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A NOTE ABOUT SHELLABLE PLANAR POSETS

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Abstract. We will show that shellability, Cohen-Macaulayness and vertex-de composability of a graded, planar poset P are all equivalent with the fact that P has the maximal possible number of edges. Also, for a such poset we will find an $R-$labelling with $\{1,2\}$ as the set of labels. Using this, we will obtain all essential linear inequalities for the flag $h-$vectors of shellable planar posets from [1].

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1. Introduction

A *graded* poset P is a finite partially ordered set with a unique minimum element $\widehat{0}$, a unique maximum element $\widehat{1}$, and a rank function $r : P \to \mathbb{N}$ where $r(\widehat{0}) = 0$, and whenever $x < y$, $\{z \in P : x < z < y\} = \emptyset$ (we say then that y *covers* x and denote $x \prec y$) then $r(y) = r(x) + 1$. We call $r(\widehat{1})$ the rank of the poset P. In a graded poset P of rank $n+1$ all maximal (unrefinable) chains have the same length $n+1$.

For a graded poset P of rank $n+1$ and $S \subseteq [n] = \{1,2,...,n\}$ we define $f_S(P)$ as the number of chains $x_1 < x_2 < \cdots < x_{|S|}$ in P such that $\{r(x_1), r(x_2), \ldots, r(x_{|S|})\} = S$. The sequence $(f_S(P))_{S\subseteq[n]}$ is called *the flag $f-$vector* of P. The first step in the characterization of flag $f-$ vectors of a class of posets is to determine the linear equations that they must satisfy. As the second step, we are looking for the essential linear inequalities that hold for all flag $f-$vectors of all posets in this class. This is equivalent with the description of the closure of the convex cone that those vectors generate.

The flag $h-$vector of P, i.e. the sequence $(h_S)_{S\subseteq[n]}$, is obtained as the following linear transformation of $(f_S)_{S\subseteq[n]}$:

$$h_S(P) = \sum_{T\subseteq S} (-1)^{|S\setminus T|} f_T(P)$$

An (abstract) *simplicial complex* is a collection Δ of finite nonempty subsets such that $\sigma \subseteq \tau \in \Delta \Rightarrow \sigma \in \Delta$. The element σ of Δ is called *face* (or *simplex*)

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of Δ and its dimension is $|\sigma| - 1$. A good source of general references for the simplicial complexes and their combinatorial properties is [3].

A simplicial complex Δ is *vertex decomposable* (see [2], [11]) if it is pure $d-$dimensional (all maximal faces of Δ have the same cardinality $d+1$) and either Δ is a simplex, or there exists a vertex x such that $\Delta \setminus \{x\}$ is $d-$dimensional and vertex decomposable, and $lk_{\Delta}(x) = \{\sigma \in \Delta : x \notin \sigma, \{x\} \cup \sigma \in \Delta\}$ is $(d-1)-$dimensional and vertex decomposable. If we use the previous definition inductively, we get that for a vertex-decomposable complex Δ there exists a linear (shedding) order of vertices $v_1, v_2, \ldots, v_n$ such that both $\Delta \setminus \{v_i, v_{i+1}, \ldots, v_n\}$ and $lk_{\Delta \setminus \{v_{i+1}, \ldots, v_n\}}(v_i)$ are vertex-decomposable, for all $i = 1, 2, \ldots, n$.

A finite dimensional simplicial complex Δ is said to be *Cohen-Macaulay* (see [3]) if for all $\sigma \in \Delta$, the reduced simplicial homology of $lk_{\Delta}(\sigma) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta\}$ is trivial ($\widetilde{H}_i(lk_{\Delta}(\sigma)) = 0$) for $i < \dim lk_{\Delta}(\sigma)$. For a definition of reduced simplicial homology see Chapter 1 in [8].

The *order-complex* $\Delta(P)$ of a graded poset P is the simplicial complex on vertex set P whose faces are the chains in P. The definition of $\Delta(P)$ (goes back to Aleksandrov, 1937) is a passage between combinatorics and topology. We say that a graded poset P is vertex decomposable (Cohen-Macaulay) if its order complex $\Delta(P)$ is vertex decomposable (Cohen-Macaulay).

Shelling of simplicial and cell complexes (see [5],[6]) is a very basic and useful technique with many geometric and combinatorial applications. The concept of shellability gives us a combinatorial description of the $h-$vector of shellable simplicial complexes, a simple proof and notation of Dehn Sommerville equations for the $f-$vector of simplicial polytopes, the upper bound theorem for simplicial polytopes ... (see [10]). For our purposes, we use the definition of shellability for graded posets from [6].

Definition 1. *A finite graded poset P is said to be shellable if all maximal chains can be ordered $C_1, C_2, \ldots, C_t$ in such a way that if $1 \le i < j \le t$ then there exist $1 \le k < j$ and x in chain C_j such that $C_i \cap C_j \subseteq C_k \cap C_j = C_j \setminus \{x\}$. Such an ordering of the maximal chains is called shelling order.*

Many examples of shellable posets can be found in [4] and [5]. Given a shelling order define *the restriction* of the maximal chain C_i by $\mathcal{R}(C_i) = \{x \in C_i : C_i \setminus \{x\} \subset C_j$ for some $j < i\}$. If we draw the Hasse diagram of the poset P chain by chain (according to given shelling order), then the restriction $\mathcal{R}(C)$ is the unique minimal new chain that appears when we draw the maximal chain C.

For a graded poset P, the following implications are strict (see [3]):

$$P \text{ is vertex decomposable} \Rightarrow P \text{ is shellable} \Rightarrow P \text{ is Cohen-Macaulay}$$

2. Shellable planar posets

For any graded poset P, embedding of its Hasse diagram in the plane defines the linear ordering $<_i$ at every level $P_i = \{x \in P : r(x) = i\}$

$$x <_i y \text{ iff the vertex } x \text{ is left from } y$$

For $x \in P$, we define $U(x) = \{y \in P : x \prec y\}$, i.e. the set of all elements of P that covers x. A poset P is *planar* if its Hasse diagram can be drawn in the plane with straight, non-crossing edges, such that whenever $x \prec y$ in P, the vertex representing y appears above the vertex representing x. Then, if P is a planar graded poset, we have that for all $x <_i x'$ holds $\max_{<_{i+1}} U(x) \leq_{i+1} \min_{<_{i+1}} U(x')$.

Remark 2. *A graded planar poset is always a lattice, see* [7].

We say that a planar graded poset is *saturated* if its Hasse diagram has the highest possible number of edges. More precisely, a graded planar poset P of rank $n+1$ is saturated iff

$$\text{(1)} \qquad \forall i \in [n], \text{ and for all } x \prec_i x', \quad \max{}_{<_{i+1}} U(x) = \min{}_{<_{i+1}} U(x')$$

Remark 3. *A simple counting of the edges between P_i and P_{i+1} gives us that the Hasse diagram of a saturated planar poset P of rank $n+1$ has $2|P| - n - 3$ edges.*

The next lemma will be useful for the characterization of shellable planar posets.

Lemma 4. *Let P be a saturated poset, and let $[x, y]$ be an interval in P such that $r(x) = i, r(y) = j, j - i \geq 2$. Let $x = x_i \prec x_{i+1} \prec \cdots \prec x_{j-1} \prec x_j = y$ be a maximal chain in $[x, y]$ such that for all k, $i < k < j$ there exist $y_k \in [x, y]$, $y_k \prec_k x_k$ (x_k is not contained in the most left maximal chain in $[x, y]$). Then, there exists k_0, $i < k_0 < j$ such that $x_{k_0-1} \prec y_{k_0} \prec x_{k_0+1}$.*

Proof. We will use the induction on $j - i$. If $j - i = 2$, then we get $y_{i+1} = y_{j-1}$ and $k_0 = i + 1 = j - 1$. For $j - i > 2$, we consider y_{i+1}. If $y_{i+1} \prec x_{i+2}$, then we get $k_0 = i + 1$. Otherwise, if $y_{i+1} \not\prec x_{i+2}$ then, from (1) we have that $x_{i+1} \prec y_{i+2}$. Now, from the inductive assumption for $[x_{i+1}, y]$, follows that there exists k_0, $i + 1 < k_0 < j$ such that $x_{k_0-1} \prec y_{k_0} \prec x_{k_0+1}$. □

Theorem 5. *For a graded, planar poset the following statements are equivalent:*

1. *P is saturated*
2. *P is shellable*
3. *P is Cohen-Macaulay*

Proof. First, we will show that $1 \Rightarrow 2$. Let P be a saturated poset of rank $n+1$. For $C\colon\widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \widehat{1}$ and $C'\colon\widehat{0} = x'_0 \prec x'_1 \prec \cdots \prec x'_n \prec x'_{n+1} = \widehat{1}$, two maximal chains in P, let $j_0 = \max\{i : x_i \neq x'_i\}$. We define a linear order $<_E$ for maximal chains of P:

$$C <_E C' \Leftrightarrow x_{j_0} <_{j_0} x'_{j_0} \tag{2}$$

i.e. C is before C' in $<_E$ iff at the level j_0 (the highest level where C and C' are different) the chain C is left from C' . If we let $i_0 = \max\{i < j_0 : x_i = x'_i\}$ (i_0 is the highest level below j_0 where C and C' are equal), then the maximal chain $x_{i_0} = x'_{i_0} \prec x'_{i_0+1} \prec \cdots \prec x'_{j_0} \prec x'_{j_0+1} = x_{j_0+1}$ in $[x_{i_0}, x_{j_0+1}]$ satisfies the conditions of Lemma 4. So, there exist k, $i_0 < k < j_0 + 1$ and $z \in [x_{i_0}, x_{j_0+1}]$ such that $x'_{k-1} \prec z \prec x'_{k+1}$. If we let $C''\colon\widehat{0} = x'_0 \prec x'_1 \prec \cdots \prec x'_{k-1} \prec z \prec x'_{k+1} \prec \cdots \prec x'_n \prec x'_{n+1} = \widehat{1}$, then C'' is before C' in $<_E$. Also

$$C \cap C' \subseteq C'' \cap C' = C' \setminus \{x'_k\}$$

and $<_E$ is a shelling order in the sense of the definition 1.

As any shellable poset is also Cohen-Macaulay (see [3], [4]), then $2 \Rightarrow 3$ is obvious.

Now, we will prove that $3 \Rightarrow 1$. Suppose that a planar, graded poset P is a Cohen-Macaulay, but not saturated. Then, there exist x and x' at the same level P_i such that $x \prec_i x'$, and $\max_{<_{i+1}} U(x) < \min_{<_{i+1}} U(x)$. Then (by remark 2) there exist $y = x \wedge x'$ and $z = x \vee x'$. If we choose two arbitrary maximal chains C_1 in $[\widehat{0}, y]$, and C_2 in $[z, \widehat{1}]$, link of the face $\sigma = C_1 \cup C_2$ in $\Delta(P)$ is exactly the order complex for the interval (y, z) in P. Since $lk_{\Delta(P)}(\sigma)$ is not connected, we have that $\widetilde{H}_0(lk_{\Delta(P)}(\sigma)) \neq 0$. This is in contradiction with the assumption that P is a Cohen-Macaulay poset. □

Remark 6. *Let P be a saturated poset of rank $n+1$. For a maximal chain $C\colon\widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \widehat{1}$, the restriction of C in the shelling order $<_E$ is $\mathcal{R}(C) = \{x_i : \exists x'_i \prec_i x_i, x_{i-1} \prec x'_i \prec x_{i+1}\}$. Then, for any $x \in P$ that is not contained in the most left maximal chain in P (x is not minimal in $<_{r(x)}$), there exists the unique maximal chain C_x whose restriction in the shelling order $<_E$ is $\{x\}$. We obtain the chain C_x as the concatenation of the most left chains in $[\widehat{0}, x]$ and $[x, \widehat{1}]$. In this way, from the shelling order defined in (2), we get the following linear order of the vertices of P:*

The most left chain $\widehat{0} = v_1 \prec v_2 \prec \cdots \prec v_{n+2} = \widehat{1}$ in P contains the first $n+2$ vertices in this order. Shelling order $<_E$ induces the linear ordering $C_1, C_2, \cdots, C_{|P|-n-2}$ of the set of maximal chains in P whose restrictions are singletons. If $\mathcal{R}(C_i) = \{x\}$, then we label x as v_{i+n+2}.

Corollary 7. *All saturated posets are vertex-decomposable.*

Proof. We will use the induction by the cardinality and the rank. The case in which $r(P) = 1$ is trivial. Let P be a saturated poset of rank $n+1$. Suppose that the statement is true for all saturated posets whose rank is less than $n+1$, and for all saturated posets of rank $n+1$ with fewer elements than $|P|$. If $|P| = n+2$ (P is a chain), then $\Delta(P)$ is a simplex. If $|P| > n+2$, we consider $v_{|P|}$, the last vertex in the order of vertices defined in remark 6. $P \setminus \{v_{|P|}\}$ is a saturated poset with fewer vertices than P, and vertex decomposable by the assumption. From remark 6, we see that $v_{|P|}$ covers and is covered by exactly one element, and so $[\widehat{0}, v_{|P|}) \cup (v_{|P|}, \widehat{1}]$ is a saturated poset, whose rank is n. Since $lk_{\Delta(P)}(v_{|P|})$ is the order complex for $[\widehat{0}, v_{|P|}) \cup (v_{|P|}, \widehat{1}]$, then P is vertex-decomposable. Note that the reverse order of vertices from the order defined in remark 6 is a shedding-order for $\Delta(P)$. □

3. Flag $h-$vectors of shellable planar posets

For any finite graded poset P we let $\mathcal{E}(P)$ denote its covering relation, $\mathcal{E}(P) = \{(x, y) \in P \times P : x \prec y\}$. An *edge-labelling* of P is a map $\lambda : \mathcal{E}(P) \to \Lambda$, where Λ is a poset (usually $\Lambda = (\mathbb{Z}, \leq)$). This corresponds to the assignment of elements of Λ to the edges of the Hasse diagram of P. Given an edge labelling λ, each unrefinable chain $C \colon x = x_0 \prec x_1 \prec \cdots \prec x_{k-1} \prec x_k = y$ of length k can be associated with a $k-$tuple $\lambda(C) = (\lambda(x_0 \prec x_1), \lambda(x_1 \prec x_2), \ldots, \lambda(x_{k-1} \prec x_k))$. We say that C is a *rising* chain if $\lambda(x_0 \prec x_1) \leq \lambda(x_1 \prec x_2) \leq \ldots \leq \lambda(x_{k-1} \prec x_k)$. The edge labelling λ of P is said to be an $R-$labelling if in every interval $[x, y]$ of P there is a unique rising maximal chain C in $[x, y]$. For a maximal chain $C \colon \widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \widehat{1}$ we define its *descent set* $D(C) = \{i \in [n] : \lambda(x_{i-1} \prec x_i) > \lambda(x_i \prec x_{i+1})\}$. If a poset P admits an $R-$labelling then the following result from [9] gives us the combinatorial interpretation of the flag $h-$vectors.

Theorem 8. *Let P be a finite bounded graded poset of rank $n+1$ with an $R-$labeling λ. Then, for all $S \subseteq [n]$, $h_S(P)$ is equal to the number of maximal chains of P with the descent set S.*

As a consequence of this theorem, it follows that for any graded poset P that admits an $R-$labelling it holds that $h_S(P) \geq 0$.

Theorem 9. *Let P be a saturated poset. Then P admits an $R-$labeling with $\{1, 2\}$ as the set of labels.*

Proof. Let P be a saturated poset of rank $n+1$ with the shelling order $<_E$ as in theorem 5. If we draw the most left chain $\widehat{0} = v_1 \prec v_2 \prec \cdots \prec v_{n+2} = \widehat{1}$ and all maximal chains $C_{i_1}, C_{i_2}, \cdots, C_{i_{|P|-n-2}}$ whose restrictions are singletons (in the order defined in Remark 6), then by Remark 3, we reconstruct Hasse-diagram of poset P. Using this, we define $\lambda : \mathcal{E}(P) \to \{1,2\}$ as follows. We label all the edges contained in the most left chain of P with 1. When we draw the chain C_i, then we add a new vertex v_{i+n+2} and two edges $a \prec v_{i+n+2}$, $v_{i+n+2} \prec b$. If we let $\lambda(a \prec v_{i+n+2}) = 2$ and $\lambda(v_{i+n+2} \prec b) = 1$, then from Remark 3, all the edges of the Hasse diagram of P are labelled. Note that

$$\lambda(x \prec y) = \begin{cases} 1; & \text{for } y = \min_{<_{r(x)+1}} U(x) \\ 2; & \text{otherwise} \end{cases}$$

Now, in any interval $[x,y]$, the unique chain without descents is the most left chain $C \colon x = x_0 \prec x_1 \prec \cdots \prec x_{k-1} \prec x_k = y$. If 2 appears as the label of the edge $x_{i-1} \prec x_i$, then there exists x_i' such that $x_{i-1} \prec x_i'$, and $x_i' \prec_{r(x_i)} x_i$ in $P_{r(x_i)}$. As C is the most left chain in $[x,y]$, we have that $x_i' \not\prec x_{i+1}$. Then, from (1) it follows that there exists $w <_{r(x_{i+1})} x_{i+1}$ such that $x_i \prec w$. So, the label of the edge $x_i \prec x_{i+1}$ is 2, and chain C is without descent.

Let $C' \colon x = x_0' \prec x_1' \prec \cdots \prec x_{k-1}' \prec x_k' = y$ be any other maximal chain in $[x,y]$. Let $i_0 = \min\{i : x_i \neq x_i'\}$ and $j_0 = \min\{j > i_0 : x_j = x_j'\}$. Then, $\lambda(x_{i_0-1} \prec x_{i_0}) = 2$, and $\lambda(x_{j_0-1} \prec x_{j_0}) = 1$, so chain C' has a descent. □

Obviously, there are no consecutive descents in the sequence $\lambda(x_0 \prec x_1), \lambda(x_1 \prec x_2), \ldots, \lambda(x_n \prec x_{n+1}) \in \{1,2\}^{n+1}$ and Theorem 8 gives us the following result from [1].

Corollary 10. *Let P be a planar shellable poset of rank $n+1$. Then, $h_S(P) \geq 0$ for all $S \subseteq [n]$. If S contains two consecutive integers, then $h_S(P) = 0$.*

From this corollary follows that the dimension of the vector space generated by the flag $f-$vectors of shellable planar posets of rank $n+1$ is the Fibonacci number c_n ($c_0 = c_1 = 1$, $c_{n+1} = c_n + c_{n-1}$). It is not difficult to prove (see [1]) that the closure of the cone generated by the flag $f-$vectors of all saturated posets of rank $n+1$ is a simplicial cone.
Also, we can note that if a graded poset P of rank $n+1$ admits an $R-$labeling then its Hasse diagram has exactly $2|P| - n - 3$ edges.

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