Novi Sad J. Math.
Vol. 34, No. 1, 2004, 71-77

ON CONHARMONICALLY AND SPECIAL WEAKLY RICCI SYMMETRIC SASAKIAN MANIFOLDS

Quddus Khan[1]

Abstract. We have studied some geometric properties of conharmonically flat Sasakian manifold and an Einstein-Sasakian manifold satisfying $R(X,Y).N = 0$. We have also obtained some results on special weakly Ricci symmetric Sasakian manifold and have shown that it is an Einstein manifold.

AMS Mathematics Subject Classification (2000): 53C21, 53C25

Key words and phrases: Sasakian manifold, conharmonic curvature tensor, Einstein manifold

1. Introduction

Let (M^n, g) be a contact Riemannian manifold with a contact form η, the associated vector field ξ, a $(1,1)$ tensor field $\varnothing$ and the associated Riemannian metric g. If ξ is a killing vector field, then M^n is called a $K-$contact Riemannian manifold ([2], [1]). A $K-$contact Riemannian manifold is called a Sasakian manifold [1] if

$$(1) \qquad (D_X\varnothing)(Y) = g(X,Y)\xi - \eta(Y)X$$

holds, where D denotes the operator of covariant differentiation with respect to g. This paper deals with a type of Sasakian manifold in which

$$(2) \qquad R(X,Y).N = 0\,,$$

where N is the conharmonic curvature tensor [4] defined by

$$(3) \qquad \begin{aligned} N(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y \\ &\quad + g(Y,Z)r(X) - g(X,Z)r(Y)]\,, \end{aligned}$$

and R is the Riemannian curvature tensor. Here Ric and r are the Ricci tensors of type $(0,2)$ and $(1,1)$, respectively, and $R(X,Y)$ is considered as derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. In this connection we mention the works of K. Sekigawa [3] and Z.L. Szabo [6]

[1]Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (Central University), New Delhi -110025, India, e–mail: dr_quddus_khan@rediffmail.com

who studied Riemannian manifold satisfying the conditions similar to it. It is easy to see that $R(X,Y).R = 0$ implies $R(X,Y).N = 0$. So it is meaningful to undertake the study of manifolds satisfying the condition (2).

In a Sasakian manifold M^n, besides the relation (1), the following also hold (see [2], [1]):

$$\varnothing(\xi) = 0 \tag{4}$$
$$\eta(\xi) = 1 \tag{5}$$
$$g(\varnothing X, \varnothing Y) = g(X,Y) - \eta(X)\eta(Y) \tag{6}$$
$$g(\xi, X) = \eta(X) \tag{7}$$
$$Ric(\xi, X) = (n-1)\eta(X) \tag{8}$$
$$D_X\xi = -\varnothing X \tag{9}$$
$$K(\xi, X)Y = g(X,Y)\xi - \eta(Y)X \tag{10}$$
$$K(\xi, X)\xi = -X + \eta(X)\xi \tag{11}$$
$$g(K(\xi, X)Y, \xi) = g(X,Y) - \eta(X)\eta(Y) \tag{12}$$
$$\eta(\varnothing X) = 0 \tag{13}$$

for any vector fields X, Y.

2. Sasakian manifold satisfying $N(X,Y)Z = 0$

We have the following:

Theorem 2.1. *A conharmonically flat Einstein Sasakian manifold M^n $(n \geq 3)$ is locally isometric with a unit sphere S^n (1).*

Proof. Let us suppose that in a Sasakian manifold M^n,

$$N(X,Y)Z = 0\,. \tag{14}$$

Then, it follows from (3) that

$$R(X,Y)Z = \frac{1}{n-2}\,[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)r(X) - g(X,Z)r(Y)]\,. \tag{15}$$

Let the manifold be Einstein, i.e. $Ric(X,Y) = kg(X,Y)$, where k is a constant. Then (15) reduces to

$$R(X,Y)Z = \frac{2k}{n-2}\,[g(Y,Z)X - g(X,Z)Y] \tag{16}$$

or,

$$g(R(X,Y)Z, V) = \frac{2k}{n-2}\,[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)]\,. \tag{17}$$

Taking $X = V = \xi$ in (17) and then using (5), (7) and (12), we get

$$g(Y,Z) - \eta(Y)\eta(Z) = \frac{2k}{n-2}\,[g(Y,Z) - \eta(Y)\eta(Z)]$$

or,

$$\left[\frac{2k}{n-2} - 1\right][g(Y,Z) - \eta(Y)\eta(Z)] = 0\,.$$

This shows that either $2k = (n-2)$ or, $g(Y,Z) = \eta(Y)\eta(Z)$.

Now, if $g(Y,Z) = \eta(Y)\eta(Z)$, then from (6), we get $g(\varnothing X, \varnothing Y) = 0$, which is not possible. Therefore, $2k = (n-2)$. Now, putting $2k = (n-2)$ in (16), we get that the manifold is of constant curvature unity, whereby proving the result. □

3. An Einstein-Sasakian manifold satisfying $R(X,Y).N = 0$

We have the following:

Theorem 3.1. *If in an Einstein Sasakian manifold the relation $R(X,Y).N = 0$ holds, then it is locally isometric with a unit sphere S^n (1).*

Proof. Let a Sasakian manifold M^n be an Einstein manifold. Then (3) gives

$$N(X,Y)Z = R(X,Y)Z - \frac{2k}{n-2}\,[g(Y,Z)X - g(X,Z)Y]\,. \tag{18}$$

We have,

$$\begin{aligned} \eta(N(X,Y)Z) &= g(N(X,Y)Z,\xi) \\ &= g(R(X,Y)Z - \frac{2k}{n-2}\,[g(Y,Z)X - g(X,Z)Y],\xi) \\ &= \eta(X)g(Z,Y) - \eta(Y)g(Z,X) \\ &\quad - \frac{2k}{n-2}\,[\eta(X)g(Z,Y) - \eta(Y)g(Z,X)] \end{aligned}$$

or,

$$\eta(N(X,Y)Z) = \left[\frac{2k}{n-2} - 1\right][\eta(Y)g(Z,X) - \eta(X)g(Z,Y)]\,. \tag{19}$$

Putting $X = \xi$ in (19) and using (5) and (7), we get

$$\eta(N(\xi,Y)Z) = \left[\frac{2k}{n-2} - 1\right][\eta(Y)\eta(Z) - g(Z,Y)]\,. \tag{20}$$

Again, putting $Z = \xi$ in (19) and using (5) and (7), we get

$$\eta(N(X,Y)\xi) = 0\,. \tag{21}$$

Now,

$$\begin{aligned}(R(X,Y)N)(U,V)W &= R(X,Y)N(U,V)W - N(R(X,Y)U,V)W \\ &\quad - N(U,R(X,Y)V)W - N(U,V)R(X,Y)W\,.\end{aligned}$$

By virtue of (2), we get

$$\begin{aligned}R(X,Y)N(U,V)W - N(R(X,Y)U,V)W \\ - N(U,R(X,Y)V)W - N(U,V)R(X,Y)W = 0.\end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned}g[R(\xi,Y)N(U,V)W,\xi] - g(N(R(\xi,Y)U,V)W,\xi] \\ - g(N(U,R(\xi,Y)V)W,\xi] - g[N(U,V)R(\xi,Y)W,\xi] = 0\,.\end{aligned}$$

From this it follows that

$$\begin{aligned}&'N(U,V,W,Y) - \eta(Y)\eta(N(U,VW) + \eta(U)\eta(N(Y,V)W) \\ &\quad + \eta(V)\eta(N(U,Y)W) + \eta(W)\eta(N(U,V)Y) - g(Y,U)\eta(N(\xi,V)W) \\ &\quad - g(Y,V)\eta(N(U,\xi)W) - g(Y,W)\eta(N(U,V)\xi) = 0\,,\end{aligned} \tag{23}$$

where $g(N(U,V)W,Y) ='N(U,V,W,Y)$.

Putting $Y = U$ in (23), we get

$$\begin{aligned}&'N(U,V,W,U) - \eta(U)\eta(N(U,V)W) + \eta(U)\eta(N(U,V)W) \\ &\quad + \eta(V)\eta(N(U,U)W) + \eta(W)\eta(N(U,V)U) - g(U,U)\eta(N(\xi,V)W) \\ &\quad - g(U,V)\eta(N(U,\xi)W) - g(U,W)\eta(N(U,V)\xi) = 0\,.\end{aligned} \tag{24}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (24) for $U = e_i$ gives

$$\begin{aligned}\eta(N(\xi,V)W) &= \frac{1}{n-1}\Big[Ric(V,W) - \frac{r}{n}g(V,W) \\ &\quad + \left(\frac{r}{n(n-1)} - 1\right)(n-1)\eta(W)\eta(V)\Big]\,.\end{aligned} \tag{25}$$

Using (19) and (25) in (23), we get

$$\begin{aligned}'N(U,V,W,Y) + \frac{r}{n(n-1)}[g(Y,U)g(V,W) - g(Y,V)g(U,W)] \\ + \frac{1}{n-1}[Ric(U,W)g(Y,V) - Ric(V,W)g(Y,U)] = 0\,.\end{aligned} \tag{26}$$

By virtue of $Ric(W,V) = kg(W,V)$ and $r = nk$, relation (26) reduces to

$$'N(U,V,W,Y) = \left(\frac{2k}{n-2} - 1\right)[g(Y,V)g(U,W) - g(Y,U)g(V,W)]\,. \tag{27}$$

From (18) and (27), we get

$$'R(U,V,W,Y) = [g(Y,U)g(V,W) - g(Y,V)g(U,W)]\,,$$

where $'R(U,V,W,Y) = g(R(U,V)W,Y)$, which proves the result. □

For a conharmonically symmetric Sasakian manifold, we have $DN = 0$. Hence for such a manifold $R(X,Y).N = 0$ holds. Thus we have the following:

Corollary 3.1. *A conharmonically symmetric Sasakian manifold is locally isometric with a unit sphere S^n (1).*

4. On special weakly Ricci symmetric Sasakian manifold

The notion of a special weakly Ricci symmetric manifold was introduced and studied by Singh and Quddus [4].

An $n-$dimensional Riemannian manifold (M^n, g) is called a special weakly Ricci symmetric manifold (SWRS)n if

$$(D_X Ric)(Y,Z) = 2\alpha(X)Ric(Y,Z) + \alpha(Y)Ric(X,Z) + \alpha(Z)Ric(Y,X)\,, \tag{28}$$

where α is a $1-$form and is defined by

$$\alpha(X) = g(X,\rho)\,, \tag{29}$$

where ρ is the associated vector field.

Let (28) and (29) be satisfied in a Sasakian manifold M^n. Taking cyclic sum of (28), we get

$$\begin{aligned}(D_X Ric)(Y,Z) &+ (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) \\ &= 4[\alpha(X)Ric(Y,Z) + \alpha(Y)Ric(Z,X) + \alpha(Z)Ric(X,Y)]\,.\end{aligned} \tag{30}$$

Let M^n admits a cyclic Ricci tensor. Then (30) reduces to

$$\alpha(X)Ric(Y,Z) + \alpha(Y)Ric(Z,X) + \alpha(Z)Ric(X,Y) = 0\,. \tag{31}$$

Taking $Z = \xi$ in (31) and then using (8) and (29), we get

$$\alpha(X)(n-1)\eta(Y) + \alpha(Y)(n-1)\eta(X) + \eta(\rho)Ric(X,Y) = 0\,. \tag{32}$$

Again, taking $Y = \xi$ in (32) and then using (5), (8) and (29), we get

$$\alpha(X) + \eta(\rho)\eta(X) + \eta(\rho)\eta(X) = 0\,. \tag{33}$$

Taking $X = \xi$ in (33) and using (5) and (29), we get

$$\eta(\rho) = 0\,. \tag{34}$$

Using (34) in (33), we have $\alpha(X) = 0$, $\forall X$.

This leads us to the following:

Theorem 4.1. *If a special weakly Ricci symmetric Sasakian manifold admits a cyclic Ricci tensor then the* $1-$*form* α *must vanish.*

Next, we have:

Theorem 4.2. *A special weakly Ricci symmetric Sasakian manifold can not be an Einstein manifold if the* $1-$*form* $\alpha \neq 0$.

Proof. For an Einstein manifold, $(D_X Ric)(Y,Z) = 0$ and $Ric(Y,Z) = kg(Y,Z)$, then (28) gives

$$2\alpha(X)g(Y,Z) + \alpha(Y)g(X,Z) + \alpha(Z)g(Y,X) = 0\,. \tag{35}$$

Taking $Z = \xi$ in (35) and then using (7) and (29), we get

$$2\alpha(X)\eta(Y) + \alpha(Y)\eta(X) + \eta(\rho)g(Y,X) = 0\,. \tag{36}$$

Again, taking $X = \xi$ and using (5), (7) and (29), we get

$$3\eta(\rho)\eta(Y) + \alpha(Y) = 0\,. \tag{37}$$

Taking $Y = \xi$ in (28) and using (5) and (29), we get

$$\eta(\rho) = 0\,. \tag{38}$$

Using (38) in (37), we get $\alpha(Y) = 0$, $\forall Y$, which completes the proof. □

Finally, we have the following:

Theorem 4.3. *A special weakly Ricci symmetric Sasakian manifold is an Einstein manifold.*

Proof. Taking $Z = \xi$ in (28), we have

$$(D_X Ric)(Y,\xi) = 2\alpha(X)Ric(Y,\xi) + \alpha(Y)Ric(X,\xi) + \alpha(\xi)Ric(Y,X)\,. \tag{39}$$

The left-hand side can be written in the form

$$(D_X Ric)(Y,\xi) = XRic(Y,\xi) - Ric(D_X Y,\xi) - Ric(Y, D_X\xi)\,.$$

Then, in view of (7), (8), (9) and (29), equation (39) becomes

$$\begin{aligned} &-(n-1)g(Y,\Phi X) + Ric(Y,\Phi X) \\ &\qquad = (n-1)[2\alpha(X)\eta(Y) + \alpha(Y)\eta(X)] + \eta(\rho)Ric(Y,X)\,. \end{aligned} \tag{40}$$

Taking $Y = \xi$ in (40) and then using (5), (7), (8) and (29), we get

$$-(n-1)\eta(\Phi X) + (n-1)\eta(\Phi X) = (n-1)[2\alpha(X) + \eta(\rho)\eta(X)] + (n-1)\eta(\rho)\eta(X)$$

or,

$$\alpha(X) + \eta(\rho)\eta(X) = 0\,. \tag{41}$$

Putting $X = \xi$ and in view of (2) and (29), equation (41) gives

$$\eta(\rho) = 0\,. \tag{42}$$

Using (42) in (41), we get

$$\alpha(X) = 0\,. \tag{43}$$

Using (43) in (28), we get $(D_X Ric) = 0$, which proves the result. □

Acknowledgement. This work was supported by the Department of Science and Technology, Government of India under SERC Fast Track Fellowship for Young Scientist Scheme No.SR/FTP/MS-17, 2001.

References

[1] Blair, D. E., Contact manifold in Riemannian Geometry. Lecture Notes in Mathematics, 509, Springer Verlag, 1976.

[2] Sasaki, S., Lecture notes on almost contact manifolds, part 1. Tohoku University, 1975.

[3] Sekigawa, K., Almost Hermitian manifold satisfying some curvature conditions. Kodai Math. J. 2(1979), 384–405.

[4] Singh, H., Khan, Q., On special weakly symmetric Riemannian manifolds. Publ. Math. Debrecen, Hungary 58/3(2001), 523–536.

[5] Singh, H., Khan, Q., On symmetric Riemannian manifolds. Novi Sad J. Math. 29(3)(1999), 301–308.

[6] Szabo, Z. L., Structure theorem on Riemannian space satisfying $K(X,Y).K = 0$, The local version. J. Diff. Geom. 17(1982), 531–582.

[7] Tamassy, L., Binh, T. Q., On weak symmetrics of Einstein and Sasakian manifolds. Tensor, N.S. 53(1993), 140–148.

[8] Tamassy, L., Binh, T. Q., Weakly symmetric and weakly projective symmetric Riemannian manifold. Calloq. Math. Soc. J. Bolyai 56(1989), 663–670.

Received by the editors August 25, 2003