherwise.} \end{cases}$$

It is obvious that f preserves ϱ. We are going to show that f does not preserve R_Θ.

For the proof, take the mapping λ. It takes 0 to $\overrightarrow{0_\lambda}$ and a to $\overrightarrow{a_\lambda}$. Since $\overrightarrow{0_\lambda}, \overrightarrow{a_\lambda} \in \{1, \dots, h\}^m$, we obtain, by Lemma 3.2, that there exist $\overrightarrow{e^2{}_\lambda}, \dots, \overrightarrow{e^h{}_\lambda}$ such that $\overrightarrow{a_\lambda} \notin \{\overrightarrow{e^2{}_\lambda}, \dots, \overrightarrow{e^h{}_\lambda}\}$, $(\overrightarrow{a_\lambda}, \overrightarrow{e^2{}_\lambda}, \dots, \overrightarrow{e^h{}_\lambda}) \in \Psi_{h,m}$ and $(\overrightarrow{0_\lambda}, \overrightarrow{e^2{}_\lambda}, \dots, \overrightarrow{e^h{}_\lambda}) \notin \Psi_{h,m}$. In particular, $\overrightarrow{0_\lambda} \neq \overrightarrow{e^i_\lambda}$, $2 \leqslant i \leqslant h$.

Now, let $e^2, \dots, e^h$ be such that $\overrightarrow{e^i_\lambda} = \lambda(e^i)$. Then $e^i \in B$, $2 \leqslant i \leqslant h$, since $\lambda(e^i) \neq \lambda(0)$. Now we have that $(a, e^2, \dots, e^h) \in R_\Theta$, but $(f(a),\ f(e^2), \dots, f(e^h)) = (0,\ e^2, \dots, e^h) \notin R_\Theta$. □

Lemma 4.26. *Let R_Θ be an h–regular relation defined by an h–regular family $\Theta = \{\theta\}$ and let a be an irreducible element in $(A, \leqslant_\varrho)$. If* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$, *then either $[a]_\theta = \{a\}$ or a and its primary neighbors are in the same equivalence class of θ.*

Proof. Suppose to the contrary that $[a]_\theta \neq \{a\}$, but there exists a primary neighbor b of a such that $b \notin [a]_\theta$. Now, take the mapping

$$f(x) = \begin{cases} b, & \text{if } x = a, \\ x, & \text{otherwise.} \end{cases}$$

It is clear that f preserves ϱ. We will show that f does not preserve R_Θ. Let $T_1, \ldots, T_h$ be the equivalence classes of θ, where $[b]_\theta = T_1$ and $[a]_\theta = T_2$. Further, let $c_2, \ldots, c_h$ be such that $c_i \in T_i$, $2 \leqslant i \leqslant h$, and $c_2 \neq a$. Then $(a, c_2, \ldots, c_h) \in R_\Theta$, but $(f(a), f(c_2), \ldots, f(c_h)) = (b, c_2, \ldots, c_h) \notin R_\Theta$ – contradiction. □

Using Lemma 4.26, we obtain the following partial characterization:

Proposition 4.27. *Let $|A| = h + 1$, let ϱ be a bounded partial order on A with the least element* 0 *and the greatest element* 1 *and let R_Θ be an h–regular relation defined by an h–regular family $\Theta = \{\theta\}$, where θ has just one nontrivial equivalence class that consists of 0 and 1. Then* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$ *if and only if 0 and 1 are not irreducible.*

Proof. ($\Leftarrow$) Let $f \in \mathrm{End}\{\varrho\}$. If $f(0) = 0$ and $f(1) = 1$, then it is clear that $f \in \mathrm{End}\{R_\Theta\}$. So, suppose that $f(0) \neq 0$ or $f(1) \neq 1$, say $f(1) \neq 1$. We are going to prove that then $|\mathrm{im}(f)| \leqslant h - 1$.

First, note that $1 \notin \mathrm{im}(f)$, since 1 is the greatest element. Further, 1 is not irreducible, so there are at least two elements a_1, a_2 such that $a_1 \prec 1$ and $a_2 \prec 1$. Now, if $a_1 \leqslant_\varrho f(1)$ and $a_2 \leqslant_\varrho f(1)$, then it follows that f(1)=1 – contradiction.

So, we have that $a_1 \not\leqslant_\varrho f(1)$ or $a_2 \not\leqslant_\varrho f(1)$. But, then at least one of a_1, a_2 does not belong to $\mathrm{im}(f)$, say $a_1 \notin \mathrm{im}(f)$. Since $|A| = h + 1$ and $a_1, 1 \notin \mathrm{im}(f)$, we obtain that $|\mathrm{im}(f)| \leqslant h - 1$. Having in mind that θ has h equivalence classes, it follows that $f \in \mathrm{End}\{R_\Theta\}$.

($\Rightarrow$) Suppose that $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$ and that at least one of 0, 1 is irreducible, say 0. Then, by Lemma 4.26 either $[0]_\theta = \{0\}$ or for every primary neighbor a of 0 we have $[0]_\theta = [a]_\theta$. It follows that 1 is the primary neighbor of 0 – contradiction. □

The proof of the following result can be found in [5].

Proposition 4.28. *Let ϱ be a nontrivial equivalence relation and let R_Θ be the h–regular relation defined by an h–regular family $\Theta = \{\theta_1, \ldots, \theta_m\}$. Then* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$ *if and only if $m = 1$, all nontrivial equivalence classes of θ_1 are also nontrivial equivalence classes of ϱ and every nontrivial class of ϱ which is a union of trivial classes of θ_1, if any, is of greater cardinality than all nontrivial classes of θ_1.*

Proposition 4.29. *Let ϱ be a permutational relation arising from a p–regular permutation α on A and let σ be a nontrivial equivalence relation on the same set. Then* $\mathrm{Aut}\{\varrho\} \subseteq \mathrm{Aut}\{\sigma\}$ *if and only if every cycle of α is an equivalence class of σ.*

Proof. ($\Leftarrow$) Suppose that every cycle of α is an equivalence class of σ and let $f \in \mathrm{Aut}\{\varrho\}$. Since f preserves ϱ, f maps each cycle of α onto a cycle of α. But, then f maps every equivalence class of σ onto an equivalence class of σ, so σ is also preserved by f. Hence, $\mathrm{Aut}\{\varrho\} \subseteq \mathrm{Aut}\{\sigma\}$.

($\Rightarrow$) Suppose that $\mathrm{Aut}\{\varrho\} \subseteq \mathrm{Aut}\{\sigma\}$. By Lemma 4.8, it follows that all equivalence classes of σ are nontrivial.

First, we are going to show that for every cycle C there exists an equivalence class S such that either C is a subset of S (as a set) or S is a subset C. Suppose that this is not the case. Then there exists a cycle C_j such that for every equivalence class S we have that C_j is not contained in S and S is not contained in C_j. Then C_j contains elements of at least two equivalence classes, say S_1 and S_2. Since S_1 is not contained in C_j, it follows that there exists an $a \in S_1 \setminus C_j$, so there exists a cycle C_i such that $a \in C_i$. Let $b \in S_1 \cap C_j$ and $c \in S_2 \cap C_j$. Since $b, c \in C_j$ it follows that $c = \alpha^m(b)$ for some m. Now, take the mapping

$$f(x) = \begin{cases} \alpha^m(x), & \text{if } x \in C_j, \\ x, & \text{otherwise.} \end{cases}$$

It is obviously an automorphism of ϱ and it is not an automorphism of σ, since $(a, b) \in \sigma$ and $(f(a), f(b)) = (a, c) \notin \sigma$.

Suppose that there exists a cycle C which consists of $k \geqslant 2$ equivalence classes of σ. Since the permutation α preserves σ, it follows that all equivalence classes of C contain the same number of elements, say $n \geqslant 2$; then C contains $k \cdot n = p$ elements, where $k, n \geqslant 2$ and p is a prime number – contradiction.

Similarly, assume that there exists an equivalence class of σ, say S, which contains at least two cycles of α, say $C_1, C_2, \ldots$. Then there is a cycle $C_0 \nsubseteq S$, so a mapping that takes C_1 to C_0 and C_0 to C_1 in an appropriate way and leaves everything else fixed obviously preserves ϱ and does not preserve σ. Therefore, every cycle of α is an equivalence class of σ. $\square$

Using Proposition 4.29, we obtain the following result:

Proposition 4.30. *Let ϱ be a permutational relation arising from a p–regular permutation α on A and let R_Θ be the h–regular relation defined by an h–regular family $\Theta = \{\theta_1, \ldots, \theta_m\}$. Then* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$ *if and only if $m = 1$ and every cycle of α is an equivalence class of θ_1.*

Proof. ($\Rightarrow$) From Lemma 3.5, we have $\mathrm{Aut}\{R_\Theta\} \subseteq \mathrm{Aut}\{\Phi\}$, where $\Phi = \cap_{i=1}^{m} \theta_i$. Note that either $\Phi = \Delta_A$, or we obtain that classes of Φ are exactly the cycles of α. (If $\Phi \neq \Delta_A$, then $\mathrm{Aut}\{\varrho\} \subseteq \mathrm{Aut}\{R_\Theta\} \subseteq \mathrm{Aut}\{\Phi\}$, so $\mathrm{Aut}\{\varrho\} \subseteq \mathrm{Aut}\{\Phi\}$ and from Proposition 4.29 it follows that the classes of Φ are exactly the cycles of α.)

Suppose, first, that $\Phi = \Delta_A$. Then, by Lemma 3.6, we may assume that $A = A/\theta_1 \times \cdots \times A/\theta_m$, so $|A| = h^m$. Denote by $T_1^i, \ldots T_h^i$ the equivalence classes of θ_i, $1 \leqslant i \leqslant m$. Then every $a \in A$ can be represented as a column-vector

$$\begin{bmatrix} j_1 \\ \vdots \\ j_m \end{bmatrix},$$

where j_i is such that $a \in T^i_{j_i}$. By Lemma 3.1, every automorphism of R_Θ has the form

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \mapsto \begin{bmatrix} \sigma_1(a_{\pi(1)}) \\ \vdots \\ \sigma_m(a_{\pi(m)}) \end{bmatrix}, \tag{$*$}$$

where $\pi : \{1, \ldots, m\} \to \{1, \ldots, m\}$ and $\sigma_j : \{1, \ldots, h\} \to \{1, \ldots, h\}$, $1 \leqslant j \leqslant m$, are bijections.

A nontrivial automorphism of the form $(*)$ has at most $(h-2)h^{m-1}$ fixed points, since moving at least one point induces moving at least $2 \cdot h^{m-1}$ points (If a_1 is moved to a_1', then all h^{m-1} elements containing a_1 as the first coordinate are moved to h^{m-1} elements containing a_1' as the first coordinate and, since this mapping is bijective, h^{m-1} elements containing a_1' as the first coordinate are also moved), so the maximal number of fixed points of such a bijection is $h^m - 2 \cdot h^{m-1} = (h-2)h^{m-1}$. We know that $p \mid |A| = h^m$, so $p \mid h$.

Further, it is clear that $\mathrm{Aut}\{\varrho\}$ contains p–cycles. (Every cycle of α belongs to $\mathrm{Aut}\{\varrho\}$). In other words, $\mathrm{Aut}\{\varrho\}$ contains a mapping which moves exactly p points. Now, if $m \geqslant 2$, then every nontrivial permutation in $\mathrm{Aut}\{R_\Theta\}$ moves at least $h^m - (h-2)h^{m-1} = 2 \cdot h^{m-1} \geqslant 2h > h \geqslant p$ points, so $\mathrm{Aut}\{R_\Theta\}$ does not contain any p–cycle, thus $\mathrm{Aut}\{\varrho\} \nleqslant \mathrm{Aut}\{R_\Theta\}$ and it follows that $m = 1$ in this case.

If Ψ consists of the cycles of α, then both $\mathrm{Aut}\{\varrho\}$ and $\mathrm{Aut}\{R_\Theta\}$ are contained in $\mathrm{Aut}\{\Phi\}$, so every $f \in \mathrm{Aut}\{\varrho\} \cup \mathrm{Aut}\{R_\Theta\}$ preserves equivalence classes of Φ. Now consider A/Φ and two permutation groups G_ϱ and G_Θ on the same set. We define G_ϱ as the group of all permutations $\hat{f} : A/\Phi \to A/\Phi$, where $\hat{f}(B) = f[B]$ and $f \in \mathrm{Aut}\{\varrho\}$, and G_Θ as the group of all permutations $\hat{f} : A/\Phi \to A/\Phi$, where $\hat{f}(B) = f[B]$ and $f \in \mathrm{Aut}\{R_\Theta\}$. The first group is the full symmetric group, since we can arbitrarily permute the cycles of α. However, if $m \geqslant 2$, the second group is not the full symmetric group. (By definition, $G_\Theta = \mathrm{Aut}\{R_{\Theta/\Phi}\}$, where $\Theta/\Phi = \{\theta_1/\Phi, \ldots, \theta_m/\Phi\}$. Take $(x_1, \ldots, x_h) \in R_{\Theta/\Phi}$, $(y_1, \ldots, y_h) \notin R_{\Theta/\Phi}$ and the mapping φ, defined by $\varphi(x_i) = y_i$, $1 \leqslant i \leqslant h$. Since φ is a partial injective mapping, it can be extended to a permutation π_φ. It is clear that π_φ does not preserve $R_{\Theta/\Phi}$, so G_Θ is not a full symmetric group.)

Since $G_\varrho \nsubseteq G_\Theta$, it follows that $\mathrm{Aut}\{\varrho\} \nsubseteq \mathrm{Aut}\{R_\Theta\}$, so $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{R_\Theta\}$.

Hence, $m = 1$. Note that $\mathrm{Aut}\{\theta_1\} = \mathrm{Aut}\{R_\Theta\}$, so θ_1 has only nontrivial equivalence classes, by Lemma 4.8.

Let us now show that every cycle of α is an equivalence class of θ_1.

First, we are going to show that for every cycle C there exists an equivalence class T of θ_1 such that either C is a subset of T (as a set) or T is a subset of C. Suppose that this is not the case. Then there exists a cycle C_j such that for every equivalence class T of θ_1 we have that C_j is not contained in T and T is not contained in C_j. Denote by $T_1, \dots, T_h$ the equivalence classes of θ_1. Then C_j contains elements of at least two equivalence classes, say T_1 and T_2. Since T_1 is not contained in C_j, it follows that there exists an element $a \in T_1 \setminus C_j$, so there exists a cycle C_i such that $a \in C_i$. Let $b \in T_1 \cap C_j$ and $c \in T_2 \cap C_j$. Since $b, c \in C_j$ it follows that $c = \alpha^m(b)$ for some m. Now, take the mapping

$$f(x) = \begin{cases} \alpha^m(x), & \text{if } x \in C_j, \\ x, & \text{otherwise} \end{cases}$$

and the tuple $(a, b, a_3, \dots, a_h) \in R_\Theta$, where $a_i \in T_i \setminus C_j$ (such element always exists, since $T_i \not\subseteq C_j$ for every $1 \leqslant i \leqslant h$). We have

$$(f(a), f(b), f(a_3), \dots, f(a_h)) = (a, c, a_3, \dots, a_h) \notin R_\Theta.$$

Further, if there exists a cycle C which consists of $k \geqslant 2$ equivalence classes of θ_1, since the permutation α preserves θ_1, it follows that all equivalence classes of C contain the same number of elements, say $n \geqslant 2$; then C contains $k \cdot n = p$ elements, where $k, n \geqslant 2$ and p is a prime number − contradiction.

Similarly, assume that there exists an equivalence class of θ_1, say T_1, which contains at least two cycles of α, say $C_1, C_2, \dots$. Then there is a cycle $C_0 \not\subseteq T_1$ such that $C_0 \cap T_2 \neq \emptyset$ for some other equivalence class T_2 of θ_1, so a mapping that takes C_1 to C_0 in an appropriate way and leaves everything else fixed obviously preserves ϱ. To see that f does not preserve R_Θ just take the tuple $(a, b, a_3, \dots, a_h) \in R_\Theta$, where $a_i \in T_i$, $3 \leqslant i \leqslant h$, $b \in C_2$ and $a \in C_1$ is such that $f(a) = c \in C_0 \cap T_2$. Then $(f(a), f(b), f(a_3), \dots, f(a_h)) = (c, b, a_3, \dots, a_h) \notin R_\Theta$.

From this it follows that every cycle of α is an equivalence class of θ_1.

($\Leftarrow$) Suppose, now, that $m = 1$ and every cycle of α is an equivalence class of θ_1 and let $f \in \mathrm{End}\{\varrho\}$. Since f preserves ϱ, f maps each cycle of α onto a cycle of α. But, then f maps every equivalence class of θ_1 into an equivalence class of θ_1, so θ_1 is also preserved by f. Hence, $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{\theta_1\}$. Now, by Lemma 3.3, it follows that $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$. □

Proposition 4.31. *If ϱ is an affine relation, then* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$ *if and only if R_Θ is a trivial regular relation.*

Proof. If R_Θ is a trivial regular relation, then $\mathrm{End}\{R_\Theta\} = A^A$ whence trivially $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$. Suppose that R_Θ is a nontrivial regular relation. Then the claim can be proved by observing that $\mathrm{Aut}\{\varrho\} = \mathrm{Aut}\{\lambda_\oplus\} \cong \mathrm{AGL}(d, p)$, is 2–transitive, but $\mathrm{Aut}\{R_\Theta\}$ is 2–transitive only in the trivial case, so $\mathrm{Aut}\{\varrho\} \not\subseteq \mathrm{Aut}\{R_\Theta\}$. From this it follows immediately that $\mathrm{End}\{\varrho\} \not\subseteq \mathrm{End}\{R_\Theta\}$.

To see that $\mathrm{Aut}\{\varrho\}$ is 2-transitive, note that each $f \in \mathrm{Aut}\{\varrho\}$ is of the form $f(x) = \varphi(x) + b$, where φ is a regular (i.e. invertible) linear function on A

(considered as vector space over p–element field GF(p)). Let $x, y, x', y' \in A$ be such that $x \neq y$, $x' \neq y'$. Hence, $y - x \neq 0$ and $y' - x' \neq 0$. Now, there exists regular linear function φ on A that maps $y - x$ to $y' - x'$. Defining $f(z) = \varphi(z - x) + x'$ we obtain an affine permutation such that $\varphi(x) = x'$ and $\varphi(y) = y'$. In other words, $\mathrm{Aut}\{\varrho\}$ is 2–transitive. □

The proofs of the following partial results can be found in [8] and [5].

Proposition 4.32. *If R_Θ is a h–regular relation defined by the h–regular family $\Theta = \{\theta_1, \dots, \theta_m\}$, where $m \geqslant 2$, and ϱ is central, then* $\mathrm{End}\{\varrho\} \nsubseteq \mathrm{End}\{R_\Theta\}$.

Proposition 4.33. *Let ϱ be a k–ary central relation with no tail whose center is C_ϱ, let $\Theta = \{\theta_1, \dots, \theta_m\}$ be an h–regular family and let R_Θ be the corresponding h–regular relation. Then* $\mathrm{End}\{\varrho\} \subseteq \mathrm{End}\{R_\Theta\}$ *if and only if $k < h$, $m = 1$ and C_ϱ is the only nontrivial equivalence class of θ_1.*

Proposition 4.34. *Let R_Θ be the h_1-regular relation defined by an h_1-regular family $\Theta = \{\theta\}$ and let $R_{\Theta'}$ be the h_2-regular relation defined by an h_2-regular family $\Theta' = \{\theta'\}$. Then* $\mathrm{End}\{R_\Theta\} \subseteq \mathrm{End}\{R_{\Theta'}\}$ *if and only if all nontrivial equivalence classes of θ' are also nontrivial equivalence classes of θ and every nontrivial class of θ which is a union of trivial classes of θ' is of greater cardinality than any nontrivial class of θ'.*

5. Conclusions

The results presented in Section 4 shed some light on the rather involved structure of the poset of traces of maximal clones on a finite set.

First it is possible to construct nontrivial chains which contain endomorphism monoids of distinct types of Rosenberg relations, e.g. one can find a unary central relation ϱ, a k–ary central relation σ, an equivalence relation τ and an h–regular relation R_θ such that $\mathrm{End}\{\varrho\} \subset \mathrm{End}\{\sigma\} \subset \mathrm{End}\{\tau\} \subset \mathrm{End}\{R_\Theta\}$. For example, let $A = \{0, 1, 2, 3, 4\}$ and let $\varrho = \{0, 1, 2\}$ be a unary central relation. Let σ be the binary central relation with no tail whose center is $C_\sigma = \{0, 1, 2\}$ and let τ be the equivalence relation whose blocks are $012|3|4$. We take for R_Θ the h–regular relation defined by the h–regular family $\Theta = \{\tau\}$. Then we have $\mathrm{End}\{\varrho\} \subset \mathrm{End}\{\sigma\} \subset \mathrm{End}\{\tau\} \subset \mathrm{End}\{R_\Theta\}$.

Further, this poset contains chains of endomorphism monoids of central relations of the length $|A| - 1$, where A is the underlying set. The construction of these chains can be found in [8]. There are also long chains of endomorphism monoids of h–regular relations and they were constructed in [5]. However, the structure of the poset of traces of maximal clones is richer than expected. In [6] it was shown that for every given natural number n, there exists the underlying set A, such that a Boolean algebra with n atoms can be embedded into the poset of traces of maximal clones over A. The same paper shows that the height of this poset is the size of the underlying set and that the width of the poset is doubly exponential in $|A|$.

References

[1] Aichinger, E., Mašulović, D., Pöschel, R., Wilson, J.S., Completeness for concrete near-rings. Journal of Algebra 279 (2004), 61-78.

[2] Gibson, P., Zaguia, I., Endomorphism Classes of Ordered Sets, Graphs and Lattices. Order 15 (1) (1998), 21-34.

[3] Lau, D., Bestimmung der Ordnung maximaler Klassen vov Funktionen der k-wertigen Logik. Z. Math. Logik u. Grundl. Math. 24 (1978) 79-96.

[4] Mašulović D., On a completeness criterion for 0-symmetric near-rings. TU Dresden, Preprint MATH-AL-17-2002, November 2002

[5] Pech M., Mašulović D., On the height of the poset of endomorphism monoids of regular relations. (submitted)

[6] Ponjavić M., On the structure of the poset of endomorphism monoids of central relations. Contributions to General Algebra 16 (to appear)

[7] Ponjavić, M., On traces of maximal clones. MSc Thesis, University of Novi Sad, Serbia and Montenegro (2003)

[8] Ponjavić M., Mašulović D., On chains and antichains in the partially ordered set of traces of maximal clones. Contributions to General Algebra 15 (2004), 119-134

[9] Pöschel, R., Kalužnin L., Funktionen- und Relationenalgebren. Berlin: Deutscher Verlag der Wissenschaften 1979; Basel: Birkhäuser Verlag , Math. Reihe Bd. 67 (1979).

[10] Rosenberg, I.G., Über die funktionale Vollständigkeit in den mehrwertigen Logiken. Rozpr. ČSAV Řada Mat. Přir. Věd., Praha 80, 4 (1970), 3-93.

[11] Szendrei, Á., Clones in Universal Algebra. Les Presses de L'Université de Montreál, (1986)

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