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CONGRUENCES OF CONCEPT ALGEBRAS[1]

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Abstract. Concept algebras are concept lattices enriched with two unary operations, a weak negation and a weak opposition. In this contribution we provide a characterization of congruences of concept algebras.

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1. Introduction

Concept algebras arise from the need to develop a *Contextual Boolean Logic*, based on “concept as unit of thought”. A concept is considered to be determined by its extent and its intent. The extent consists of all entities belonging to the concept, whilst the intent is the set of properties shared by these entities. The notion of concept has been formalized at the early 80s ([Wi82]) and led to the theory of *Formal Concept Analysis* ([GW99]). A **formal context** is a triple (G, M, I) of sets such that $I \subseteq G \times M$. The members of G are called **objects** and those of M **attributes**. If $(g, m) \in I$ the object g is said to have m as an attribute. For subsets $A \subseteq G$ and $B \subseteq M$, A' and B' are defined by

$$A' := \{m \in M \mid \forall g \in A \quad gIm\}$$

$$B' := \{g \in G \mid \forall m \in B \quad gIm\}.$$

The operation $'$, usually called **derivation**, induces a Galois connection between the powersets of G and of M. If different relations are defined on the same sets we use other notations to avoid confusion.

A **formal concept** of the context (G, M, I) is a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ such that $A' = B$ and $B' = A$. A is called the **extent** and B the **intent** of the concept (A, B). $\mathfrak{B}(G, M, I)$ denotes the set of all formal concepts

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of the formal context (G, M, I). For $g \in G$ and $m \in M$, we set $g' := \{g\}'$ and $m' := \{m\}'$. The concepts

$$\gamma g := (g'', g') \text{ and } \mu m := (m', m'')$$

are respectively called **object concept** and **attribute concept**. They play an important rôle in the representation of complete lattices as concept lattices.

The hierarchy on concepts is captured by the **subconcept-superconcept relation**. The concept (A, B) is called a **subconcept** of the concept (C, D) provided that $A \subseteq C$ (which is equivalent to $D \subseteq B$). In this case, (C, D) is a **superconcept** of (A, B) and we write $(A, B) \leq (C, D)$. $(B(G, M, I); \leq)$ is a complete lattice and is called the **concept lattice** of the context (G, M, I), and usually denoted by $\underline{\mathfrak{B}}(G, M, I)$. Conversely, each complete lattice is isomorphic to some concept lattice ([GW99]).

To introduce a notion of negation[3] on concepts, Rudolf Wille introduced two unary operations $^\triangle$ and $^\triangledown$ called **weak negation** and **weak opposition** defined for each concept (A, B) by

$$(A, B)^\triangle := (\overline{A}'', \overline{A}') \quad \text{and} \quad (A, B)^\triangledown := (\overline{B}', \overline{B}''),$$

where $\overline{A} := G \setminus A$ and $\overline{B} := M \setminus B$. A concept lattice enriched with a weak negation and a weak opposition is called a **concept algebra**. Here the motto is that *the negation of a concept should be a concept*. There are other approaches driven by the aim to *keep the correspondence between set-complementation and negation*. For this purpose Rudolf Wille et al. extended the notion of concepts to that of semiconcepts ([HLSW01]), protoconcepts ([Wi00]) and preconcepts ([Wi04]). These extensions are not in the scope of this paper. Our main interest is the structure theory of concept algebras. In this contribution we focus on congruences of concept algebras.

The concept algebra of the context (G, M, I) will be denoted by $\mathfrak{A}(G, M, I)$. Each concept algebra is a complete lattice and satisfies the following equations (see [Wi00]):

(1) $x^{\triangle\triangle} \leq x$, (1') $x^{\triangledown\triangledown} \geq x$,

(2) $x \leq y \Rightarrow x^\triangle \geq y^\triangle$, (2') $x \leq y \Rightarrow x^\triangledown \geq y^\triangledown$,

(3) $(x \wedge y) \vee (x \wedge y^\triangle) = x$, (3') $(x \vee y) \wedge (x \vee y^\triangledown) = x$.

Definition 1.1 *Each bounded lattice equipped with two unary operations* $^\triangle$ *and* $^\triangledown$ *(respectively called* **weak complementation** *and* **dual weak complementation***) satisfying the equations (1)-(3') above, will be called a* **weakly dicomplemented lattice***.*

[3] The problem of negation is one of the oldest problems in the scientific and philosophic community, and still attracts the attention of many researchers (see [Wa96]).

Finite distributive weakly dicomplemented lattices are (isomorphic to) concept algebras (see [GK04]). It is still an open problem whether these equations are enough to describe the equational theory (if there is one) of concept algebras. Working on the class of concept algebras, we would like, for example, to know if this is closed under complete homomorphisms. At the present this is now not clear. Observe that the kernels of complete homomorphisms are complete congruences (see below). In Section 2 we give two characterizations of concept algebra congruences. In Section 3 we consider the problem of describing the lattice of concept algebra congruences.

2. Congruences of Concept Algebras

Concept algebras are concept lattices with additional operations. Therefore, each concept algebra congruence is a concept lattice congruence with some additional properties.

Definition 2.1 *A* **complete congruence** *relation on a complete lattice L is an equivalence relation θ on L such that $x_t \theta y_t$ for all $t \in T$ implies*

$$\bigwedge_{t\in T} x_t \theta \bigwedge_{t\in T} y_t \text{ and } \bigvee_{t\in T} x_t \theta \bigvee_{t\in T} y_t.$$

In the rest of this contribution, we will usually use the term congruence to mean complete congruence. Note that for a congruence θ, we have $x\theta y$ if and only if $x \wedge y \theta x \vee y$. The congruence classes are intervals of L. For an element $x \in L$, we denote by $[x]_\theta$ its congruence class. We denote by x_θ the least element of $[x]_\theta$ and by x^θ its greatest element. Thus $[x]_\theta$ is the interval $[x_\theta, x^\theta]$. The **factor lattice** $L_{/\theta}$ is a complete lattice with respect to the order relation defined by:

$$[x]_\theta \leq [y]_\theta :\iff x\theta(x \wedge y).$$

The following proposition gives a characterization of complete congruence relations.

Proposition 2.1 *[GW99, pp. 106-107] An equivalence relation θ on a complete lattice L is a complete congruence relation if and only if every equivalence class of θ is an interval of L, the lower bounds of these intervals being closed under suprema and the upper bounds being closed under infima.*

Concept lattice congruences are described by **compatible subcontexts**. For a formal context (G, M, I), a subcontext $(H, N, I \cap H \times N)$, usually denoted by (H, N), is said to be compatible if for all (A, B) in $B(G, M, I)$, the pair $(A \cap H, B \cap N)$ is a formal concept of (H, N). The compatible subcontexts are characterized by their induced **projections**.

Proposition 2.2 *[GW99, p. 100] The subcontext (H,N) of (G,M,I) is compatible if and only if the mapping*

$$\begin{array}{llll}\Pi_{H,N}\colon & B(G,M,I) & \to & B(H,N) \\ & (A,B) & \mapsto & (A\cap H, B\cap N)\end{array}$$

is a surjective complete homomorphism.

The kernel of $\Pi_{H,N}$ is a complete congruence of $\underline{\mathfrak{B}}(G,M,I)$. We denote it by $\theta_{H,N}$. We get

$$\underline{\mathfrak{B}}(H,N) \cong \underline{\mathfrak{B}}(G,M,I)_{/\theta_{H,N}}$$

with

$$(A_1,B_1)\theta_{H,N}(A_2,B_2) \iff A_1\cap H = A_2\cap H \iff B_1\cap N = B_2\cap N.$$

The bounds of congruence classes can be easily identified. In fact for a concept (A,B), the least element of $[(A,B)]_{\theta_{H,N}}$ is the concept $((A\cap H)'',(A\cap H)')$ and the greatest element is $((B\cap N)',(B\cap N)'')$. A **complete congruence** θ is said to be **induced by a subcontext** if there is a compatible subcontext (H,N) such that $\theta=\theta_{H,N}$. In the case of a doubly founded[4] concept lattice every complete congruence is induced by a subcontext. If in addition the context is reduced then this subcontext is uniquely determined by the congruence.

Definition 2.2 *A concept lattice congruence θ of $\underline{\mathfrak{B}}(G,M,I)$ is said to be* **$^\triangle$-compatible** *(resp.* **$^\triangledown$-compatible***) if for all concepts x and y in $B(G,M,I)$, $x\theta y \Rightarrow x^\triangle\theta y^\triangle$ (resp. $x\theta y\Rightarrow x^\triangledown\theta y^\triangledown$). A* **concept algebra congruence** *is a $^\triangle$-compatible and $^\triangledown$-compatible concept lattice congruence.*

If θ is a congruence of the concept algebra $\underline{\mathfrak{A}}(\mathbb{K})$ then θ is a congruence of the concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ and therefore corresponds to a compatible subcontext of $\mathbb{K}$. Which of these subcontexts enable the preservation of the unary operations? We are going to examine under which conditions a congruence induced by a compatible subcontext preserves the operation $^\triangle$ and dualize to get the result for the operation $^\triangledown$.

We adopt the following notations for $m,n,m_0\in M$ and $N\subseteq M$.

$$m\underline{\perp}n :\iff m_0\perp n \text{ for all } m_0\in m''\cap N,$$

where $m_0\perp n$ stands for $m_0'\cup n'=G$. The relation $\perp$ is called "the orthogonal relation". The attributes m and n are orthogonal if they (their extents m' and n') cover G. Note that if m and n are in N, and $m\underline{\perp}n$ (all elements of $m''\cap N$ are orthogonal to n) then trivially m is orthogonal to n.

[4] A complete lattice $\underline{L}$ is **doubly founded**, if for any two elements $x<y$ of L, there are elements $s,t\in L$ with: s is minimal with respect to $s\le y$, $s\not\le x$, as well as t is maximal with respect to $t\ge x$, $t\not\ge y$.

Theorem 2.3 *The lattice congruence induced by a compatible subcontext* (H, N) *is* $^{\triangle}$*-compatible if and only if*

$$\forall_{m\in M}\,\forall_{n\in N}\quad m\underline{\perp}n \Rightarrow m \perp n. \qquad (*)$$

Before we come to the proof let us rephrase this result. The theorem states that if a subcontext (H, N) induces a $^{\triangle}$-compatible congruence, then from $m \in M$ and $n \in N$ such that all elements of $m'' \cap N$ are orthogonal to n, it follows that m and n automatically cover G. Moreover, the condition $(*)$ is sufficient.
Proof. ($\Leftarrow$) We assume that the condition $(*)$ holds. We prove that if x and y are concepts such that $x\theta_{H,N}y$ then automatically $x^{\triangle}\theta_{H,N}y^{\triangle}$. Since $x\theta_{H,N}y$ is equivalent to $(x \wedge y)\theta_{H,N}(x \vee y)$, it is enough to prove the assertion only for $x \leq y$. We can even restrict to pairs (x, y) such that x is minimal and y is maximal in their congruence class. Recall that $x\theta_{H,N}y$ means

$$ext(x) \cap H = ext(y) \cap H =: A \text{ and } int(x) \cap N = int(y) \cap N =: B,$$

where $ext(x)$ denotes the extent of the concept x and $int(x)$ its intent. As we assume x to be minimal and y maximal, we have $x = (A'', A')$ and $y = (B', B'')$. Reformulating the problem, we have to prove that

$$\left((G \setminus A'')'' , (G \setminus A'')'\right) \theta_{H,N} \left((G \setminus B')'' , (G \setminus B')'\right).$$

This is equivalent to the equality

$$(G \setminus A'')' \cap N = (G \setminus B')' \cap N.$$

The inclusion

$$(G \setminus A'')' \cap N \subseteq (G \setminus B')' \cap N$$

is immediate since A'' is a subset of B'. Note that for all $n \in N$,

$$n \in (G \setminus B')' \iff n'' \subseteq (G \setminus B')' \iff G \setminus B' \subseteq n' \iff G \setminus n' \subseteq B'.$$

Therefore it suffices to show that

$$\forall_{n\in N}\quad [G \setminus n' \subseteq B' \Rightarrow G \setminus n' \subseteq A''] .$$

We know that $B = A' \cap N = \{n \in N \mid A \subseteq n'\}$. To get the above assertion we need to demonstrate that

$$\forall_{n\in N}\quad \left[G \setminus n' \subseteq \bigcap_{m_0\in N,\, A\subseteq m_0'} m_0' \Rightarrow G \setminus n' \subseteq \bigcap_{m\in M,\, A\subseteq m'} m'\right].$$

i.e.

$$\forall_{n\in N}\, [m_0 \perp n \quad \forall_{m_0\in N} \text{ with } A \subseteq m_0' \Rightarrow m \perp n \quad \forall_{m\in M} \text{ with } A \subseteq m'] .$$

Equivalently, we do prove that for all $n \in N$ the assertion

$$\exists_{m\in M} \text{ such that } A \subseteq m' \text{ and } m \not\swarrow n$$

implies

$$\exists_{m_0\in N} \text{ such that } A \subseteq m_0' \text{ and } m_0 \not\swarrow n.$$

If this implication were not true for a certain n in N there would exist an attribute m with $A \subseteq m'$ and $m \not\swarrow n$ such that for any attribute $m_0 \in N$ with $A \subseteq m_0'$, we have $m_0 \perp n$. All attributes from $m'' \cap N$ belong particularly[5] to these attributes. Therefore $m_0 \perp n$ for all $m_0 \in m'' \cap N$. This is exactly $m \underline{\perp} n$. From $(*)$ we would get $m \perp n$, which would be a contradiction. Thus $x^{\triangle}\theta_{H,N}y^{\triangle}$. Since x and y were chosen arbitrary we obtain that $\theta_{H,N}$ is $^{\triangle}$-compatible.

($\Rightarrow$) For the converse we assume that $\theta_{H,N}$ is $^{\triangle}$-compatible and want to prove the condition $(*)$. We consider $m \in M$ and $n \in N$ with $m \underline{\perp} n$. When do we have $m \perp n$? The congruence class $[(m', m'')]_{\theta_{H,N}}$ of the attribute concept (m', m'') is the interval

$$[((m' \cap H)'', (m' \cap H)'), ((m'' \cap N)', (m'' \cap N)'')].$$

From the $^{\triangle}$-compatibility of $\theta_{H,N}$ it follows that

$$((m' \cap H)'', (m' \cap H)')^{\triangle}\theta_{H,N}((m'' \cap N)', (m'' \cap N)'')^{\triangle}.$$

i.e.

$$((G \setminus (m' \cap H)'')'', (G \setminus (m' \cap H)'')')\theta_{H,N}((G \setminus (m'' \cap N)')'', (G \setminus (m'' \cap N)')').$$

Thus

$$(G \setminus (m' \cap H)'')' \cap N = (G \setminus (m'' \cap N)')' \cap N.$$

This is equivalent to

$$\forall_{n\in N} \quad G \setminus n' \subseteq (m' \cap H)'' \iff G \setminus n' \subseteq (m'' \cap N)'$$

which is the same as

$$\forall_{n\in N} \quad G \setminus n' \subseteq (m'' \cap N)' \Rightarrow G \setminus n' \subseteq (m' \cap H)''$$

since $m' \cap H \subseteq (m'' \cap N)'$. From $m \underline{\perp} n$ we get

$$\forall_{m_0\in N} \quad m_0' \supseteq m' \Rightarrow m_0 \perp n$$

and furthermore

$$(m'' \cap N)' = \bigcap_{m_0 \in m'' \cap N} m_0' \supseteq G \setminus n'.$$

If $m \not\swarrow n$ then $G \setminus n' \not\subseteq m'$. But $m' \supseteq (m' \cap H)''$; it follows that $G \setminus n' \not\subseteq (m' \cap H)''$. This is a contradiction since $G \setminus n' \subseteq (m'' \cap N)'$ implies $G \setminus n' \subseteq (m' \cap H)''$. This completes the proof. $\square$

[5] $m_0 \in m'' \cap N \Rightarrow m_0' \supseteq m' \supseteq A$.

Corollary 2.4 *A compatible subcontext* (H, N) *of* (G, M, I) *induces a concept algebra congruence if and only if the conditions* (*i*) *and* (*ii*) *below hold.*

(i) $\forall_{m\in M}\forall_{n\in N} \quad m \underline{\perp} n \Rightarrow m \perp n.$

(ii) $\forall_{g\in G}\forall_{h\in H} \quad g \underline{\perp} h \Rightarrow g \perp h.$[6]

From Proposition 2.2 compatible subcontexts correspond to projections that are surjective homomorphisms. Another way to look for concept algebra congruences is to examine compatible subcontexts (H, N) for which the projection $\Pi_{H,N}$ preserves the unary operations. We denote by ^{j} the derivation[7] in the subcontext (H, N). Let $x = (A'', A')$ be a concept of (G, M, I).

$$\Pi_{H,N}(x^{\triangle}) = ((G \setminus A'')'' \cap H, (G \setminus A'')' \cap N)$$

and

$$\Pi_{H,N}(x)^{\triangle} = ((H \setminus A'')^{jj}, (H \setminus A'')^{j}).$$

Thus $\Pi_{H,N}(x^{\triangle}) = \Pi_{H,N}(x)^{\triangle}$ if and only if

$$(G \setminus A'')' \cap N = (H \setminus A'')^{j}.$$

This means that for all $n \in N$

$$\left[G \setminus A'' \subseteq n' \iff H \setminus A'' \subseteq n^{j}\right].$$

Thus $\Pi_{H,N}(x^{\triangle}) = \Pi_{H,N}(x)^{\triangle}$ if and only if for all $n \in N$

$$\left[G \setminus n' \subseteq A'' \iff H \setminus n^{j} \subseteq A''\right].$$

The above equivalence can be rewritten as

$$\left[G \setminus n' \subseteq \bigcap_{A\subseteq m'} m' \iff H \setminus n^{j} \subseteq \bigcap_{A\subseteq m'} m'\right].$$

$$i.e. \qquad \forall_{m\in M} \quad A \subseteq m',\, n' \cup m' = G \iff n^{j} \cup m' \supseteq H.$$

Since x was taken arbitrarily in $B(G, M, I)$, we obtain the equality $\Pi_{H,N}(x^{\triangle}) = \Pi_{H,N}(x)^{\triangle}$ if and only if for every subset A of G, for all $n \in N$ and for all $m \in M$ with $A \subseteq m'$, the equivalence

$$n \perp_G m \iff n \perp_H m$$

holds. This is equivalent to

$$\forall_{n\in N}\,\forall_{m\in M} \quad (n \perp_G m \iff n \perp_H m)\,.$$

Thus, the following theorem holds.

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[6] $g \perp h :\iff g' \cup h' = M$ and $g \underline{\perp} h :\iff g_0 \perp h \quad \forall g_0 \in g'' \cap H$

[7] see Section 1

Theorem 2.5 *A compatible subcontext (H,N) of (G,M,I) induces a congruence of the concept algebra $\underline{\mathfrak{A}}(\mathbb{K})$ if and only if the following assertions are valid:*

(i) $\forall_{n\in N}\forall_{m\in M} \quad n \perp_G m \iff n \perp_H m,$

(ii) $\forall_{h\in H}\forall_{g\in G} \quad h \perp_M g \iff h \perp_N g.$

We denote by M_{irr} the set of irreducible attributes of a context (G,M,I). The test of compatibility of subcontexts can just be done on the irreducible elements, as we can see in the next proposition.

Proposition 2.6 *The following assertions are equivalent:*

(i) $\forall m \in M, \forall n \in N, n \perp_G m \iff n \perp_H m.$

(ii) $\forall m \in M_{irr}, \forall n \in N \cap M_{irr}, n \perp_G m \iff n \perp_H m.$

Proof. The implication $(i) \Rightarrow (ii)$ is obviously true. We are going to prove $(ii) \Rightarrow (i)$. We assume that (ii) holds. We need only to prove that for $m \in M$ and $n \in N$, $m \perp_H n \Rightarrow m \perp_G n$ since the reverse implication is trivial. Let $m \in M$ and $n \in N$ such that $m \perp_H n$. We want to prove that $m \perp_G n$. If m and n are irreducible then we are done. Else we get

$$m' = \bigcap_{i=0}^{k} m_i' \quad \text{and} \quad n' = \bigcap_{s=0}^{l} n_s' \qquad \text{for } 0 \le i \le k \text{ and } 0 \le s \le l$$

where m_i and n_s are irreducible. Therefore

$$\begin{aligned}
m \perp_H n \quad &\Rightarrow \quad (m' \cap H) \cup (n' \cap H) = H \\
&\Rightarrow \quad (m_i' \cup n_s') \cap H = H \quad \forall (i,s) \in \{0,\dots,k\} \times \{0,\dots,l\} \\
&\Rightarrow \quad m_i \perp_H n_s \quad \forall (i,s) \in \{0,\dots,k\} \times \{0,\dots,l\} \\
&\Rightarrow \quad m_i \perp_G n_s \quad \forall (i,s) \in \{0,\dots,k\} \times \{0,\dots,l\} \\
&\Rightarrow \quad m_i' \cup n_s' = G \quad \forall (i,s) \in \{0,\dots,k\} \times \{0,\dots,l\} \\
&\Rightarrow \quad \bigcap_{i=0}^{k} m_i' \cup \bigcap_{s=0}^{l} n_s' = G \\
&\Rightarrow \quad m' \cup n' = G \\
&\Rightarrow \quad m \perp_G n.
\end{aligned}$$

And (i) is proved. □

This result is a little bit surprising, since the concept algebra structure does not live on the irreducible elements, but on the $^{\triangle}$-compatible and $^{\triangledown}$-compatible elements. Some of them are from the lattice point of view reducible, but not from the concept algebra point of view.

Let us have a look at the problem of complete homomorphic images of concept algebras. We consider a concept algebra $\underline{\mathfrak{A}}(G,M,I)$, a weakly dicomplemented lattice L and a surjective complete homomorphism $f:\mathfrak{A}(G,M,I)\to L$. The kernel of f is a complete congruence of $\underline{\mathfrak{A}}(G,M,I)$. We assume that (G,M,I) is doubly founded. Then there exists a compatible subcontext (H,N) of (G,M,I) such that $\theta_{H,N}$ is $ker f$. Thus (H,N) is a compatible subcontext of (G,M,I) and (H,N) induces a concept algebra congruence. By the first isomorphism theorem we would obtain that

$$L \cong \underline{\mathfrak{A}}(G,M,I)_{/ker f} \cong \underline{\mathfrak{A}}(G,M,I)_{/\theta_{H,N}} \cong \underline{\mathfrak{A}}(H,N). \qquad i.e.$$

Theorem 2.7 *Homomorphic images of doubly founded concept algebras are (isomorphic to) concept algebras.*

All finite lattices are complete and doubly founded. Thus the class of finite concept algebras is stable under homomorphic images.

3 Congruence lattices of concept algebras.

The set of concept algebra congruences of a formal context $\mathbb{K}$ is a sublattice of the lattice of all equivalence relations on $B(\mathbb{K})$. We denote it by $Con\underline{\mathfrak{A}}(\mathbb{K})$. It is a sublattice of the distributive lattice $Con\underline{\mathfrak{B}}(\mathbb{K})$, the congruence lattice of the concept lattice of $\mathbb{K}$. Thus $Con\underline{\mathfrak{A}}(\mathbb{K})$ is a distributive lattice. By Birkhoff's theorem there is an ordered set $(P,\leq)$ such that $Con\underline{\mathfrak{A}}(\mathbb{K})$ is isomorphic to $\underline{\mathfrak{B}}(P,P,\not\geq)$. **Finding a good description of the poset** $(P,\leq)$ **is a problem worthy to be considered**. In the case of $Con\underline{\mathfrak{B}}(\mathbb{K})$, the poset $(P,\leq)$ is a copy of $(G_{irr},\leq)$[8] and of $(M_{irr},\leq)$ with $g\leq h \iff g''\subseteq h''$ for g and h in G_{irr}, where G_{irr} is the set of join irreducible elements and M_{irr} the set of meet irreducible elements.

Compatible subcontexts can be determined by means of arrow relations.

Definition 3.1 *Let (G,M,I) be a formal context with $g\in G$ and $m\in M$. The* **arrow relations** *are defined by:*

$$g \swarrow m :\iff m\notin g' \text{ and } g' \text{ maximal with respect to } m\notin g',$$

$$g \nearrow m :\iff g\notin m' \text{ and } m' \text{ maximal with respect to } g\notin m'.$$

A subcontext (H,N) of a clarified context (G,M,I) is **arrow-closed** *if the following holds: $h\nearrow m$ and $h\in H$ together imply $m\in N$, and dually, $g\swarrow n$ and $n\in N$ together imply $g\in H$.*

Proposition 3.1 *[GW99, p. 101] Every compatible subcontext is arrow-closed. Every arrow-closed subcontext of a doubly founded context is compatible.*

[8] $(G_{irr},\leq)\cong(M_{irr},\leq)$ with $m\leq n \iff m''\subseteq n''$ for $m,n\in M_{irr}$.

If the context (G, M, I) is reduced, arrow-closed subcontexts can be elegantly described in terms of the concepts of a context. For this purpose the transitive closure of the arrow relations is needed.

Definition 3.2 *For $g \in G$ and $m \in M$ we write $g \mathrel{\swarrow\!\!\!\nearrow} m$ if there are objects $g = g_1, g_2, \dots, g_k \in G$ and attributes $m_1, m_2, \dots, m_k = m \in M$ with $g_i \swarrow m_i$ for $i \in \{1, \dots, k\}$ and $g_j \nearrow m_{j-1}$ for $j \in \{2, \dots, k\}$. The relation $\mathrel{\swarrow\!\!\!\nearrow}$ is called the transitive closure of the arrow relation and is also denoted by $trans(\swarrow, \nearrow)$. The complement of this relation is denoted by $\mathrel{\not\swarrow\!\!\!\nearrow}$.*

Proposition 3.2 *[GW99, p. 102] Let (G, M, I) be a reduced doubly founded context. Then (H, N) is an arrow-closed subcontext if and only if $(G \setminus H, N)$ is a concept of the context $(G, M, \mathrel{\not\swarrow\!\!\!\nearrow})$.*

Thus the congruence lattice of $\underline{\mathfrak{B}}(G, M, I)$ is isomorphic to the concept lattice of the context $(G, M, \mathrel{\not\swarrow\!\!\!\nearrow})$ if (G, M, I) is reduced and doubly founded. This isomorphism exists even if the context is not assumed to be reduced. Our aim is to find a similar description for the congruence lattice of concept algebras. Complete sublattices of concept lattices are described by closed subrelations. We consider a reduced finite context (G, M, I). The congruence lattice of $\underline{\mathfrak{B}}(\mathbb{K})$ is isomorphic to $\underline{\mathfrak{B}}(G, M, \mathrel{\not\swarrow\!\!\!\nearrow})$. The congruence lattice of $\underline{\mathfrak{A}}(\mathbb{K})$ is a sublattice of the congruence lattice of $\underline{\mathfrak{B}}(\mathbb{K})$. Thus there is a closed subrelation $\bowtie$ of $\mathrel{\not\swarrow\!\!\!\nearrow}$ such that $Con\underline{\mathfrak{A}}(\mathbb{K}) \cong \underline{\mathfrak{B}}(G, M, \bowtie)$. Is there any characterization of the relation $\bowtie$? **When does (g, m) belong to $\bowtie$?** It is enough to find out when (g, m) does not belong to $\bowtie$; (i.e. the relation $\not\bowtie$).

Observe that

$$\not\bowtie \;= (\not\bowtie \cap \mathrel{\not\swarrow\!\!\!\nearrow}) \cup (\not\bowtie \cap \mathrel{\swarrow\!\!\!\nearrow}) = (\not\bowtie \cap \mathrel{\not\swarrow\!\!\!\nearrow}) \cup \mathrel{\swarrow\!\!\!\nearrow}$$

since $\bowtie \leq \mathrel{\not\swarrow\!\!\!\nearrow}$ implies that $\not\bowtie \;\supseteq \mathrel{\swarrow\!\!\!\nearrow}$. The problem now is to find $\not\bowtie \cap \mathrel{\not\swarrow\!\!\!\nearrow}$.

Note that

$$\bowtie = \bigcup_{(A, A') \in B(G, M, \bowtie)} A \times A'.$$

i.e. (g, m) is in $\bowtie$ if and only if there exists (A, A') in $B(G, M, \bowtie)$ such that $g \in A$ and $m \in A'$. Therefore $(g, m) \notin \bowtie$ if and only if for any concept (A, A') in $B(G, M, \bowtie)$, it holds $g \notin A$ or $m \notin A'$. This is equivalent to $g \in A \Rightarrow m \notin A'$ for any concept $(A, A') \in B(G, M, \bowtie)$. Thus $(g, m) \notin \bowtie$ if and only if for any compatible subcontext (H, N) $^{\triangle}$- and $^{\triangledown}$-compatible, $g \notin G \setminus H$ or $m \notin N$. i.e. $(g, m) \notin \bowtie$ if and only if for any compatible subcontext (H, N) $^{\triangle}$- and $^{\triangledown}$-compatible, $m \in N \Rightarrow g \in H$.

Remark 3.1 *Let (G, M, I) be a reduced doubly founded context. Any complete congruence θ of $\underline{\mathfrak{B}}(G, M, I)$ is induced by (H, N) with*

$$H = \{g \in G_{irr} \mid \gamma g \not\theta \gamma g_*\}$$

and

$$N = \{m \in M_{irr} \mid \mu m \between \mu m^*\}.$$

Proposition 3.3 *We consider* (G, M, I) *to be a reduced context. Let* $g \in G$ *and* $m \in M$*. The following holds:*

$$\gamma g^{\triangle} \leq \mu m \text{ and } \gamma g_*^{\triangle} \not\leq \mu m \Rightarrow (g, m) \notin \bowtie \text{ and } (g, m) \notin \mathrm{I}\ .$$

Proof. If $\gamma g^{\triangle} \leq \mu m$ and $g \mathrel{\mathrm{I}} m$ then $\gamma g \leq \mu m$ and $1 = \gamma g^{\triangle} \vee \gamma g \leq \mu m$, which is in contradiction with $\gamma g_*^{\triangle} \not\leq \mu m$. Let (H, N) be a compatible and $\{^{\triangle}, ^{\triangledown}\}$-compatible subcontext and $\theta_{H,N}$ be the corresponding congruence. We consider $m \in N$ and want to show that g must be in H.

$$\gamma g \theta \gamma g_* \Rightarrow \gamma g^{\triangle} \theta \gamma g_*^{\triangle} \Rightarrow \gamma g^{\triangle} \vee \mu m \theta \gamma g_*^{\triangle} \vee \mu m \Rightarrow \mu m \theta \mu m^*.$$

$m \in N$ implies $\mu m \between \mu m^*$ and $\gamma g \between \gamma g_*$. Thus $g \in H$. □

The above proposition gives a sufficient condition for membership of the relation $\not\bowtie$. But it is not necessary. It is possible to have $(g, m) \in \not\bowtie$ and $(g, m) \in \mathrm{I}$. From the proof we also have

$$\gamma g^{\triangledown} \leq \mu m \text{ and } \gamma g_*^{\triangledown} \not\leq \mu m \Rightarrow (g, m) \notin \bowtie\ .$$

Is the condition "$\gamma g^{\triangledown} \leq \mu m$ and $\gamma g_*^{\triangle} \not\leq \mu m$" enough to get $(g, m) \in \not\bowtie$?

This leaves us with the open problem to find a necessary and sufficient condition. An answer is given for finite distributive concept algebras ([Ga04]).

4 Conclusion

We have found a condition under which a complete congruence of a concept lattice is a concept algebra congruence. The lattice of all congruences is still to be described. Although there is an evidence that this lattice is isomorphic to a lattice $\underline{\mathfrak{B}}(P, P, \not\geq)$ for some poset $(P, \leq)$, a "good description" of this poset is still to be found.

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