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NOTE ON THE FACTORS OF GRAPHS

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Abstract. We show that if G is a connected graph of order at least four with an 1-factor F, and k a positive integer such that $G - V(e)$ has a k-factor for each $e \in F$, then G has a k-factor, where $V(e)$ denotes the set of endvertices of edge e.

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Graphs, considered in this paper, are finite, undirected and simple (without loops or multiple edges). We denote the vertex set of a graph G by $V(G)$ and its edge set by $E(G)$. For a positive integer k, a *k-factor* of a graph G is a k-regular spanning subgraph of G. In particular, an 1-factor of G is its complete matching. We often identify a k-factor with its edge set. So, if F is a k-factor, we write $e \in F$, instead of $e \in E(F)$.

In [2], the following has been proved

Theorem 1. *Let G be a graph of order at least three, and let k be a positive integer. If $G - \{x, y\}$ has a k-factor for any pair of adjacent vertices x and y, then G has a k- factor.*

Obviously, the number of subgraphs required by the condition of Theorem 1, is equal to $|E(G)|$. However, not all subgraphs $G - -\{x, y\}$, $xy \in E(G)$, may be necessary, when we ensure the existence of an 1-factor.

In this paper, we show that if G has an 1-factor F, then we have only to check $G - \{x, y\}$ for $xy \in F$ and, hence, the number of required subgraphs is reduced to $|V(G)|/2$.

Let G be an arbitrary graph. For $e \in E(G)$, we denote by $V(e)$ the set of endvertices of e, and the degree of x in G by $d(G, x)$. In the sequel, we consider the set of vertices $\Gamma(G, x)$, defined by

$$\Gamma(G, x) = \cup_{e \in E(G), x \in V(e)} (V(e) - \{x\}).$$

So, $\Gamma(G, x)$ is the set of vertices which are adjacent to x. We write $K(G)$ for the set of connected components of G. For a set S, we denote by $|S|$ the cardinality of S, and, for a graph G, we write $|G|$ instead of $|V(G)|$. If $X, Y \subseteq V(G)$ and $X \cap Y = \emptyset$, we denote by $e(G, X, Y)$ the number of edges joining X and Y, in G.

For the notation not defined here, we refer to [1].

The main result of this paper consists of the following

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Theorem 2. *Let G be a graph with an 1-factor F and of order at least four. Let k be a positive integer. If $G - V(e)$ has a k-factor for each $e \in F$, then G has a k-factor.*

Our proof heavily depends on Tutte's criterion for the existence of k-factors. Let S and T be disjoint subsets of $V(G)$. We define

$$H(G,S,T) = \{C \in K(G-(S \cup T)) : k|C| + e(G,C,T) \text{ is odd } \},$$

$$h(G,S,T) = |H(G,S,T)|,$$

$$\sigma(G,S,T) = k|S| + \sum_{t \in T} d(G-S,t) - k|T| - h(G,S,T).$$

Then, from [3], we have the following

Theorem 3.

(1) A graph G has a k-factor if and only if $\sigma(G,S,T) \geq 0$ for all pairs (S,T) of disjoint subsets of $V(G)$.

(2) $\sigma(G,S,T) \equiv k|G| (\text{mod} 2)$ for all pairs (S,T) of disjoint subsets of $V(G)$.

First, we estimate the difference between $\sigma(G,S,T)$ and $\sigma(G-X, S-X, T-X)$ for a certain $X \subseteq V(G)$. Obviously, we can prove the following lemma by simple calculations, but it can be found in [2].

Lemma 1.

(1) If $x \in S$, then $\sigma(G-x, S-x, T) = \sigma(G,S,T) - k$.

(2) If $x \in T$, then $\sigma(G-x, S, T-x) \leq \sigma(G,S,T) + k$.

(3) If $X \subseteq C$ for some $C \in K(G-(S \cup T))$, then $\sigma(G-X,S,T) \leq \sigma(G,S,T) + 1$.

Proof of Theorem 2. Without loss of generality, we may assume that G is connected. Suppose that G has no k-factor. Then, $\sigma(G,S,T) < 0$ for some $S,T \subseteq V(G)$, $S \cap T = \emptyset$. Take S, T so that $\sigma(G,S,T) < 0$ and $|T|$ is as small as possible. Let $U = V(G) - (S \cup T)$. Note that $h(G,S,T) \leq |U|$. Since G has an 1-factor F, then, clearly, $|G|$ is even. Thus, by Theorem 3(2), we have $\sigma(G,S,T) \leq -2$.

For $X \subseteq V(G)$, we use the abbreviation $\sigma(X)$, in order to denote $\sigma(G-X, S-X, T-X)$. Obviously, according to this abbreviation, $\sigma(X) < 0$ implies that $G - X$ has no k-factor. Furthermore, $\sigma(G,S,T) = \sigma(\emptyset)$.

Now, we formulate the following claim.

Claim. If $x \in V(G)$ and $|\Gamma(G,x)|d \geq 2$, then $d(G,x) \geq k+1$.

Proof. Since $|G| \geq 4$ and $G - V(e)$ has a k-factor for any $e \in F$, then $d(G,x) \geq k$ for all $x \in V(G)$. Assume, for some $x \in V(G)$, that $|\Gamma(G,x)| \geq 2$ and $d(G,x) = k$. Let $y \in V(G)$ such that $xy \in F$ and take $z \in \Gamma(G,x) - \{y\}$. Thus, for some $u \in V(G)$, we have $zu \in F$. Since $d(G - \{z,u\}, x) \leq k-1$, then $G - \{z,u\}$ has no k-factor. This contradicts the above assumption and, consequently, the claim is proved.

Now, we show that $S = \emptyset$. Suppose that $S \neq \emptyset$, say $x \in S$. Thus, for some $y \in V(G)$, $xy \in F$. If $y \in S$, then, by the above Lemma, we have

$$\sigma(\{x, y\}) = \sigma(\{x\}) - k = \sigma(\emptyset) - 2k < 0.$$

Similarly, if $y \in T$, then we have

$$\sigma(\{x, y\}) \leq \sigma(\{x\}) + k = \sigma(\emptyset) - k + k = \sigma(\emptyset) < 0,$$

and if $y \in U$, then we have

$$\sigma(\{x, y\}) \leq \sigma(\{x\}) + 1 \leq \sigma(\emptyset) - k + 1 < 0.$$

Therefore, $\sigma(\{x, y\}) < 0$ for every y, $xy \in F$. This contradicts our assumption and, hence, $S = \emptyset$.

Since $S = \emptyset$, then $\sigma(\emptyset) = \sum_{t \in T}(d(G, t) - k) - h(G, \emptyset, T)$. Suppose that $xy \in F$ and $x, y \in U$. Then, $\{x, y\} \subseteq C$ for some $C \in K(G - T)$. However, by the above lemma, this implies that $\sigma(\{x, y\}) \leq \sigma(\emptyset) + 1 \leq -1$. This is also a contradiction and, consequently, if $xy \in F$ and $x \in U$, then $y \in T$. Since F is an 1-factor of G, this implies that $|U| \leq |T|$.

If $d(G, t) \geq k + 1$ for all $t \in T$, then

$$-2 \geq \sigma(\emptyset) = \sum_{t \in T}(d(G, t) - k) - h(G, \emptyset, T) \geq |T| - |U|.$$

This is a contradiction and, hence, $d(G, t) = k$ for some $t \in T$. By the above claim, we have $|\Gamma(G, t)| = 1$. Next, consider $\sigma(G, \emptyset, T - \{t\})$, for which we have

$$\begin{aligned} \sigma(G, \emptyset, T - \{t\}) &= \sum_{x \in T - \{t\}} d(G, x) - k|T - \{t\}| - h(G, \emptyset, T - \{t\}) \\ &= \sum_{x \in T} d(G, x) - d(G, t) - k|T| + k - h(G, \emptyset, T - \{t\}) \\ &= \sigma(G, \emptyset, T) + h(G, \emptyset, T) - h(G, \emptyset, T - \{t\}). \end{aligned}$$

Since $|\Gamma(G, t)| = 1$, then $h(G, \emptyset, T - \{t\}) \geq h(G, \emptyset, T) - 1$ and, therefore, $\sigma(G, \emptyset, T - \{t\}) \leq \sigma(G, \emptyset, T) + 1 \leq -1$. This contradicts the minimality of T and, now, the theorem is proved.

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References

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About paper

Note on the factors of graphs

by Danut Marcu

The editorial board has recently become aware of the following review published on MathSciNet:

MR2201402 (2006i:05130)
Marcu, Dănuţ
Note on the factors of graphs. (English summary)
Novi Sad J. Math. 35 (2005), no. 1, 11–13. 05C70

The author has plagiarized this paper from A. Saito, Discrete Math. 91 (1991), no. 3, 323–326; MR1129996 (92k:05100). It is almost word-for-word identical, except for the introduction of typographical errors. Although Marcu lists in his references an earlier paper by the true author, which the original paper had cited as well, he fails to acknowledge the prior publication of this work.

Reviewed by Jerrold W. Grossman

We wish to apology to Saito Akira for this unattended mistake.

Editorial board