Novi Sad J. Math.
Vol. 35, No. 1, 2005, 151-159

SOME CLASSES OF POSITIVE DEFINITE COLOMBEAU GENERALIZED FUNCTIONS

Zagorka Lozanov-Crvenković[1], Radoslava Mirkov[2]

Abstract. Positivity and positive definiteness in algebra of generalized functions are studied. Basic definitions and notions of Colombeau algebra of generalized functions are given and some special classes of positive definite generalized functions on those algebras are introduced. Their relation to distributions is also investigated.

AMS Mathematics Subject Classification (2000):

Key words and phrases: Colombeau generalized functions, positive definite, evenly positive definite, conditionaly positive definite

1. Basic Notions

1.1. The Special Colombeau Algebra

The special Colombeau algebra is a simplified version of the full Colombeau algebra. Omitting the general construction, we give only the definition of the special Colombeau algebra, denoted by $\mathcal{G}^s(\Omega)$ (see [2]):

Definition 1.1. *Let Ω be an open subset of $\mathbb{R}^n, I = (0,1]$. Set*

$$\begin{aligned}
\mathcal{E}^s(\Omega) &= (\mathcal{C}^\infty(\mathbb{R}^n))^I \\
\mathcal{E}^s_M(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) : \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n\ \exists p \in \mathbb{N} \text{ with} \\
&\quad \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\} \\
\mathcal{N}^s(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) : \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n\ \forall q \in \mathbb{N} \text{ with} \\
&\quad \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{q}) \text{ as } \varepsilon \to 0\}.
\end{aligned}$$

Elements of $\mathcal{E}^s_M(\Omega)$ and $\mathcal{N}^s(\Omega)$ are called moderate, respectively negligible functions. The special Colombeau algebra on Ω is defined as

$$\mathcal{G}^s(\Omega) = \mathcal{E}^s_M(\Omega)/\mathcal{N}^s(\Omega).$$

In the definition the Landau symbol $a_\varepsilon = \mathcal{O}(b_\varepsilon)$ appears, having the following meaning: $\exists C > 0, \exists \varepsilon_0 \in I, \forall \varepsilon < \varepsilon_0, a_\varepsilon \leq C b_\varepsilon$. If $(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega)$ is a representative of an element $u \in \mathcal{G}^s(\Omega)$, we write $u = [(u_\varepsilon)_\varepsilon]$. This emphasizes that u is the class of $(u_\varepsilon)_\varepsilon$. One should note that $(u_\varepsilon)_\varepsilon$ stands for $(u_\varepsilon)_{\varepsilon\in I}$, and

[1] University of Novi Sad, Department of Mathematics and Informatics, Trg D. Obradovića 4, 21000 Novi Sad, Serbia and Montenegro
[2] Department of Statistics and Decision Support System. University of Vienna, Austria

$I = (0, 1]$ throughout. The space of all moderate sequences $\mathcal{E}^s_M(\Omega)$ is a differential algebra with pointwise multiplication and differentiation $(f_\varepsilon)^{(\alpha)}_\varepsilon = (f^{(\alpha)}_\varepsilon)_\varepsilon$. Clearly, it is the largest differential subalgebra of $\mathcal{E}^s(\Omega)$ in which $\mathcal{N}^s(\Omega)$ is a differential ideal. Thus, $\mathcal{G}^s(\Omega)$ is an associative, commutative differential algebra.

Elements of $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ are embedded into $\mathcal{G}^s(\Omega)$ via the convolution with an appropriate mollifier φ_ε, $\varphi \in \mathcal{S}'(\mathbb{R}^n)$, $\varepsilon \in I$ satisfying

$$\int \varphi(x)\,dx = 1, \qquad \int x^k \varphi(x)\,dx = 0, \;\; k \in \mathbb{N}^n,$$

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right) \tag{1}$$

Then a linear embedding of $\mathcal{E}'(\Omega)$ into $\mathcal{G}^s(\Omega)$ is given by

$$T \rightsquigarrow Cd(T) = [((T * \varphi_\varepsilon)|_\Omega)_\varepsilon] = ((T * \varphi_\varepsilon)|_\Omega)_\varepsilon + \mathcal{N}^s(\Omega).$$

The sheaf properties of $\mathcal{D}'$ and $\mathcal{G}^s$ enable the extension of this embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}^s(\Omega)$ for any open subset $\Omega \subset \mathbb{R}^n$.

1.2. Point Values and Generalized Numbers

Within the classical distribution theory, distributions cannot be characterized by their point values in any way similar to classical functions. On the other hand, there is a very natural and direct way of obtaining the point values of the elements of Colombeau's algebra: points are simply inserted into representatives. The objects so obtained are sequences of numbers, and as such are not the elements in the field $\mathbb{C}$ or $\mathbb{R}$. Instead, they are the representatives of generalized numbers. We give the exact definition of these "numbers" [2]:

Definition 1.2. *Set*

$$\begin{aligned} \mathcal{E}^s_{M0} &= \{(z_\varepsilon)_\varepsilon \in \mathbb{C}^I \,:\, \exists p \in \mathbb{N} \;\; |z_\varepsilon| = \mathcal{O}(\varepsilon^{-p}) \;\; as \;\; \varepsilon \to 0\} \\ \mathcal{N}^s_0 &= \{(z_\varepsilon)_\varepsilon \in \mathbb{C}^I \,:\, \forall q \in \mathbb{N} \;\; |z_\varepsilon| = \mathcal{O}(\varepsilon^{q}) \;\; as \;\; \varepsilon \to 0\}. \end{aligned}$$

The quotient space

$$\widetilde{\mathbb{C}} = \mathcal{E}^s_{M0}/\mathcal{N}^s_0$$

is called the ring of generalized complex numbers. If we replace $\mathbb{C}$ with $\mathbb{R}$, we obtain the ring of generalized real numbers $\widetilde{\mathbb{R}}$.

The canonical embedding of $\mathbb{C}$ into the ring $\widetilde{\mathbb{C}}$ is given by $z \rightsquigarrow (z)_\varepsilon + \mathcal{N}^s_0$. Also, $\widetilde{\mathbb{C}}$ is the subring of $\mathcal{G}^s(\Omega)$. We have the same for $\widetilde{\mathbb{R}}$.

Definition 1.3. *For $u \in \mathcal{G}^s(\Omega)$ and $x_0 \in \Omega$, the point value of u at the point x_0, $u(x_0)$, is defined as the class of $(u_\varepsilon(x_0))_\varepsilon$ in $\widetilde{\mathbb{C}}$, i.e. $\widetilde{\mathbb{R}}$.*

$\widetilde{\mathbb{C}}$ and $\widetilde{\mathbb{R}}$ are actually the ring of constants of the $\mathcal{G}^s(\Omega)$

Theorem 1.1. *Let Ω be a connected open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}^s(\Omega)$. Then $(\partial/\partial x)u \equiv 0$ if and only if $u \in \widetilde{\mathbb{C}}$, i.e. $u \in \widetilde{\mathbb{R}}$.*

The proofs of the stated theorems may be found in [2].

We say that an element $r \in \widetilde{\mathbb{R}}$ is strictly nonzero if there exists a representative $(r_\varepsilon)_\varepsilon$ and a $q \in \mathbb{N}$ such that $|r_\varepsilon| \geq \varepsilon^q$ for ε sufficiently small. If r is strictly nonzero, then it is also invertible with the inverse $[(1/r_\varepsilon)_\varepsilon]$. The converse is true as well. The same is true in $\widetilde{\mathbb{C}}$.

Treating the elements of Colombeau algebras as a generalization of classical functions, the question arises whether the definition of point values can be extended in such a way that each element is characterized by its values. Such an extension is indeed possible. The basic idea is to introduce an analogue to nonstandard numbers on Ω into the theory [2].

Definition 1.4. *Let Ω be an open subset of $\mathbb{R}^n$. On*

$$\begin{aligned} \Omega_M &= \{(x_\varepsilon)_\varepsilon \in \Omega^I \,:\, \exists p > 0 \;\; |x_\varepsilon| = \mathcal{O}(\varepsilon^{-p})\} \\ &= \{(x_\varepsilon)_\varepsilon \in \Omega^I \,:\, \exists p > 0 \; \exists \varepsilon_0 > 0 \;\; |x_\varepsilon| \leq \varepsilon^{-p} \;\; \textit{for } 0 < \varepsilon < \varepsilon_0\} \end{aligned}$$

we introduce an equivalence relation:

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow \forall q \in > 0 \; \exists \varepsilon_0 > 0 \;\; |x_\varepsilon - y_\varepsilon| \leq \varepsilon^q \;\; \textit{for } 0 < \varepsilon < \varepsilon_0$$

and denote by $\widetilde{\Omega} = \Omega_M / \sim$ the set of generalized points. The set of points with compact support is

$$\widetilde{\Omega}_c = \{\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega} \,:\, \exists K \subset\subset \Omega \; \exists \varepsilon_0 > 0 \;\; \textit{such that } x_\varepsilon \in K \;\; \textit{for } 0 < \varepsilon < \varepsilon_0\}.$$

2. Positive and positive definite Colombeau Generalized Functions

I this paper we will give the characterization of conditionally positive-definite and evenly positive-definite Colombeau generalized functions motivated by the results on positive and positive definite Colombeau generalized functions obtained in [4]. We now quote definitions and results obtained in [4].

Definition 2.1. *We say that $z \in \widetilde{\mathbb{C}}$, with a representative $(z_\varepsilon)_\varepsilon$, is a positive generalized complex number, and we write $z \geq 0$, if for every $q > 0$ there exists $\varepsilon_0 \in I = (0, 1]$ satisfying the inequality*

$$z_\varepsilon + \varepsilon^q \geq 0 \qquad \textit{for } \varepsilon < \varepsilon_0. \tag{2}$$

If the inequality (2) is true for one representative, it is true for any representative of $z \in \widetilde{\mathbb{C}}$. In [2] is proved that $z \geq 0$ if and only if there exists a positive representative $(z_\varepsilon)_\varepsilon$ such that $z_\varepsilon \geq 0$ for $\varepsilon \in I$.

Further, we observe generalized functions with values in $\widetilde{\mathbb{R}}$, keeping in mind that $\widetilde{\mathbb{R}}$ is a partially ordered ring.

In order to introduce the definition of Colombeau positive generalized function, we recall the definition of positive distributions, i.e. of generalized function from $\mathcal{D}'(\Omega)$, [1].

Definition 2.2. *$T \in \mathcal{D}'(\Omega)$ is positive, we write $T \geq 0$, if $\langle T, \varphi \rangle \geq 0$ for each positive test function.*

The definition of positive Colombeau generalized function is given in the following way:

Definition 2.3. *Colombeau generalized function $u \in \mathcal{G}^s(\Omega)$ is positive, $u \geq 0$ on Ω, if, for every compact subset K of Ω,*

$$\inf_{x \in K} u(x) \geq 0,$$

where $\inf_{x \in K} u(x) \geq 0$ means that there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that

$$\inf_{x \in K} u_\varepsilon(x) + \varepsilon^q \geq 0, \tag{3}$$

for every $q > 0$, $K \subset\subset \Omega$, and some $\varepsilon_0 \in I$ such that $\varepsilon < \varepsilon_0$.

Note, if $u \geq 0$ then every representative $(u_\varepsilon)_\varepsilon$ of u satisfies (3). It can be shown that $u \geq 0$ if and only if there exists a positive representative $(u_\varepsilon)_\varepsilon$ of u (see [2]). The representative $(u_\varepsilon)_\varepsilon$ is said to be positive if $u_\varepsilon \geq 0$ for $\varepsilon < \varepsilon_0$.

Example 2.1. *For $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\Omega)$, we have that $u^2 = [(u_\varepsilon^2)_\varepsilon]$ is a positive Colombeau generalized function, which is not associated with a Schwartz distribution.*

The proof of the following theorem may be found in [4].

Theorem 2.1. *([2]) Colombeau generalized function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\Omega)$ is positive if and only if for every $\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c$, the generalized number $[(u_\varepsilon(x_\varepsilon))_\varepsilon]$ is positive.*

Definition 2.4. *$u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\Omega)$ is $\mathcal{D}'(\Omega)$-weak positive if for every $\phi \in \mathcal{C}_c^\infty(\Omega), \phi > 0$,*

$$z_\phi = \left[\left(\int_{\mathbb{R}^n} u_\varepsilon(t) \phi(t) \, dt \right)_\varepsilon \right] \geq 0.$$

Generalized function $[(\frac{1}{\varepsilon}\varphi(\frac{\cdot}{\varepsilon}))_\varepsilon]$, where $(\varphi_\varepsilon)_\varepsilon$ is a net of mollifiers (given by (1)), is $\mathcal{D}'(\mathbb{R})$-weak positive but not positive.

We mention the following property of Colombeau generalized functions. If $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\Omega)$ satisfies $u \geq 0$ and $u \leq 0$, then $u = 0$. Also, if $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\Omega)$ satisfies $u \geq 0$ and $u \leq 0$ in the $\mathcal{D}'$-weak sense, then $u = 0$ in the sense of $\mathcal{D}'(\Omega)$, i.e. for each $\phi \in \mathcal{C}_c^\infty(\Omega)$, $\phi \geq 0$, we have

$$\langle u_\varepsilon(x), \phi(x) \rangle \in \mathcal{N}_0^s.$$

The following theorem gives the compatibility between Colombeau generalized functions and classical generalized functions, i.e. distributions. The proof is given in [4].

Theorem 2.2. *Let $T \in \mathcal{D}'(\Omega)$. $Cd(T) \in \mathcal{G}^s(\Omega)$ is $\mathcal{D}'$-weak positive if and only if T is positive.*

As already mentioned, the elements of Colombeau algebras are characterized by their point values. This implies the following definition of positive-definite Colombeau generalized functions.

Definition 2.5. *Colombeau generalized function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\mathbb{R}^n)$ is positive-definite on $\mathbb{R}^n$, if there exists a representative $(u_\varepsilon)_\varepsilon$ of u, such that for any of the real numbers $x_1, \dots, x_m$ and complex numbers $z_1, \dots, z_m$, and for every $q > 0$ there exists $\varepsilon_0 \in I$, such that*

$$\sum_{i,j=1}^{m} u_\varepsilon(x_i - x_j) z_i \bar{z}_j + \varepsilon^q \geq 0, \ \ \textit{for } \varepsilon < \varepsilon_0.$$

It can be shown that there exists a representative $(u_\varepsilon)_\varepsilon$ satisfying the condition

$$\sum_{i,j=1}^{m} u_\varepsilon(x_i - x_j) z_i \bar{z}_j \geq 0, \quad \varepsilon \in I,$$

for arbitrary numbers $x_i, x_j \in \mathbb{R}, z_i, z_j \in \mathbb{C}, i, j = 1, \dots, m$.

Theorem 2.3. *The following conditions are equivalent:*

(a) u is positive-definite on $\mathbb{R}^n$.

(b) There exists a representative $(u_\varepsilon)_\varepsilon$ of u such that

$$(\forall q > 0)(\exists \varepsilon_0 \in I)(\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n))$$
$$\langle u_\varepsilon(x) + \varepsilon^q, \phi * \check{\phi}(x) \rangle \geq 0, \ \ \varepsilon < \varepsilon_0.$$

(c) For every $q > 0$ there exists $\varepsilon_0 \in I$ and the neighborhood of zero $\mathcal{U}$ in $\mathcal{C}_c^\infty(\Omega)$ such that

$$\inf_{\phi \in \mathcal{U}} \int_{\mathbb{R}^n} (u_\varepsilon(t) + \varepsilon^q) \phi * \check{\phi}(t) \, dt \geq 0, \ \ \varepsilon < \varepsilon_0.$$

The conditions of the previous theorem are equivalent to the condition of the the next definition [4]:

Definition 2.6. *Colombeau generalized function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\mathbb{R}^n)$ is $\mathcal{D}'(\mathbb{R}^n)$-weak positive-definite if, for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,*

$$z_\phi = \left[\left(\int_{\mathbb{R}^n} u_\varepsilon(t) \phi * \check{\phi}(t) \, dt \right)_\varepsilon \right] \geq 0.$$

The proof of the next theorem may be found in [4].

Theorem 2.4. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$. $Cd(T) \in \mathcal{G}^s(\mathbb{R}^n)$ is $\mathcal{D}'(\mathbb{R}^n)$-weak positive-definite if and only if T is positive-definite distribution.*

3. Conditionally Positive-Definite Colombeau Generalized Functions

We give the definitions of conditionally positive-definite distribution (generalized function) [1]:

Definition 3.1. *Distribution T is conditionally positive-definite of order s, if the inequality*

$$\langle D\bar{D}T, \varphi * \check{\varphi}\rangle \geq 0$$

holds for each test function φ and all linear homogeneous differential operators D with constant coefficients of order s.

One can see that a distribution T is conditionally positive-definite if the inequality

$$\langle T, \varphi * \check{\varphi}\rangle \geq 0$$

holds for all test functions of the form $\varphi(x) = D\psi(x), x \in \mathbb{R}^n$, where D denotes a linear homogeneous differential operator with constant coefficients of order s, and ψ is a test function.

We define now conditionally $\mathcal{D}'(\mathbb{R}^n)$-weak positive-definite Colombeau generalized functions.

Definition 3.2. *Colombeau generalized function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\mathbb{R}^n)$ is called conditionally $\mathcal{D}'(\mathbb{R}^n)$-weak positive-definite if*

$$z_\phi = \left[\left(\int_{\mathbb{R}^n} u_\varepsilon(t)\phi * \check{\phi}(t)\, dt\right)_\varepsilon\right] \geq 0$$

for each $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ of the form $\phi = D\psi$, where D stands for a linear homogeneous differential operator with constant coefficients of order s, and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

We have the theorem:

Theorem 3.1. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$. $Cd(T) \in \mathcal{G}^s(\mathbb{R}^n)$ is conditionally $\mathcal{D}'(\mathbb{R}^n)$-weak positive-definite if and only if T is conditionally positive-definite distribution.*

Proof. Using the partition of unity (see [6, Ch.16]) we assume that $T \in \mathcal{E}'(\mathbb{R}^n)$.

Let T be conditionally positive-definite distribution with compact support. For each $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ having the form $\phi = D\psi$, where D is a linear homogeneous differential operator with constant coefficients of order s and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle T * \varphi_\varepsilon, \phi * \check{\phi}\rangle &= \langle T * \varphi_\varepsilon, [D\psi] * [\check{D\psi}]\rangle \\
&= \langle T, ([D\psi] * [\check{D\phi}]) * \check{\varphi}_\varepsilon\rangle \\
&= \langle T(x), \int ([D\psi] * [\check{D\psi}])(x - \varepsilon t)\varphi(-t)\, dt\rangle.
\end{aligned}$$

Taylor formula implies

$$\begin{aligned}\langle T(x), \quad &\int \Big(([D\psi] * [\check{D\psi}])(x) + (-\varepsilon t)([D\psi] * [\check{D\psi}])'(x) + \cdots + \\ &(-\varepsilon t)^q([D\psi] * [\check{D\psi}])(x - \xi(t))\Big)\, \varphi(-t)\; dt\rangle \\ &\geq \langle T(x), [D\psi] * [\check{D\psi}](x)\rangle + C(T, D\psi, \varphi)\varepsilon^q,\end{aligned}$$

for every q, where the constant $C(T, D\psi, \varphi)$ depends on $T, D\psi$ i φ. This inequality tells us that $T * \varphi_\varepsilon$, $\varepsilon \in I$, satisfies the conditions of Definition 3.2, i.e. $Cd(T)$ is conditionally $\mathcal{D}'(\mathbb{R}^n)$-weak positive-definite.

Conversely, if $Cd(T) = [(T * \varphi_\varepsilon|_\Omega)_\varepsilon]$ fulfils the requirements of Definition 3.2, we have

$$\langle T * \varphi_\varepsilon, \phi * \check{\phi}\rangle \geq 0,$$

for each test function ϕ on $\mathbb{R}^n$ of the form $\phi = D\psi$. Again D is the linear homogeneous differential operator with constant coefficients of order s, and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. In other words, we have

$$\langle T * \varphi_\varepsilon, [D\psi] * [\check{D\psi}]\rangle \geq 0.$$

This means that for every $q > 0$, and any test function $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, there exists $\varepsilon_0 \in I$ such that, for $\varepsilon < \varepsilon_0$, holds

$$\langle T * \varphi_\varepsilon, [D\psi] * [\check{D\psi}]\rangle + \varepsilon^q \geq 0, \qquad \text{i.e.}$$

$$\langle T, ([D\psi] * [\check{D\psi}]) * \check{\varphi}_\varepsilon\rangle + \varepsilon^q \geq 0.$$

Letting $\varepsilon \to 0$, we obtain

$$\langle T, [D\psi] * [\check{D\psi}]\rangle \geq 0, \text{ for any } \psi \in \mathcal{C}_c^\infty(\Omega),$$

i.e. T is conditionally positive-definite distribution. □

4. Evenly Positive-Definite Colombeau Generalized Functions

Let us recall the definitions of even and evenly positive-definite distributions [1]:

Definition 4.1. *A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is even (with respect to each argument), if*

$$T(\pm x_1, \ldots, \pm x_n) = T(x_1, \ldots, x_n),$$

for any combination of signs.

Definition 4.2. *Even distribution such that*

$$\langle T, \varphi * \check{\varphi}\rangle \geq 0,$$

only for even test functions $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, is called evenly positive-definite distribution.

We give a definition of evenly $\mathcal{D}'(\mathbb{R}^n)$-weak positive-definite Colombeau generalized function.

Definition 4.3. *Colombeau generalized function* $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\mathbb{R}^n)$ *is evenly* $\mathcal{D}'(\mathbb{R}^n)$*-weak positive-definite if*

$$z_\phi = \left[\left(\int_{\mathbb{R}^n} u_\varepsilon(t)\phi * \check{\phi}(t)\, dt \right)_\varepsilon \right] \geq 0,$$

for any even test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

We have the following result:

Theorem 4.1. *Let* $T \in \mathcal{D}'(\mathbb{R}^n)$. $Cd(T) \in \mathcal{G}^s(\Omega)$ *is evenly* $\mathcal{D}'$*-weak positive-definite Colombeau generalized function if and only if* T *is evenly positive-definite distribution.*

Proof. In the same way as before, using the partition of unity (see [6, Ch.16]), we assume that $T \in \mathcal{E}'(\mathbb{R}^n)$.

Let T be evenly positive-definite distribution with compact support. For each $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ even, we have

$$\begin{aligned} \langle T * \varphi_\varepsilon, \phi * \check{\phi} \rangle &= \langle T, (\phi * \check{\phi}) * \check{\varphi}_\varepsilon \rangle \\ &= \langle T(x), \int (\phi * \check{\phi})(x - \varepsilon t)\varphi(-t)\, dt \rangle. \end{aligned}$$

Taylor formula implies

$$\begin{aligned} &\langle T(x), \int \big((\phi * \check{\phi})(x) + (-\varepsilon t)(\phi * \check{\phi})'(x) + \cdots \\ &\cdots + (-\varepsilon t)^q (\phi * \check{\phi})(x - \xi(t)) \big)\, \varphi(-t)\, dt \rangle \\ &\geq \langle T(x), (\phi * \check{\phi})(x) \rangle + C(T, \phi, \varphi)\varepsilon^q, \end{aligned}$$

for every q, where the constant $C(T, \phi, \varphi)$ depends on T, ϕ i φ. This inequality tells us that $T * \varphi_\varepsilon,\ \varepsilon \in I$, satisfies the conditions of Definition 4.3, i.e. $Cd(T)$ is evenly $\mathcal{D}'(\mathbb{R}^n)$-weak positive-definite.

Conversely, if $Cd(T) = [(T * \varphi_\varepsilon|_\Omega)_\varepsilon]$ fulfils the requirements of Definition 4.3, we have

$$\langle T * \varphi_\varepsilon, \phi * \check{\phi} \rangle \geq 0.$$

This means that for every $q > 0$, and any even test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, there exists $\varepsilon_0 \in I$ such that, for $\varepsilon < \varepsilon_0$, holds

$$\langle T * \varphi_\varepsilon, \phi * \check{\phi} \rangle + \varepsilon^q \geq 0, \qquad \text{i.e.}$$

$$\langle T, (\phi * \check{\phi}) * \check{\varphi}_\varepsilon \rangle + \varepsilon^q \geq 0.$$

Letting $\varepsilon \to 0$, we obtain

$$\langle T, \phi * \check{\phi} \rangle \geq 0, \text{ for every even } \phi \in \mathcal{C}_c^\infty(\Omega),$$

i.e. T is evenly positive-definite distribution. □

References

[1] Gel'fand I., M., Vilenkin, N., Ya., Generalized Functions, vol. 4. New York: Academic Press 1964.

[2] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R., Geometric Theory of Generalized Functions with Applications to General Relativity. Boston, Dordrecht, London: Kluwer Academic Publishers 2001.

[3] Kunzinger, M., Lie Transformation Groups in Colombeau Algebras. Dissertation zur Erlangung des akademischen Grades Doktor der Naturwissenschaften an der Formal-und naturwissenschaflichen Fakultaet der Universitaet Wien

[4] Oberguggenberger, M., Pilipović, S., Scarpalezos, D., Positive and Positive Definite Colombeau Generalized Functions, Positivity, 2004, in press

[5] Pilipović, S., Colombeau's Generalized Functions and Pseudo-differential Operators. Univ. Tokyo, 1993.

[6] Treves, F., Topological Vector Spaces, Distributions and Kernels. New York, London: Academic Press 1967.

Received by the editors March 23, 2005