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THE ALGEBRA OF STRONGLY FULL TERMS

K. Denecke,[1], L. Freiberg[1]

Abstract. The well-known connection between hyperidentities of an algebra and identities satisfied by the clone of this algebra is studied here in a restricted setting, that of n-ary strongly full hyperidentities and identities of the n-ary clone of term operations of an algebra induced by strongly full terms, both of a type consisting only of n-ary operation symbols. We call such a type an n-ary type. Using the concept of a weakly invariant congruence relation we characterize varieties of n-ary type whose identities consist of strongly full terms which are closed under taking of isomorphic copies of their clones of all strongly full n-ary term operations. Finally, we show that a variety of n-ary type defined by identities consisting of strongly full terms has this property if and only if it is $\mathcal{O}_{SF}$-solid for the monoid $\mathcal{O}_{SF}$ of all strongly full hypersubstitutions which have surjective extensions.

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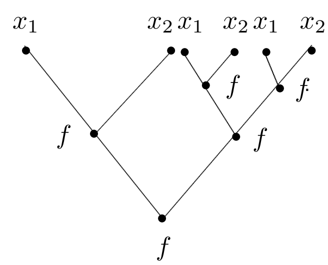
1. Preliminaries

In this paper we consider algebras whose fundamental operations have the same arities n. The type τ_n of such an algebra is a sequence $(n, \ldots, n, \ldots)$. Let $(f_i)_{i\in I}$ be an indexed set of operation symbols of arity n. We denote by $X := \{x_1, \ldots, x_n, \ldots\}$ a countably infinite set of individual variables, and for each $m \geq 1$ let $X_m := \{x_1, \ldots, x_m\}$. Then the union $W_{\tau_n}(X) = \cup_{m\geq 1} W_{\tau_n}(X_m)$ is the set of all (finitary) terms of type τ_n. The set $W_{\tau_n}(X)$ of all terms is the universe of the absolutely free algebra $\mathcal{F}_{\tau_n}(X) := (W_{\tau_n}(X); (\overline{f}_i)_{i\in I})$ of type τ_n on the alphabet X where the operations are defined by $\overline{f}_i(t_1, t_2, \ldots, t_n)$ $:= f_i(t_1, t_2, \ldots, t_n)$ for every n-tuple $(t_1, t_2, \ldots, t_n)$ of terms. Similarly, the set $W_{\tau_n}(X_m)$ of m-ary terms is the universe of the free algebra $\mathcal{F}_{\tau_n}(X_m) :=$ $(W_{\tau_n}(X_m); (\overline{f}_i)_{i\in I})$ on the alphabet X_m. Terms can be visualized as trees,

[1]University of Potsdam, Institute of Mathematics, Am Neuen Palais, 14415 Potsdam, Germany, kdenecke@rz.uni-potsdam.de

where the vertices are labelled by operation symbols and the leaves are labelled by variables. For instance, the following tree corresponds to the term $f(f(x_1, x_2), f(f(x_1, x_2), f(x_1, x_2)))$.



Now we consider the concept of a term in a restricted setting. *Strongly full* terms are inductively defined by the following steps:

(i) $f_i(x_1, \ldots, x_n), i \in I$, is a strongly full term,

(ii) If $t_1, \ldots, t_n$ are strongly full terms, then $f_i(t_1, \ldots, t_n)$ is strongly full.

The set $W^{SF}_{\tau_n}(X_n)$ of strongly full n-ary terms is the universe of an algebra $\mathcal{F}^{SF}_{\tau_n}(X_n) := (W^{SF}_{\tau_n}(X_n); (\overline{f}_i)_{i\in I})$ of our type τ_n. This algebra is generated by the set $F_n := \{f_i(x_1, \ldots x_n) \mid i \in I\}$. It is clearly a subalgebra of the absolutely free algebra $\mathcal{F}_{\tau_n}(X) := (W_{\tau_n}(X); (\overline{f}_i)_{i\in I})$ of type τ_n generated by the alphabet X. On the set $W^{SF}_{\tau_n}(X_n)$ we define an $(n+1)$-ary operation S^n_n as follows:

(i) $S^n_n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n) := f_i(t_1, \ldots, t_n)$,

(ii) $S^n_n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) := f_i(S^n_n(s_1, t_1, \ldots, t_n), \ldots,$
$S^n_n(s_n, t_1, \ldots, t_n))$ for $s_1, \ldots, s_n, t_1, \ldots, t_n \in W^{SF}_{\tau_n}(X_n)$.

Then $clone_{SF}\tau_n := (W^{SF}_{\tau_n}(X_n); S^n_n)$ is an algebra of type $\tau = (n+1)$ with F_n as a generating system. The algebra $clone_{SF}\tau_n$ is called *the clone of strongly full terms* of type τ_n.
If $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ is an algebra of type τ_n (n-ary algebra), then every strongly full term t of type τ_n induces a term operation $t^{\mathcal{A}}$ on $\mathcal{A}$ via the following steps:

(i) $[f_i(x_1, \ldots, x_n)]^{\mathcal{A}} := f_i^{\mathcal{A}}$

(ii) If $t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}$ are the n-ary term operations which are induced by the strongly full terms $t_1, \ldots, t_n \in W^{SF}_{\tau_n}(X_n)$, then $(f_i(t_1, \ldots, t_n))^{\mathcal{A}} :=$
$f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$ is the n-ary term operation induced by $f_i(t_1, \ldots, t_n)$.

Here the right hand side of the equation in (ii) means the n-ary operation defined by $f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})(a_1, \ldots, a_n) := f_i^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_n^{\mathcal{A}}(a_1, \ldots, a_n))$

for every $(a_1, \ldots, a_n) \in A^n$. Let $T_{SF}^{(n)}(\mathcal{A})$ be the set of all these term operations. On the set $T_{SF}^{(n)}(\mathcal{A})$ we define inductively an $(n+1)$-ary superposition operation $S_n^{n,A}$, by

(i) $S_n^{n,A}(f_i^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) := f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$ for $t_1, \ldots, t_n \in W_{\tau_n}^{SF}(X_n)$,

(ii) $S_n^{n,A}(f_i^{\mathcal{A}}(s_1^{\mathcal{A}}, \ldots, s_n^{\mathcal{A}}), t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) := f_i^{\mathcal{A}}(S_n^{n,A}(s_1^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}), \ldots,$ $S_n^{n,A}(s_n^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}))$.

This gives an algebra $\mathcal{T}_{SF}^{(n)}(\mathcal{A}) := (T_{SF}^{(n)}(\mathcal{A}); S_n^{n,A})$, called the n-ary strongly full (term) clone of the n-ary algebra $\mathcal{A}$.
In the case $n = 1$, this unary strongly full term clone forms a semigroup called the transition semigroup of $\mathcal{A}$, which has been intensively studied; see for instance [3]. In the next section we will find out that the clone of strongly full n-ary terms of type τ_n and the n-ary strongly full (term) clone of the n-ary algebra $\mathcal{A}$ belong to the same variety.

2. The Variety of Strongly Full Clones

Using a new set of variables $\mathcal{X} = (Y_i)_{i \in I}$ indexed by I, and an $(n+1)-$ary operation symbol $\widetilde{S}_n^n$ we define a new language of type $\tau = (n+1)$ and consider equations formulated in this new language.

Proposition 2.1 *The algebra clone*$_{SF}\tau_n$ *satisfies the following identity*
(C) $\quad \widetilde{S}_n^n(X_0, \widetilde{S}_n^n(X_{i_1}, X_2, \ldots, X_{n+1}), \ldots, \widetilde{S}_n^n(X_{i_n}, X_2, \ldots, X_{n+1})) \approx$
$\widetilde{S}_n^n(\widetilde{S}_n^n(X_0, X_{i_1}, \ldots, X_{i_n}), X_2, \ldots, X_{n+1})$.

Proof. We will give a proof by induction on the complexity of the strongly full term which is substituted for X_0. If we substitute for X_0 the strongly full term $f_i(x_1, \ldots, x_n)$ and for $X_{i_1}, \ldots, X_{i_n}, X_2, \ldots, X_{n+1}$ the n-ary terms $t_{i_1}, \ldots, t_{i_n}, t_2, \ldots, t_{n+1}$, then we obtain
$S_n^n(f_i(x_1, \ldots, x_n), S_n^n(t_{i_1}, t_2, \ldots, t_{n+1}), \ldots, S_n^n(t_{i_n}, t_2, \ldots, t_{n+1}))$
$= f_i(S_n^n(t_{i_1}, t_2, \ldots, t_{n+1}), \ldots, S_n^n(t_{i_n}, t_2, \ldots, t_{n+1}))$
$= S_n^n(f_i(t_{i_1}, \ldots, t_{i_n}), t_2, \ldots, t_{n+1})$
$= S_n^n(S_n^n(f_i(x_1, \ldots, x_n), t_{i_1}, \ldots, t_{i_n}), t_2, \ldots, t_{n+1})$
using the definition of S_n^n.
If we substitute for X_0 the term $t = f_i(s_1, \ldots, s_n)$ and assume inductively that (C) is satisfied for $s_1, \ldots, s_n$, then
$S_n^n(f_i(s_1, \ldots, s_n), S_n^n(t_{i_1}, t_2, \ldots, t_{n+1}), \ldots, S_n^n(t_{i_n}, t_2 \ldots, t_{n+1}))$

$$\begin{aligned}
&= f_i(S^n_n(s_1, S^n_n(t_{i_1}, t_2, \ldots, t_{n+1}), \ldots, S^n_n(t_{i_n}, t_2 \ldots, t_{n+1})), \ldots,\\
&\quad S^n_n(s_n, S^n_n(t_{i_1}, t_2, \ldots, t_{n+1}), \ldots, S^n_n(t_{i_n}, t_2 \ldots, t_{n+1})))\\
&= f_i(S^n_n(S^n_n(s_1, t_{i_1}, \ldots, t_{i_n}), t_2, \ldots, t_{n+1}), \ldots,\\
&\quad S^n_n(S^n_n(s_n, t_{i_1}, \ldots, t_{i_n}), t_2, \ldots, t_{n+1}))\\
&= S^n_n(f_i(S^n_n(s_1, t_{i_1}, \ldots, t_{i_n}), \ldots, S^n_n(s_n, t_{i_1}, \ldots, t_{i_n})), t_2, \ldots, t_{n+1}))\\
&= S^n_n(S^n_n(f_i(s_1, \ldots, s_n), t_{i_1}, \ldots, t_{i_n}), t_2, \ldots, t_{n+1})\\
&= S^n_n(S^n_n(t, t_{i_1}, \ldots, t_{i_n}), t_2, \ldots, t_{n+1}).
\end{aligned}$$

This shows that the algebra $clone_{SF}\tau_n$ satisfies (C). □

Algebras $(M; S)$ of type $n+1$ which satisfy (C) are called Menger algebras of rank n; see in [8], [4].

In a similar way one shows that also $\mathcal{T}^{(n)}_{SF}(\mathcal{A})$ satisfies (C). Let $V^{SFC}_{\tau_n}$ be the variety of type $(n+1)$ generated by the identity (C). Both algebras belong to this variety.

Now we consider the free algebra $\mathcal{F}_{V^{SFC}_{\tau_n}}(\{Y_i \mid i \in I\})$ in the variety $V^{SFC}_{\tau_n}$, generated by a special alphabet $\{Y_i \mid i \in I\}$. The fact that this alphabet is in bijection with the set of fundamental operations $(f_i)_{i\in I}$ of type τ_n, and hence with the set F_{τ_n} of fundamental terms which generates $clone_{SF}\tau_n$, will give us an isomorphism between this free algebra and the $clone_{SF}\tau_n$.

Theorem 2.2 *The algebra $clone_{SF}\tau_n$ is isomorphic to $\mathcal{F}_{V^{SFC}_{\tau_n}}(\{Y_i \mid i \in I\})$, and therefore free with respect to the variety $V^{SFC}_{\tau_n}$, and freely generated by the set*
$\{f_i(x_1, \ldots, x_n) \mid i \in I\}$.

Proof. We define a mapping $\varphi : W^{SF}_{\tau_n}(X_n) \to F_{V^{SFC}_{\tau_n}}(\{Y_i \mid i \in I\})$ inductively as follows:

(i) $\varphi(f_i(x_1 \ldots x_n)) := Y_i$ for every $i \in I$,

(ii) $\varphi(f_i(t_1, \ldots, t_n)) := \widetilde{S}^n_n(Y_i, \varphi(t_1), \ldots, \varphi(t_n))$.

Since φ maps the generating system of $clone_{SF}\tau_n$ onto the generating system of $\mathcal{F}_{V^{SFC}_{\tau_n}}(\{Y_i \mid i \in I\})$ it is surjective. We prove the homomorphism property $\varphi(S^n_n(t_0, t_1, \ldots, t_n)) = \widetilde{S}^n_n(\varphi(t_0), \ldots, \varphi(t_n))$ by induction on the complexity of the term t_0. If $t_0 = f_i(x_1, \ldots, x_n)$, then $\varphi(S^n_n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n))$ $= \varphi(f_i(t_1, \ldots, t_n)) = \widetilde{S}^n_n(Y_i, \varphi(t_1), \ldots, \varphi(t_n)) = \widetilde{S}^n_n(\varphi(f_i(x_1, \ldots, x_n)), \varphi(t_1), \ldots,$ $\varphi(t_n))$. Inductively, assume that $t_0 = f_i(s_1, \ldots, s_n)$ and that $\varphi(S^n_n(s_j, t_1, \ldots, t_n))$ $= \widetilde{S}^n_n(\varphi(s_j), \ldots, \varphi(t_n))$ for all $1 \leq j \leq n$. Then

$$\begin{aligned}
&\varphi(S^n_n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n))\\
&= \varphi(f_i(S^n_n(s_1, t_1, \ldots, t_n), S^n_n(s_2, t_1, \ldots, t_n), \ldots, S^n_n(s_n, t_1, \ldots, t_n))\\
&= \widetilde{S}^n_n(Y_i, \varphi(S^n_n(s_1, t_1, \ldots, t_n)), \varphi(S^n_n(s_2, t_1, \ldots, t_n)), \ldots, \varphi(S^n_n(s_n, t_1, \ldots, t_n)))\\
&= \widetilde{S}^n_n(Y_i, \widetilde{S}^n_n(\varphi(s_1), \varphi(t_1), \ldots, \varphi(t_n)), \ldots, \widetilde{S}^n_n(\varphi(s_n), \varphi(t_1), \ldots, \varphi(t_n)))\\
&= \widetilde{S}^n_n(\widetilde{S}^n_n(Y_i, \varphi(s_1), \ldots, \varphi(s_n)), \varphi(t_1), \ldots, \varphi(t_n)))
\end{aligned}$$

$$= \widetilde{S}_n^n(\varphi(f_i(s_1,\ldots,s_n)),\varphi(t_1),\ldots,\varphi(t_n)))$$
$$= \widetilde{S}_n^n(\varphi(t_0),\varphi(t_1),\ldots,\varphi(t_n)).$$

Thus φ is a homomorphism. The mapping φ is bijective since $\{Y_i \mid i \in I\}$ is a free independent set and therefore we have

$$Y_i = Y_j \quad \Rightarrow \quad i = j \quad \Rightarrow \quad f_i(x_1,\ldots,x_n) = f_j(x_1,\ldots,x_n).$$

Thus φ is a bijection between the generating sets of $clone_{SF}\tau_n$ and $\mathcal{F}_{V_{\tau_n}^{SFC}}(\mathcal{X})$, and hence it is bijective on $W_{\tau_n}^{SF}(X_n)$. Altogether, φ is an isomorphism. □

3. Strongly full Hypersubstitutions and Substitutions of $clone_{SF}\tau_n$

Since $clone_{SF}\tau_n = (W_{\tau_n}^{SF}(X_n); S_n^n)$ is free, freely generated by the set F_{τ_n}, any mapping η from this generating set into $W_{\tau_n}^{SF}(X_n)$ can be uniquely extended to an endomorphism $\overline{\eta}$ from $clone_{SF}\tau_n$. Such mappings are called substitutions. We will denote by $Subst_{SF}$ the set of all such clone substitutions. We introduce a binary composition operation $\odot$ on this set, by setting $\eta_1 \odot \eta_2 := \overline{\eta_1} \circ \eta_2$, where $\circ$ denotes the usual composition of functions. Denoting by id the identity mapping on $\{f_i(x_1,\ldots,x_n) \mid i \in I\}$, we see that $(Subst_{SF};\odot,id)$ is a monoid. In order to examine the connection between this monoid and the monoid of hypersubstitutions of type τ_n, we introduce some basic concepts about hyper-identities and hypersubstitutions. Note that although these concepts can be defined for arbitrary type, we define them here only for type τ_n and for strongly full terms.

Definition 3.1 *A strongly full hypersubstitution of n-ary type τ_n is a mapping from the set $\{f_i \mid i \in I\}$ of n-ary operation symbols of the type τ_n to the set $W_{\tau_n}^{SF}(X_n)$ of all strongly full n-ary terms of type τ_n.*

Any strongly full hypersubstitution σ induces a mapping $\hat{\sigma}$ defined on the set $W_{\tau_n}^{SF}(X_n)$ of all n-ary terms of the type τ_n, as follows.

Definition 3.2 *Let σ be a strongly full hypersubstitution of type τ_n. Then σ induces a mapping $\hat{\sigma} : W_{\tau_n}^{SF}(X_n) \longrightarrow W_{\tau_n}^{SF}(X_n)$, by setting*

(i) $\hat{\sigma}[f_i(x_1,\ldots,x_n)] := \sigma(f_i)$, $i \in I$,

(ii) $\hat{\sigma}[f_i(t_1,\ldots,t_n)] := \sigma(f_i)(\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n]) := S_n^n(\sigma(f_i),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])$.

Let $Hyp^{SF}(\tau_n)$ be the set of all strongly full hypersubstitutions of type τ_n. By setting $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, we define a binary operation $\circ_h$ on $Hyp^{SF}(\tau_n)$. This operation is associative, and together with the identity hypersubstitution σ_{id}

defined by $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_n)$ we have a monoid $(Hyp^{SF}(\tau_n); \circ_h, \sigma_{id})$. Let $\mathcal{M}$ be any submonoid of $Hyp^{SF}(\tau_n)$. If $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ is an n-ary algebra, then an identity $s \approx t$ in $\mathcal{A}$ is said to be an M-hyperidentity in $\mathcal{A}$ if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in $\mathcal{A}$ for every hypersubstitution $\sigma \in M$. In the special case that M is all of $Hyp^{SF}(\tau_n)$, an M-hyperidentity is usually called a strongly full hyperidentity. An identity is an M-hyperidentity of a variety V if it is an M-hyperidentity of every algebra in V. A variety in which every identity of the variety holds as an M-hyperidentity is called an M-solid variety, or a SF-solid variety in the special case $M = Hyp^{SF}(\tau_n)$. For more detailed information on hyperidentities we refer the reader to [1].
Between strongly full hypersubstitutions and substitutions of $clone_{SF}\tau_n$ there is a close interconnection.

Proposition 3.3 *The monoids* $(Subst_{SF}; \odot, id)$ *and* $(Hyp^{SF}(\tau_n); \circ_h, \sigma_{id})$ *are isomorphic.*

Proof. We define a mapping $\psi : Subst_{SF} \longrightarrow Hyp^{SF}(\tau_n)$ by $\psi(\eta) := \eta \circ \sigma_{id}$. This gives a well-defined mapping between $Subst_{SF}$ and $Hyp^{SF}(\tau_n)$. The mapping ψ is surjective, since any strongly full hypersubstitution σ can be obtained as $\psi(\eta)$ for $\eta = \sigma \circ \sigma_{id}^{-1}$. The mapping ψ is also injective, since

$$\psi(\eta_1) = \psi(\eta_2) \quad \Rightarrow \quad \eta_1 \circ \sigma_{id} = \eta_2 \circ \sigma_{id} \quad \Rightarrow \quad \eta_1 = \eta_2,$$

since σ_{id} is a bijection. To show that ψ is a homomorphism, we first verify the following additional property:

$$(\eta \circ \sigma_{id})\hat{}\,[t] \;=\; \overline{\eta}(t), \qquad (*)$$

where $\overline{\eta}$ is the unique extension of η. For the fundamental terms $t = f_i(x_1, \ldots, x_n)$ we have
$(\eta \circ \sigma_{id})\hat{}\,[f_i(x_1, \ldots, x_n)] \;=\; (\eta \circ \sigma_{id})(f_i)$
$= \eta(f_i(x_1, \ldots, x_n)) \;=\; \overline{\eta}(f_i(x_1, \ldots, x_n)),$
by (C) and the definition of the extension of a hypersubstitution. The claimed property then follows by induction. Now for the homomorphism property for ψ we have
$\psi(\eta_1) \circ_h \psi(\eta_2) \;=\; (\eta_1 \circ \sigma_{id}) \circ_h (\eta_2 \circ \sigma_{id})$
$= (\eta_1 \circ \sigma_{id})\hat{} \circ (\eta_2 \circ \sigma_{id})$
$= \overline{\eta}_1 \circ (\eta_2 \circ \sigma_{id})$, by property (*) above,
$= (\overline{\eta_1} \circ \eta_2) \circ \sigma_{id}$, by associativity
$= (\eta_1 \odot \eta_2) \circ \sigma_{id}$, by definition of $\odot$,
$= \psi(\eta_1 \odot \eta_2)$. □

The condition (*) shows that extensions of hypersubstitutions are endomorphisms of $clone_{SF}\tau_n$. Further it is clear that the monoid $End(clone_{SF}\tau_n)$ of all endomorphisms of $clone_{SF}\tau_n$ is isomorphic to $(Hyp^{SF}(\tau_n); \circ_h, \sigma_{id})$.

For the next proof we will need the following mapping g. Let $\mathcal{A}$ be any n-ary algebra. We define $g : \{f_i(x_1, \ldots, x_n) \mid i \in I\} \to \{f_i^{\mathcal{A}} \mid i \in I\}$, by letting $g(f_i(x_1, \ldots, x_n)) = f_i^A$, for each $i \in I$. Since $clone_{SF}\tau_n$ is free with respect to the variety $V_{\tau_n}^{SFC}$ and since $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$ is an element of this variety, this mapping g has a unique extension to a surjective homomorphism $\overline{g}$. It is clear that the mapping $\overline{g}$ assigns to each term $t \in W_{\tau_n}^{SF}(X_n)$ the induced term operation $t^{\mathcal{A}}$. We denote by $Id_n^{SF}(\mathcal{A})$ the set of all identities $s \approx t$ in $\mathcal{A}$ with $s, t \in W_{\tau_n}^{SF}(X_n)$. Such identities are called strongly full identities. Then we have:

Theorem 3.4 *Let $\mathcal{A}$ be an algebra of type τ_n, and let $s \approx t \in Id_n^{SF}\mathcal{A}$. Then $s \approx t$ is a strongly full hyperidentity in $\mathcal{A}$ iff $s \approx t$ is an identity in $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$.*

Proof. We first assume that $s \approx t$ is a strongly full hyperidentity of $\mathcal{A}$. This means that for every $\sigma \in Hyp^{SF}(\tau_n)$ we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^{SF}\mathcal{A}$, i.e. $\hat{\sigma}[s]^{\mathcal{A}} = \hat{\sigma}[t]^{\mathcal{A}}$, and hence that $\overline{g}(\hat{\sigma}[s]) = \overline{g}(\hat{\sigma}[t])$. To show that $s \approx t$ holds in $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$, we will show that $\overline{v}(s) = \overline{v}(t)$ for every valuation $v : \{f_i(x_1, \ldots, x_n\} \mid i \in I\} \to T_{SF}^{(n)}(\mathcal{A})$. Since $\overline{g}$ is surjective, there exists a clone substitution η_v such that $v = \overline{g} \circ \eta_v$, using the axiom of choice. Then $\eta_v \circ \sigma_{id}$ is a hypersubstitution, which we shall denote by σ_v. Then we have

$$\overline{v}(s) = (\overline{g} \circ \overline{\eta}_v)(s) = (\overline{g} \circ (\eta_v \circ \sigma_{id})\hat{\ })(s) = \overline{g}(\hat{\sigma}_v[s]).$$

Similarly, we have $\overline{v}(t) = \overline{g}(\hat{\sigma}_v[t])$. Since by our assumption we have $\overline{g}(\hat{\sigma}_v[s]) = \overline{g}(\hat{\sigma}_v[t])$, we get $\overline{v}(s) = \overline{v}(t)$, as required. Conversely, let $s \approx t \in Id\ T_{SF}^{(n)}(\mathcal{A})$, so that $s, t \in W_{\tau_n}^{SF}(X_n)$ and for every valuation mapping v we have $\overline{v}(s) = \overline{v}(t)$. Let σ be any SF-hypersubstitution. By the surjectivity from Proposition 3.3, there is a clone substitution η_σ such that $\eta_\sigma \circ \sigma_{id} = \sigma$. We take v to be the valuation $\overline{g} \circ \overline{\eta_\sigma}$. Then

$$\hat{\sigma}[s]^{\mathcal{A}} = \overline{g}(\hat{\sigma}[s]) = (\overline{g} \circ (\eta_\sigma \circ \sigma_{id})\hat{\ })(s) = (\overline{g} \circ \overline{\eta}_\sigma)(s) = \overline{v}(s),$$

again using Property (*). Similarly, we have $\hat{\sigma}[t]^{\mathcal{A}} = \overline{v}(t)$, and our assumption that $\overline{v}(s) = \overline{v}(t)$ gives the desired equality. □

Let $\mathcal{L}(tau_n)$ be the lattice of all varieties of type τ_n. For any variety V of type τ_n we can form the variety $SF_n^A(V)$ of type τ_n, determined by all n-ary strongly full identities of V. More precisely, if $SF_n^E(\tau_n) := W_{\tau_n}^{SF}(X_n)^2 \cup \{s \approx s \mid s \in W_{\tau_n}(X_n)\}$, then $SF_n^A(V) := Mod(SF_n^E(\tau_n) \cap IdV)$, where $SF_n^E(V) := SF_n^E(\tau_n) \cap IdV$ is a congruence relation on $\mathcal{F}_{\tau_n}(X_n)$ (and on the subalgebra $\mathcal{F}_{\tau_n}^{SF}(X_n)$). In general, $SF_n^E(V)$ is not fully invariant since it is not closed under substitutions and therefore $SF_n^E(V)$ is not an equational theory. It is easy to see that the operator

$$SF_n^E : \mathcal{P}(W_{\tau_n}(X_n) \times W_{\tau_n}(X_n)) \to \mathcal{P}(W_{\tau_n}(X_n) \times W_{\tau_n}(X_n))$$

(where $\mathcal{P}$ denotes the formation of the power set) is a kernel operator. The variety V is a subvariety of $SF_n^A(V)$. The operator $SF_n^A : \mathcal{L}(\tau_n) \rightarrow \mathcal{L}(\tau_n)$ defined by $V \mapsto SF_n^A(V)$ is a closure operator. Indeed, extensivity and monotonicity are clear. From $SF_n^E(V) \subseteq IdModSF_n^E(V)$ there follows $SF_n^E(V) \subseteq IdModSF_n^E(V) \cap W_{\tau_n}^{SF}(X_n)^2$ and $ModSF_n^E(V) \supseteq Mod(IdModSF_n^E(V) \cap W_{\tau_n}^{SF}(X_n)^2)$ and therefore, $SF_n^A(V) \supseteq Mod(SF_n^E(SF_n^A(V))) = SF_n^A(SF_n^A(V))$. The converse inclusion follows from extensivity. Therefore SF_n^A is idempotent and thus it is a closure operator. As a consequence, the class of all varieties V with $V = SF_n^A(V)$ forms a sublattice $\mathcal{L}_{SF}(\tau_n)$ of the lattice $\mathcal{L}(\tau_n)$ of all varieties of type τ_n.

We recall that a variety V of type τ_n is called M-solid if every identity in V is satisfied as an M-hyperidentity. For $M = Hyp^{SF}(\tau_n)$ we speak of SF-solid varieties. Then we have

Corollary 3.5 *Let $\mathcal{A}$ be an algebra of type τ_n. Then the variety $SF_n^A(V(\mathcal{A}))$ is SF-solid iff $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$ is free with respect to itself, freely generated by the set $\{f_i^{\mathcal{A}} \mid i \in I\}$, meaning that every mapping from $\{f_i^{\mathcal{A}} \mid i \in I\}$ to $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$ can be extended to an endomorphism of $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$.*

Proof. Using the equivalence from Theorem 3.4, we will show that $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$ is free iff every identity $s \approx t \in IdSF_n^A(V(\mathcal{A}))$ is also an identity in $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$. Suppose first that $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$ is free with respect to itself, freely generated by the set $\{f_i^{\mathcal{A}} \mid i \in I\}$. Let $s \approx t$ be any identity in $Id_n^{SF}(SF_n^A(V(\mathcal{A})))$, so that $\overline{g}(s) = \overline{g}(t)$. To show that $s \approx t$ is an identity in $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$, we will show that $\overline{v}(s) = \overline{v}(t)$ for any valuation mapping $v : F_{\tau_n} \longrightarrow T_{SF}^{(n)}(\mathcal{A})$. Given v, we define a mapping $\alpha_v : \{f_i^{\mathcal{A}} \mid i \in I\} \longrightarrow T_{SF}^{(n)}(\mathcal{A})$ by $\alpha_v(f_i^{\mathcal{A}}) = v(f_i(x_1, \ldots, x_n))$. Since

$$\begin{aligned} f_i^{\mathcal{A}} = f_j^{\mathcal{A}} &\Longrightarrow i = j \Longrightarrow f_i(x_1, \ldots, x_n) = f_j(x_1, \ldots, x_n) \\ &\Longrightarrow v(f_i(x_1, \ldots, x_n)) = v(f_j(x_1, \ldots, x_n)) \\ &\Longrightarrow \alpha_v(f_i(x_1, \ldots, x_n)) = \alpha_v(f_j(x_1, \ldots, x_n)), \end{aligned}$$

the mapping α_v is well-defined. Since the set F_{τ_n} generates the algebra $clone_{SF}\tau_n$, the mapping v can be uniquely extended to $\overline{v}$ on the set $W_{\tau_n}^{SF}(X_n)$. Then we have

$$\overline{g}(s) = \overline{g}(t) \Longrightarrow \overline{\alpha}_v(\overline{g}(s)) = \overline{\alpha}_v(\overline{g}(t)) \Longrightarrow \overline{v}(s) = \overline{v}(t),$$

showing that $s \approx t \in IdT_{SF}^{(n)}(\mathcal{A})$.

For the converse direction, we show that when $SF_n^A(V(\mathcal{A}))$ is SF-solid, any mapping $\alpha : \{f_i^{\mathcal{A}} \mid i \in I\} \longrightarrow T_{SF}^{(n)}(\mathcal{A})$ can be extended to an endomorphism of $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$. We consider the mapping $\overline{\alpha} = \overline{\alpha \circ g} : W_{\tau_n}^{SF}(X_n) \longrightarrow T_{SF}^{(n)}(\mathcal{A})$ with $\overline{\alpha}(t^{\mathcal{A}}) = \overline{\alpha \circ g}(t)$, which is a valuation of terms. Then for any terms $s, t \in W_{\tau_n}^{SF}(X_n)$, it follows from $s^{\mathcal{A}} = t^{\mathcal{A}}$ that $\overline{g}(s) = \overline{g}(t)$ and hence that $\overline{\alpha}(s^{\mathcal{A}}) = \overline{\alpha}(\overline{g}(s)) = \overline{\alpha}(\overline{g}(t)) = \overline{\alpha}(t^{\mathcal{A}})$, since $\overline{\alpha} \circ \overline{g}$ is a valuation and every identity of $SF_n^A(V(A))$ is a $clone_{SF}\tau_n$-identity. This shows that $\overline{\alpha}$ is well-defined. It is also an endomorphism since $\overline{\alpha}(S_n^{n,\mathcal{A}}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})) = \overline{\alpha}(\overline{g}(S_n^n(s, t_1, \ldots, t_n))) = (\overline{\alpha \circ g})(S_n^n(s, t_1, \ldots, t_n)) = S_n^{n,\mathcal{A}}(\overline{\alpha \circ g}(s), \overline{\alpha \circ g}(t_1), \ldots, \overline{\alpha \circ g}(t_n)) = S_n^{n,\mathcal{A}}(\overline{\alpha}(s^{\mathcal{A}}), \overline{\alpha}(t_1^{\mathcal{A}}), \ldots, \overline{\alpha}(t_n^{\mathcal{A}}))$, using the fact that $\overline{\alpha \circ g}$ is the homomorphism extending the valuation $\alpha \circ g$ defined on the generating set of the free algebra $clone_{SF}\tau_n$. Finally, $\overline{\alpha}$ extends α since $\overline{\alpha}(f_i^{\mathcal{A}}) = \overline{\alpha \circ g}(f_i(x_1, \ldots, x_n)) = (\alpha \circ g)(f_i(x_1, \ldots, x_n)) = \alpha(g(f_i(x_1, \ldots, x_n))) = \alpha(f_i^{\mathcal{A}})$, for each $i \in I$. □

Proposition 3.6 *Let $\mathcal{A}$ be an n-ary algebra. Then the set $SF_n^E(\mathcal{A}) := SF_n^E(V(\mathcal{A}))$ is a congruence on $clone_{SF}\tau_n$ and the quotient algebra $clone_{SF}\tau_n/SF_n^E(\mathcal{A})$ is isomorphic to $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$.*

Proof. The set $SF_n^E(V(\mathcal{A}))$ is an equivalence relation on $W_{\tau_n}^{SF}(X_n)$. It is easy to see that the operation S_n^n preserves the relation $SF_n^E(V(\mathcal{A}))$. Thus $SF_n^E(V(\mathcal{A}))$ is a congruence relation. The surjective homomorphism $\bar{g}$ maps $clone_{SF}\tau_n$ onto $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$. By the homomorphism theorem we have $\mathcal{T}_{SF}^{(n)}(\mathcal{A}) \cong clone_{SF}\tau_n/ker\bar{g}$. Further we have $ker\bar{g} = SF_n^E(V(\mathcal{A}))$. □

For any congruence θ on $clone_{SF}\tau_n$ we may consider the quotient algebra $\mathcal{M}^{SF}(\theta) := (W_{\tau_n}^{SF}(X_n)/\theta; (f_i^*)_{i \in I})$ since θ is also a congruence on $\mathcal{F}_{\tau_n}^{SF}(X_n)$. This algebra is called SF-Myhill algebra of θ. The congruence $SF_n^E(V(\mathcal{A}))$ is called the SF-Myhill congruence ([5]) on $\mathcal{A}$ and the corresponding quotient algebra is called SF-Myhill algebra $\mathcal{M}^{SF}(\mathcal{A})$. For any variety V we introduce $\mathcal{M}^{SF}(V)$ as quotient algebra $(W_{\tau_n}^{SF}(X_n)/Id_n^{SF}; (f_i^*)_{i \in I})$.

Proposition 3.7 *For every congruence θ on $clone_{SF}\tau_n$ we have $\mathcal{T}_{SF}^{(n)}(\mathcal{M}^{SF}(\theta)) \cong clone_{SF}\tau_n/\theta$, in particular $\mathcal{T}_{SF}^{(n)}(\mathcal{M}^{SF}(\mathcal{A})) \cong \mathcal{T}_{SF}^{(n)}(\mathcal{A})$.*

Proof. $\mathcal{T}_{SF}^{(n)}(\mathcal{M}^{SF}(\theta))$ is the strongly full clone generated by $\{f_i^* \mid i \in I\}$. We consider a mapping $\varphi : \{f_i(x_1, \ldots, x_n) \mid i \in I\} \to \{f_i^* \mid i \in I\}$ defined by $\varphi(f_i(x_1, \ldots, x_n)) = f_i^*$ for all $i \in I$. Since $clone_{SF}\tau_n$ is free in the variety $V_{\tau_n}^{SFC}$, freely generated by the set $\{f_i(x_1, \ldots, x_n) \mid i \in I\}$, the mapping φ can be extended to a homomorphism $\bar{\varphi}$ which is surjective, since φ maps the generating sets to each other. By definition $T_{SF}^{(n)}(\mathcal{M}^{SF}(\theta)) = clone_{SF}/ker\varphi$. It is easy to see that $ker\varphi = \theta$. Then the special case follows from the previous definition. □

4. I_{SF}-Closed Varieties of Type τ_n

In this section we examine the connection between a variety V of type τ_n and the class of all strongly full clones $\{T_{SF}^{(n)}(\mathcal{A}) \mid \mathcal{A} \in V\}$ of its algebras.

Definition 4.1 *Let V be a variety of type τ_n. Then $SF_n^A(V)$ is called I_{SF}-closed if whenever $\mathcal{A} \in SF_n^A(V)$ and $\mathcal{T}_{SF}^{(n)}(\mathcal{A}) \cong \mathcal{T}_{SF}^{(n)}(\mathcal{B})$, then also $\mathcal{B} \in SF_n^A(V)$.*

We consider the following set of hypersubstitutions of type τ_n:

$$\mathcal{O}_{SF} := \{\sigma \mid \sigma \in Hyp^{SF}(\tau_n) \text{ and } \hat{\sigma} \quad \text{is surjective}\}.$$

It is easy to see that $\mathcal{O}_{SF}$ is a submonoid of $Hyp_{SF}(\tau_n)$. Our aim is to show that I_{SF}-closedness is closely related to certain congruence relations. We recall the concept of a weekly invariant congruence relation.

Definition 4.2 *Let $\mathcal{A}$ be an algebra of arbitrary type. A congruence $\theta \in Con\mathcal{A}$ is said to be weakly invariant if for every $\rho \in Con\mathcal{A}$, the following condition is satisfied: if there exists a homomorphism from $\mathcal{A}/\theta$ onto $\mathcal{A}/\rho$, then $\theta \subseteq \rho$.*

Let $\mathcal{A}$ be an algebra, and let θ and ρ be any congruences on $\mathcal{A}$. By the second isomorphism theorem, it always follows from $\theta \subseteq \rho$ that there exists a surjective homomorphism $\mathcal{A}/\theta \longrightarrow \mathcal{A}/\rho$, but the converse is in general not true. Weakly invariant congruences were introduced in [6] and used for semigroup varieties in [5]. They are related to isomorpically closed principal filters in the congruence lattice.

Definition 4.3 *A set C of congruences of an algebra $\mathcal{A}$ of arbitrary type τ is said to be isomorphically closed if whenever $\theta \in C$ and $\mathcal{A}/\theta \cong \mathcal{A}/\rho$ it follows that $\rho \in C$.*

The following theorem was proved for semigroups in [5] and for algebras of arbitrary type in [2] .

Theorem 4.4 *Let $\mathcal{A}$ be an algebra of arbitrary type τ, let V be a variety of type τ and let $\mathcal{F}_V(X)$ be the free algebra with respect to V, freely generated by X. Then*
(i) A congruence θ on $\mathcal{A}$ is weakly invariant iff the principal filter $[\theta)$ generated by θ in $Con\mathcal{A}$ is isomorphically closed.
(ii) Every weakly invariant congruence on $\mathcal{A}$ is invariant under all surjective endomorphisms of $\mathcal{A}$.

Now we can characterize I_{SF}-closed varieties of type τ_n, generalizing the characterization of σ-closed varieties given in [5] for the unary case and for the n-ary case in [2].

Theorem 4.5 *Let V be a variety of type τ_n. Then $SF_n^A(V)$ is I_{SF}-closed iff $SF_n^A(V)$ satisfies the following two properties:*

(i) $SF_n^E(V)$ is weakly invariant.

(ii) $\mathcal{A} \in SF_n^A(V)$ iff $\mathcal{M}^{SF}(\mathcal{A}) \in SF_n^A(V)$.

Proof. Suppose first that $SF_n^A(V)$ is I_{SF}-closed. Property (ii) follows from I_{SF}-closure and the result from Proposition 3.7 that for any algebra $\mathcal{A}$ in $SF_n^A(V)$ we have $\mathcal{T}_{SF}^{(n)}(\mathcal{M}^{SF}(\mathcal{A})) \cong \mathcal{T}_{SF}^{(n)}(\mathcal{A})$. By Theorem 4.4, we can prove that (i) holds by showing that $[SF_n^E(V))$ is isomorphically closed. For this, let $\alpha \supseteq SF_n^E(V)$, and let θ be a congruence on $clone_{SF}\tau_n$ such that $clone_{SF}\tau_n/\alpha \cong clone_{SF}\tau_n/\theta$. Since $SF_n^E(V) = \cap\{SF_n^E(\mathcal{A}) \mid \mathcal{A} \in V\}$, the algebra $\mathcal{M}^{SF}(V)$ is isomorphic to a subdirect product of the algebras $\mathcal{M}^{SF}(\mathcal{A})$ with $\mathcal{A} \in SF_n^A(V)(\mathcal{M}^{SF}(\mathcal{A}) \in SF_n^{\mathcal{A}}(V)$ by (ii)), and thus $\mathcal{M}^{SF}(V) \in SF_n^A(V)$. The inclusion $\alpha \supseteq SF_n^E(V)$ implies that there is a surjective homomorphism from $\mathcal{M}^{SF}(V)$ onto $\mathcal{M}^{SF}(\alpha)$. Combining these two facts gives $\mathcal{M}^{SF}(\alpha) \in SF_n^A(V)$. Furthermore by Proposition 3.7 we have $\mathcal{T}_{SF}^{(n)}(\mathcal{M}^{SF}(\alpha)) \cong clone_{SF}\tau_n/\alpha \cong clone_{SF}\tau_n/\theta \cong \mathcal{T}_{SF}^{(n)}(\mathcal{M}^{SF}(\theta))$, and since $SF_n^A(V)$ is I_{SF}-closed this gives $\mathcal{M}^{SF}(\theta) \in SF_n^A(V)$. This means that $SF_n^E(V) \subseteq SF_n^E(\mathcal{M}^{SF}(\theta))$, and we can finish the proof by showing that $SF_n^E(\mathcal{M}^{SF}(\theta)) = \theta$, so that $\theta \in [SF_n^E(V))$. This equality $SF_n^E(\mathcal{M}^{SF}(\theta)) = \theta$ holds because

$$s \approx t \in SF_n^E(\mathcal{M}^{SF}(\theta)) \quad \Leftrightarrow \quad [s]_\theta = [t]_\theta \quad \Leftrightarrow \quad (s,t) \in \theta.$$

Conversely, we assume that the variety V of type τ_n satisfies (i) and (ii). From (ii) we get $\mathcal{M}^{SF}(\mathcal{A}) \in SF_n^A(V)$, for all $\mathcal{A} \in SF_n^A(V)$, and since $\mathcal{M}^{SF}(V)$ is isomorphic to a subdirect product of all these algebras, we also have $\mathcal{M}^{SF}(V) \in SF_n^A(V)$. To establish that $SF_n^A(V)$ is I_{SF}-closed, let $\mathcal{B}$ and $\mathcal{C}$ be any two n-ary algebras, and suppose that $\mathcal{T}_{SF}^{(n)}(\mathcal{B}) \cong \mathcal{T}_{SF}^{(n)}(\mathcal{C})$ and $\mathcal{B} \in SF_n^A(V)$. It follows from Proposition 3.7 that $clone_{SF}\tau_n/SF_n^E(\mathcal{B}) \cong clone_{SF}\tau_n/SF_n^E(\mathcal{C})$, and since $\mathcal{B} \in SF_n^A(V)$ we have $SF_n^E(V) \subseteq SF_n^E(\mathcal{B})$. By (ii) $SF_n^E(V)$ is weakly invariant, so we get $SF_n^E(V) \subseteq SF_n^E(\mathcal{C})$. But then $\mathcal{M}^{SF}(\mathcal{C})$ is a homomorphic image of $\mathcal{M}^{SF}(V)$, and hence $\mathcal{M}^{SF}(\mathcal{C}) \in SF_n^A(V)$. By (ii) we have $\mathcal{C} \in SF_n^A(V)$, establishing that $SF_n^A(V)$ is I_{SF}-closed. □

The next Theorem characterizes I_{SF}-closure for SF-varieties in terms of SF-solidity.

Theorem 4.6 *A SF-variety $SF_n^A(V)$ of type τ_n is I_{SF}-closed iff it is $\mathcal{O}_{SF}$-solid.*

Proof. First assume that $SF_n^A(V)$ is $\mathcal{O}_{SF}$-solid, so that every $s \approx t \in SF_n^E(V)$ is an $\mathcal{O}_{SF}$-hyperidentity in $SF_n^A(V)$. We claim that in fact $\mathcal{A} \in SF_n^A(V)$ iff $\mathcal{A}$ satisfies as an $\mathcal{O}_{SF}$-hyperidentity every identity $s \approx t$ in $SF_n^E(V)$. From the $\mathcal{O}_{SF}$-solidity of $SF_n^A(V)$ follows at first that from $\mathcal{A} \in SF_n^A(V) = ModSF_n^E(V)$ the algebra $\mathcal{A}$ satisfies every identity $s \approx t \in SF_n^E(V)$ as an $\mathcal{O}_{SF}$-hyperidentity. For the other direction we note that any $\mathcal{O}_{SF}$-hyperidentity of an algebra is also an identity, so $\mathcal{A}$ satisfies the basis identities $SF_n^E(V)$ of $SF_n^A(V)$. By Theorem 3.4 from $\mathcal{A} \underset{\mathcal{O}_{SF}\text{-}hyp}{\models} SF_n^E(V)$ there follows $\mathcal{T}_{SF}^{(n)}(\mathcal{A}) \models SF_n^E(V)$. If

$\mathcal{A} \in SF_n^A(V)$ and $\mathcal{T}_{SF}^{(n)}(\mathcal{B})$ is isomorphic to $\mathcal{T}_{SF}^{(n)}(\mathcal{A})$ then $\mathcal{B} \in SF_n^A(V)$ and thus $SF_n^A(V)$ is I_{SF}-closed.

Conversely, assume that $SF_n^A(V)$ is I_{SF}-closed. Then by Theorem 4.5 we know that $SF_n^E(V)$ is both, weakly invariant and invariant under all surjective endomorphisms of $clone_{SF}\tau_n$. Then for any identity $s \approx t \in SF_n^E(V)$, any algebra $\mathcal{A} \in SF_n^A(V)$ and any surjective endomorphism η, we have $\eta(s) \approx \eta(t) \in Id\mathcal{T}_{SF}^{(n)}(\mathcal{A})$. Using the isomorphism from Proposition 3.3 this means $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in SF_n^E(V)$ for every $\sigma \in \mathcal{O}_{SF}$ and $SF_n^A(V)$ is $\mathcal{O}_{SF}-$ solid. □

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