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FUZZY RANDOM VARIABLE IN MATHEMATICAL ECONOMICS

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Abstract. This paper deals with a new concept introducing notion of fuzzy set to one mathematical model describing a economic system. A finite pure exchange economy is considered.

Key words and phrases: Finite pure exchange economy, Equilibrium, Fuzzy random variable

1. Introduction

Mathematical theory is an invaluable analytical tool in different areas of economy (see [1], [2],[3], [8]). In [8], the theory of correspondences is used as a mathematical framework to give an interpretation and solutions in theoretical economics. In this paper, using the model from [8], a new interpretation within the framework of the fuzzy theory is presented. The well known simple economic system - finite pure exchange economy - is considered as an illustration of how such abstract and complex mathematical theory have become a useful tool of economic theory. Since the definitions which include fuzzy sets are not restricted to finite pure exchange economy, it is obvious that this approach can be used to treat more complicated economic systems such as coalition, production or large economy.

The language of fuzziness, suitably interpreted, is a very convenient component of the rigorous language of theoretical economics. The uncertainties following from the individual character of agents in pure exchange economy, can be interpreted using fuzzy random variable. In this paper it is shown how the mathematical theory of fuzzy random variable can be used for the purpose of modelling and analyzing an economic system. The economies we deal with here all stem from the theory of general economic equilibrium. In the theory of equilibrium analysis it is usual to define an economy or economic model as a system of sets, as a mapping or as a measure. In this paper, the fuzzy set approach is used to define basic notions such as the preferences or "tastes" of economic agents, and to establish relations and theorems as the consequence of that approach.

Some interesting papers related to the application of fuzzy set theory in the theory of economic system are [6], [7], [9].

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2. Preliminaries

This section deals with some well known basic definitions and notions which will be used in the next section. The definitions and notions of economic system and fuzzy random variable are used according to [8] and [10].

Throughout this paper we denote by $R^k = \{(\chi_1, \chi_2, \cdots, \chi_k), \chi_1, \chi_2, \cdots, \chi_k \in R\}$, $k \geq 2$. Further, $R^k_+ = \{(\chi_1, \chi_2, \cdots, \chi_k),\ \ \chi_1, \chi_2, \cdots, \chi_k \in R^+\}$, where $R = (-\infty, \infty)$ and $R^+ = [0, \infty)$.

If $\{A_n\}$ is a sequence of nonempty subsets of R^k, then

- a point $x \in R^k$ is a limit point of $\{A_n\}$ if, for every neighborhood U of x, there exists an n in N such that for all $m \geq n$, $A_m \cap U \neq \emptyset$,
- a point $x \in R^k$ is a cluster point of $\{A_n\}$ if, for every neighborhood U of x and every n in N there is an $m \geq n$ such that $A_m \cap U \neq \emptyset$,
- $\liminf A_n$ is the set of all limit points of $\{A_n\}$,
- $\limsup A_n$ is the set of all cluster points of $\{A_n\}$,
- if $\liminf A_n = \limsup A_n = A$, then $A = \lim A_n$.

$\mathcal{F}(R^k)$ is a set of normal, upper semicontinuous fuzzy sets $u : R^k \to [0,1]$ with usual topology (see [10], [11]). The α-level set, $\alpha \in (0,1]$, is defined by $u_\alpha = \{x \in R^k : u(x) \geq \alpha\}$ and $u_+ = \{x \in R^k : u(x) > 0\}$. If the fuzzy set u is normal and upper semicontinuous, then α-level u_α, $\alpha \in (0,1]$, is nonempty closed set. If $(A,\mathcal{A},\mu)$ is a probability space, then the mapping $X : A \to \mathcal{F}(R^k)$ such that α-level mapping X_α is measurable according to the measurability of set valued mapping defined in [4], is fuzzy random variable (see[10],[11]).

The concept of an economic system or an economy may be formalized in different ways. In this paper we use the concept used in [8]. When the theory is only concerned with the economic exchange system and process (the markets), with total independence of what happens in the production sector, then this is best represented by an economic system without production, called a **pure exchange economy** and conceived as a structure

$$\mathcal{E} = ((A, \mathcal{A}, \mu), X, R^k).$$

The triple $(A,\mathcal{A},\mu)$ is a probability space of agents (consumers) with the following economic interpretation: A is the set of agents (consumers); $\mathcal{A}$, a σ-algebra, is the set of all possible coalitions of agents; μ is a probability measure, an indicator of the totality of agents in each coalition of $\mathcal{A}$. If A is finite, then $\mathcal{E}$ is **finite pure exchange economy**.

The Euclidean space R^k is called the **commodity space** so that a point $x = (\chi_1, \chi_2, \cdots, \chi_k) \in R^k$ is a **commodity bundle**. Each axis of R^k is given the task of representing amounts of a specific commodity. In this paper we confine ourselves to bundles in the nonnegative cone R^k_+ of the space R^k.

The symbol X denotes fuzzy random variable, the measurable fuzzy function from A to $\mathcal{F}(R^k)$ called **consumption fuzzy function**. The fuzzy set $X(a)$

is called the **consumption fuzzy set** of agent $a \in A$. It represents all the consumption plans which are a priori possible for a and it may contain positive, zero or negative components. An $\chi_i(a) > 0$, $i = 1, 2, \ldots, k$ is considered to be an input to a. An $\chi_i(a) < 0, i = 1, 2, \ldots, k$ is considered to be an output of a. The grade $X(a) \in [0, 1]$ represents a preference mapping, i.e. a preference relation on commodity space purporting the consumer's tastes.

For two commodity bundles $x, y \in R^k$, $x = (\chi_1, \chi_2, \cdots, \chi_k)$, $y = (\upsilon_1, \upsilon_2, \cdots, \upsilon_k)$, we use the following inequality symbols: $x \geq y$ to mean that $\chi_i \geq \upsilon_i$ for every $i = 1, 2, \ldots, k$; $x > y$ to mean that $x \geq y$ and $x \neq y$; and $x \gg y$ to mean that $\chi_i > \upsilon_i$ for every $i = 1, 2, \ldots, k$.

Next, we make reference to the notion of a **price system**. We assume that to every commodity $i = 1, 2, \ldots, k$ there is associated a real number $\pi_i \geq 0$, its price. A vector $p = (\pi_1, \pi_2, \ldots, \pi_k)$ is called a price system. If the price system p prevails, then the real number $p \cdot x$ is called the value of the bundle $x \in R^k$.

In this paper we shall always work with nonnegative prices. Every vector of prices $p = (\pi_1, \pi_2, \ldots, \pi_k)$ can be normalized and it is convenient here to let

$$P = \{p \in R^k_+ : 0 < \pi_i < 1, i = 1, 2, \ldots, k, \sum_{i=1}^{k} \pi_i = 1\}.$$

Let $\bar{P}$ denotes the closure of P and $P_n = \{p \in P : \pi_i \geq \frac{1}{n}, i = 1, \cdots, k\}$, $n \in \{k, k+1, \cdots\}$.

To complete the consumption sector of an economy, one introduces the function $i : A \to R^k_+$ which assigns to each agent $a \in A$ the agent's initial endowment vector $i(a) \in R^k_+$. The function i is called the **initial allocation**. An agent a of the economy $\mathcal{E}$ is fully characterized by the pair $(X(a), i(a))$. If an agent a owns some amount of some commodity, and if the price system $p \in P$ prevails, then the function $w(a, p) = p \cdot i(a)$, called **wealth** of the agent a, can be used instead of $i(a)$.

In this paper a finite economy (A is finite) is considered. Then, the sum r of all initial allocations $i(a)$, $r = \sum_{a \in A} i(a)$, called the **total resources** of $\mathcal{E}$, is an element of R^k_+. The result of the exchange activity in a pure exchange economy is a redistribution of the total resources r. Each consumer $a \in A$ maximizes his satisfaction level by choosing preferable element (an element with higher preference grade) from the set $X_+(a)$. This leads to exchange of commodities among the members of $\mathcal{E}$ and hence to a redistribution of r - that is, to a new state of economy described by a new allocation $f = (f(a) : a \in A)$. Since only exchange takes place, it is clear that feasibility requires that the condition $r = \sum_{a \in A} i(a) = \sum_{a \in A} f(a)$ must be satisfied. We call such a f a **feasible allocation**.

For every price system $p \in \bar{P}$ and for every $a \in A$, two subsets of R^k are defined: the budget set $b(a, p) = \{x \in X_+(a) : p \cdot x \leq w(a, p)\}$ and the demand set $d(a, p) = \{y \in b(p, a) : X(a)(y) \geq X(a)(x),\ \ y \not< x, \text{ for all } x \in b(a, p)\}$.

A **competitive equilibrium** for an economy $\mathcal{E}$ is a pair (f, p), where f is a feasible allocation and p is a price system, such that $f(a) \in d(a, p)$.

3. Competitive equilibrium for finite pure exchange economy

In this section we assume that:

- $\mathcal{E} = ((A, \mathcal{A}, \mu), X, R^k)$ is a finite pure exchange economy,
- for every $a \in A$, the set $X_+(a) = R^k_+$,
- for every $a \in A$, the sets $X_\alpha(a) \subset R^k_+$, $\alpha \in (0, 1]$ are convex,
- for every $a \in A$, the fuzzy set $X(a)$ is nondecreasing function, that is, for every $x, y \in X_+(a)$, $x < y$, it follows that $X(a)(x) \leq X(a)(y)$,
- if the price system $p \in P$ prevails, then $\inf_{x \in X_+(a)} p \cdot x < w(a, p)$,
- the total resources $r = \sum_{a \in A} i(a) \gg 0$.

Lemma 3.1. *Let $a \in A$, $p \in P$ and $c > 0$. Then $d(a, p) = d(a, cp)$.*

Proof. If $c > 0$, from the equivalence

$$p \cdot x \leq p \cdot i(a) = w(a) \Longleftrightarrow cp \cdot x \leq cp \cdot i(a) = cw(a)$$

it follows that $b(a, p) = b(a, cp)$, which implies $d(a, p) = d(a, cp)$. □

Lemma 3.2. *If $p \in P$ and $x \in d(a, p)$, then $p \cdot x = w(a, p)$ for every $a \in A$.*

Proof. If $x \in d(a, p)$, then $p \cdot x \leq w(a, p)$, $X(a)(x) \geq X(a)(y)$ and $x \not< y$ for all $y \in b(a, p)$. Let us assume that $p \cdot x \neq w(a, p)$, i.e. that $p \cdot x < w(a, p)$. Then there exists $y \in b(a, p)$, $x < y$, which contradicts the fact that $x \in d(a, p)$. □

Lemma 3.3. *Let $a \in A$ and $p \in P$. Then the budget set $b(a, p)$ is nonempty, compact and convex.*

Proof. Since $X_+(a) = R^k_+$ and since $\inf_{x \in X_+(a)} p \cdot x < w(a, p)$, the budget set $b(a, p) = \{x \in R^k_+ : p \cdot x \leq w(a, p)\}$ is nonempty, bounded, closed and convex. □

Lemma 3.4. *Let $a \in A$ and $p \in P$. Then the demand set $d(a, p)$ is nonempty, compact and convex.*

Proof. From the last lemma, the budget set $b(a,p) \in R^k_+$ is a nonempty, compact and convex set and $X(a)$ is upper semicontinuous mapping, which means that $X(a)$ attends its maximum on the set $b(a,p)$. Hence $d(a,p) \neq \emptyset$. Further, since $d(a,p)$ is a closed subset of the compact set $b(a,p)$, it is compact too.

To prove convexity we shall use the fact that the intersection of convex sets is convex itself. In the last lemma it was proved that $b(a,p)$ is convex. On the other hand, the set $d(a,p)$ is not empty, meaning that there exists $x \in R^k_+$ which belongs to $d(a,p)$. Let the grade of x be α, i.e. $X(a)(x) = \alpha$. The set $X_\alpha(a)$ is a convex set. The set $W = \{y \in b(a,p) : y \not< x, \text{ for all } x \in b(a,p)\} = \{y \in b(a,p) : p \cdot y = w(a,p)\}$ is convex too. Hence, $d(a,p) = X_\alpha(a) \bigcap b(a,p) \bigcap W$ is convex set. □

Lemma 3.5. *If the sequence $\{p_n\}_{n\in N}$ converges to any point $p \in P$, then every sequence $\{x_n\}_{n\in N}$, $x_n \in d(a,p_n)$, is bounded.*

Proof. If $p = (\pi_1, \pi_2, \cdots, \pi_k)$, $p_n = (\pi_{n1}, \pi_{n2}, \cdots, \pi_{nk})$, then (since $\lim_{n\to\infty} p_n = p$ and $0 < \pi_i < 1$), there exist $\epsilon > 0$ and $n_0(\epsilon)$ such that for $n > n_0$

$$0 < \min\{\pi_i\} - \epsilon \leq \pi_{ni} \leq \max\{\pi_i\} + \epsilon < 1.$$

Further, if $n > n_0$, for every $i \in \{1, 2, \cdots, k\}$, the next implication holds

$$x_n \in d(a,p_n) \Longrightarrow x_n \in b(a,p_n) \Longrightarrow p_n \cdot x_n \leq p_n \cdot i(a) \leq |p_n| \cdot |i(a)| \Longrightarrow$$

$$\rho \cdot \chi_{ni} < \rho(\chi_{n1} + \chi_{n2} + \cdots + \chi_{nk}) \leq p_n \cdot x_n \Longrightarrow \chi_{ni} < \rho^{-1}|p_n||i(a)| < M \in R,$$

where $\rho = \min\{\pi_i\} - \epsilon$ and $x_n = (\chi_{n1}, \chi_{n2}, \cdots, \chi_{nk})$. It means that the sequence $\{x_n\}$ is bounded. □

Lemma 3.6. *If $\lim_{j\to\infty} p_j = p \in P$ and if $x_j \in d(a,p_j)$, $\lim_{j\to\infty} x_j = x$, then $x \in d(a,p)$.*

Proof. Since $p_j \cdot x_j \leq p_j \cdot i(a)$, letting $j \to \infty$, we get $p \cdot x \leq p \cdot i(a)$, which means $x \in b(a,p)$

In order to prove that $x \in d(a,p)$, we shall show that $X(a)(x) \geq X(a)(y)$ and $x \not< y$ for every $y \in b(a,p)$.

If $i(a) = 0$ the case is trivial, so we assume $i(a) > 0$. If $y \in b(a,p)$, then either $p \cdot y < p \cdot i(a)$ or $p \cdot y = p \cdot i(a)$.

From inequality $p \cdot y < p \cdot i(a)$, for an n big enough we get $p_n \cdot y < p_n \cdot i(a)$, which means that $y \in b(a,p_n)$. Since $x_n \in d(a,p_n)$, it is clear that $X(a)(x_n) \geq X(a)(y) = \alpha$. Knowing that α-cuts of fuzzy sets $X(a)$ are closed sets and $x_n \in X_\alpha(a)$, we obtain that $x \in X_\alpha(a)$. Therefore $X(a)(x) \geq X(a)(y)$.

If $p \cdot y = p \cdot i(a) \neq 0$, then there exists a sequence $\{y_n\} :\ y_n < y,\ \lim_{n\to\infty} y_n = y$. Then $p \cdot y_n < p \cdot i(a)$. But for that kind of elements y_n (as it was shown in previous case) $X(a)(x) \geq X(a)(y_n)$. If one would have $\beta = X(a)(x) < X(a)(y) = \alpha$, since $X_\alpha(a)$ is closed, then, for some $n \in N$, $y_n \in X_\alpha(a)$. It would imply $\beta = X(a)(x) < \alpha \leq X(a)(y_n)$, which would contradict the inequality $X(a)(x) \geq X(a)(y_n)$. So, $X(a)(x) \geq X(a)(y)$.

In order to prove that $x \not< y$ for all $y \in b(a,p)$, we suppose opposite, i.e. that there exists $y \in b(a,p)$, $x < y$. Then the next implication holds

$$x < y \Rightarrow p \cdot x < p \cdot y \leq p \cdot i(a) = w(a,p) \Rightarrow$$

$$\Rightarrow \exists\, n \in N : p_n \cdot x_n < p_n \cdot i(a) = w(a,p_n) \Rightarrow x_n \notin d(a,p_n),$$

which contradicts the fact that $x_n \in d(a,p_n)$. Hence, $x \not< y$ for all $y \in b(a,p)$. It completes the proof that $x \in d(a,p)$. □

Lemma 3.7. *The set valued mapping $d(a,\cdot)$ is upper semicontinuous for every $p \in P$.*

Proof. Mapping $d(a,\cdot) : P \to 2^{R^k} \backslash \emptyset$ is upper semicontinuous if $\limsup d(a,p_n) \subset d(a,p)$ for any $p \in P$ and any sequence $\{p_n\}$ converging toward p in P.

In order to prove upper semicontinuity of $d(a,\cdot)$, we shall show that for every sequence $\{p_n\}_{n\in N}$ converging to any point $p \in P$ and for every sequence $\{x_n\}_{n\in N}, x_n \in d(a,p_n)$, there exists a convergent subsequence $\{x_j\}_{j\in N} \subset \{x_n\}_{n\in N}$, such that $\lim_{j\to\infty} x_j = x \in d(a,p)$.

According to Lemma 3.5, the sequence $\{x_n\}$ is bounded, hence there exists a convergent subsequence $\{x_j\} \subset \{x_n\}$, $\lim_{j\to\infty} x_j = x$, and by Lemma 3.6, $x \in d(a,p)$. □

Lemma 3.8. *Let P, $\bar{P}$, P_n are sets described in Preliminaries. Then*

1. $P_k \subset P_{k+1} \subset P_{k+2} \subset \cdots$
2. $\cup_{n=k}^{\infty} P_n = P \subset \lim_{n\to\infty} P_n = \bar{P}$
3. *P_n is nonempty, compact, convex set for every $n \in \{k, k+1, \cdots\}$.*

Proof. The proof for 1, 2. and 3. is obvious, so it is omitted. □

Lemma 3.9. *Let P_n is a set described in Preliminaries. Then*

1. *there exists a bounded set $S_n \subset R^k$ such that for all $p \in P_n$*

$$\sum_{a\in A} \{x - i(a) : x \in d(a,p)\} \subset S_n,$$

2. *if $p \in P_n$, then $p \cdot y = 0$ for every $y \in \sum_{a\in A}\{x - i(a) : x \in d(a,p)\}$.*

Proof. We shall prove 1, i.e. we shall prove the uniform boundedness of the set of sets $\sum_{a\in A}\{x - i(a) : x \in d(a,p)\}$, $p \in P_n$. Let us suppose opposite, i.e. that for every ball $L(0,m) \subset R^k$, $m \in N$, there exists $p_m \in P_n$ such that $\sum_{a\in A}\{x - i(a) : x \in d(a,p_m)\} \not\subset L(0,m)$. Since the set A of agents is finite, for some $a \in A$ there exists a sequence $\{x_i\}$, $x_i \in d(a,p_i)$, $\{p_i\} \subset \{p_m\}$, such that $\lim_{m\to\infty} \|x_i\| = \infty$. Since the related sequence $\{p_i\} \subset P_n$ is bounded, there exists a convergent subsequence $\{p_j\} \subset \{p_i\}$, $\lim_{j\to\infty} p_j = p \in P_n \subset P$. As it was proved in Lemma 3.5, for every sequence $\{p_j\}_{j\in N}$ converging to any point $p \in P$ and for every sequence $\{x_j\}_{j\in N}, x_j \in d(a,p_j)$, the sequence $\{x_j\}$ is bounded. But $\{x_j\} \subset \{x_n\}$ and this contradicts the fact that $\lim_{n\to\infty} \|x_n\| = \infty$, which means that the supposition is not correct

To prove 2, let $y \in \sum_{a\in A}\{x - i(a) : x \in d(a,p)\}$. Then $y = \sum_{a\in A}(x_a - i(a))$, $x_a \in d(a,p)$. Further,

$$p \cdot y = p \cdot \sum_{a\in A}(x_a - i(a)) = \sum_{a\in A}(p \cdot x_a - p \cdot i(a)) = \sum_{a\in A}(w(a) - w(a)) = 0$$

□

Lemma 3.10. *Let P, $\bar{P}$, are sets described in Preliminaries. Let $\{p_m\} \subset P$ and $\lim_{m\to\infty} p_m = p \in \bar{P} \setminus P$. If $p \cdot i(a) > 0$, then*

$$\lim_{m\to\infty} \inf\{\|x\| \in R : x \in d(a,p_m)\} = \infty.$$

Proof. To prove the lemma, we suppose opposite, i.e. that $\lim_{m\to\infty} \inf\{\|x\| \in R : x \in d(a,p_m)\} \in R$. Then there exists a bounded set $S \subset R$ such that $d(a,p_m) \cap S \neq \emptyset$ for infinitely many m's and there exists a bounded sequence $\{x_m\}_{m\in N}$, $x_m \in d(a,p_m) \cap S$. Therefore, there exists a convergent subsequence $\{x_n\} \subset \{x_m\}$ converging to some $x \in R^k$. Since $p_n \cdot x_n \le p_n \cdot i(a)$, letting $n \to \infty$, we get $p \cdot x \le p \cdot i(a)$, which means $x \in b(a,p)$

In order to prove that $x \in d(a,p)$, we shall show that $X(a)(x) \ge X(a)(y)$ and $x \not< y$ for every $y \in b(a,p)$.

Since $i(a) > 0$ and since $y \in b(a,p)$, then either $p \cdot y < p \cdot i(a)$ or $p \cdot y = p \cdot i(a)$.

From inequality $p \cdot y < p \cdot i(a)$, for an n big enough we get $p_n \cdot y < p_n \cdot i(a)$, which means that $y \in b(a,p_n)$. Since $x_n \in d(a,p_n)$, it is clear that $X(a)(x_n) \ge X(a)(y) = \alpha$. Knowing that α-cuts of fuzzy sets $X(a)$ are closed sets and $x_n \in X_\alpha(a)$, we obtain that $x \in X_\alpha(a)$. Therefore $X(a)(x) \ge X(a)(y)$.

If $p \cdot y = p \cdot i(a) \neq 0$, then there exists a sequence $\{y_n\} :\ y_n < y,\ \lim_{n\to\infty} y_n = y$. Then $p \cdot y_n < p \cdot i(a)$. But for that kind of elements y_n (as it was shown in the previous case) $X(a)(x) \ge X(a)(y_n)$. If one would have $\beta = X(a)(x) < X(a)(y) = \alpha$, since $X_\alpha(a)$ is closed, then, for some $n \in N$, $y_n \in X_\alpha(a)$. It would imply $\beta = X(a)(x) < \alpha \le X(a)(y_n)$, which would contradict the inequality $X(a)(x) \ge X(a)(y_n)$. So, $X(a)(x) \ge X(a)(y)$.

In order to prove that $x \not< y$ for all $y \in b(a,p)$, we suppose the opposite, i.e. that there exists $y \in b(a,p)$, $x < y$. Then the next implication holds

$$x < y \Rightarrow p \cdot x < p \cdot y \le p \cdot i(a) = w(a,p) \Rightarrow$$

$$\Rightarrow \exists\, n \in N : p_n \cdot x_n < p_n \cdot i(a) = w(a, p_n) \Rightarrow x_n \notin d(a, p_n),$$

which contradicts the fact that $x_n \in d(a, p_n)$. Hence, $x \not< y$ for all $y \in b(a, p)$. It completes the proof that $x \in d(a, p)$.

On the other hand, since $p \in \bar{P} \setminus P$, it follows that $b(a, p)$ is an unbounded set. But if $x \in d(a, p)$ then $x \not< y$ for all $y \in b(a, p)$, which means that the set $d(a, p)$ is empty. This contradicts the existence of the limit $x \in d(a, p)$, $x \in R^k$. It means that the supposition that $\lim_{m\to\infty} \inf\{\|x\| \in R : x \in d(a, p_m)\} \in R$ is not correct. □

For the set S_n, $n \in \{k, k+1, \ldots\}$, from Lemma 3.9, part 1, we can choose compact set S_n. Then the convex hull of S (denoted by $co\, S_n$), is the convex, compact set with the same properties. The set valued mapping $F_n : co\, S_n \to 2^{P_n}$ is defined by

$$F_n(x) = \{p \in P_n : p \cdot x = \max_{q \in P_n} q \cdot x\}$$

Lemma 3.11. *The mapping F_n described in the preceding paragraph has the next properties:*

1. *For every $x \in co\, S_n$, $F_n(x) \subset P_n$ is nonempty,compact, convex set.*
2. *F_n has a closed graph.*

Proof. Since P_n is a compact set and since scalar product is a continuous operation, it attends maximum on P_n, which means that $F_n(x) \neq \emptyset$. $F_n(x)$ is a closed subset of the compact set P_n, thus it is compact. To prove convexity, consider $a, b \in F_n(x)$. From the definition of F_n, we get $a \cdot x = \max_{q \in P_n} q \cdot x = m$ and $b \cdot x = \max_{q \in P_n} q \cdot x = m$. Then $(\lambda a + (1-\lambda)b) \cdot x = \lambda a \cdot x + (1-\lambda) b \cdot x = m$, i.e. $(\lambda a + (1-\lambda)b \in F_n(x)$.

In order to prove 2, we have to show that, if $(x_i, p_i) \in gr\, F$, $\lim_{i\to\infty} x_i = x$, $\lim_{i\to\infty} p_i = p$, then $p \in F_n(x)$. If $(x_i, p_i) \in gr\, F$, then $x_i \cdot q \leq x_i \cdot p_i$ for every $q \in P_n$. From continuity of scalar product, letting $i \to \infty$, for every $q \in P_n$ we have $\lim_{i\to\infty}(x_i \cdot q \leq x_i \cdot p_i) \Longrightarrow x \cdot q \leq x \cdot p$. So, we have proved that $p \in F_n(x)$. □

Lemma 3.12. *Let $H_n : co\, S_n \times P_n \to 2^{co\, S_n \times P_n}$ be a set valued mapping defined by*

$$F_n(x) \times \sum_{a \in A} \{x - i(a) : x \in d(a, p)\} = H_n(x, p).$$

Then H_n has a fixed point, i.e. there exists $(x_n, p_n) \in co\, S_n \times P_n$ such that $(x_n, p_n) \in H_n(x_n, p_n)$.

Proof. Since $co\, S_n \times P_n$ is a nonempty, compact, convex subset of R^{2k} and since H_n is a set valued mapping with convex images in $2^{co\, S_n \times P_n}$ which has a closed graph, Kakutani's fixed point theorem can be applied. □

Theorem 3.1. *Let $\mathcal{E}$ be a finite pure exchange economy. Then $\mathcal{E}$ has a competitive equilibrium (f, p), where $p \in P$.*

Proof. Let (x_n, p_n), $n \in \{k, k+1, \ldots\}$, be the fixed point from the last lemma. Then $x_n \in \sum_{a \in A}\{x - i(a) : x \in d(a, p_n)\}$, $p_n \in P_n$ and $p_n \in F_n(x_n) = \{p \in P_n : p \cdot x_n = \max_{q \in P_n} q \cdot x_n\}$. Obviously, invoking Lemma 3.9 (2), we get $q \cdot x_n \le p_n \cdot x_n = 0$ for every $q \in P_n$.

Since the sequence $\{p_n\} \subset P$, it is bounded with convergent subsequence $\{p_m\} \subset \{p_n\}$, $\lim_{m \to \infty} p_m = p$. On the other hand, concerning Lemma 3.8. (1), there exists $q \in P_k \subset P_n$, $n \in N$, such that $q \cdot x_n \le 0$ for all $n \ge k$. Hence, the sequence $\{x_n\}$ is bounded. Let $\{x_j\} \subset \{x_m\}$ be a convergent subsequence and $\lim_{j \to \infty} x_j = x$. Observe next that $p \in \bar{P}$, but $p \notin \bar{P} \setminus P$ (by Lemma 3.10) would imply divergence of the sequence $\{x_j\}$). Since $\{x_j\}$ is convergent subsequence, $p \in P$. By the method used in Lemma 3.6, we can show that $x \in \sum_{a \in A}\{x - i(a) : x \in d(a, p)\}$ and, according Lemma 3.9 (2), we get $p \cdot x = 0$. From the inequality $q \cdot x_j \le 0$ for every $q \in P_j$, letting $j \to \infty$ and from Lemma 3.8. (2), we have $q \cdot x \le 0$ for every $q \in \bar{P}$. Therefore $x \le 0$. Finally, from $p \cdot x = 0$, noting that all components of p are positive, we conclude $x = 0$. Since $0 \in \sum_{a \in A}\{x - i(a) : x \in d(a, p)\}$, there exists a feasible allocation $f(a) \in d(a, p)$ so $\sum_{a \in A} f(a) = \sum_{a \in A} i(a)$, what we had to prove. □

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