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ON PRIMITIVE Γ-SEMIRINGS

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Abstract. After introducing the notions of primitive Γ-semiring and primitive ideal of a Γ-semiring we study them via operator semiring and obtain some results analogous to those of semiring theory.

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1. Introduction

We introduce the notion of Γ-semiring S-semimodule which we call ΓS-semimodule along with the ideas of irreducible, semi-irreducible and faithful ΓS-semimodules with an intention to introduce the notion of primitive Γ-semiring and in future to introduce the notion of Jacobson radical of a Γ-semiring. Here we study primitive Γ-semiring via the operator semirings of a Γ-semiring which we introduced in [1]. We show that a Γ-semiring S is primitive if and only if its right operator semiring R is a primitive semiring ([6]). Lastly, we characterize primitive h-ideal of a Γ-semiring S using the relation between the annihilator of an irreducible ΓS-semimodule M in S and that of M in the right operator semiring R of the Γ-semiring S.

2. Preliminaries

Let S and Γ be two additive commutative semigroups. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ (images to be denoted by $a\ \alpha\ b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:
(i) $a\ \alpha\ (b + c) = a\ \alpha\ b + a\ \alpha\ c$
(ii) $(a + b)\ \alpha\ c = a\ \alpha\ c + b\ \alpha\ c$
(iii) $a(\alpha + \beta)c = a\ \alpha\ c + a\ \beta\ c$
(iv) $a\ \alpha\ (b\ \beta\ c) = (a\ \alpha\ b)\ \beta\ c$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
If A and B are subsets of a Γ-semiring S and $\Delta \subseteq \Gamma$, we denote by $A\Delta B$, the subset of S consisting of all finite sums of the form $\sum a_i \alpha_i b_i$ where $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$. For the singleton subset $\{x\}$ of S we write $x\Delta B$ instead of $\{x\}\Delta B$. A *right(left)ideal* I of a Γ-semiring S is an additive subsemigroup

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of S such that $I\ \Gamma\ S \subseteq I$ ($S\ \Gamma\ I \subseteq I$). If I is both a right and a left ideal of S, then we say that I is a *two-sided ideal* or simply an *ideal* of S. An ideal I in a Γ-semiring S is called a *k-ideal* if $x + y \in I$, $x \in S$, $y \in I$ imply that $x \in I$. An ideal I in a Γ-semiring S is called an *h-ideal* if $x + y_1 + z = y_2 + z$, $x, z \in S$ and $y_1, y_2 \in I$ imply that $x \in I$. Let S be a Γ-semiring and G be the free additive commutative semigroup generated by $\Gamma \times S$. Then the relation ρ on G, defined by $\sum_{i=1}^{m}(\alpha_i, x_i)\rho\sum_{j=1}^{n}(\beta_j, y_j)$ if and only if $\sum_{i=1}^{m} a\alpha_i x_i = \sum_{j=1}^{n} a\beta_j y_j$ for all $a \in S$ ($m, n \in Z^+ =$ the set of all positive integers), is a congruence on G. Congruence class containing $\sum_{i=1}^{m}(\alpha_i, x_i)$ is denoted by $\sum_{i=1}^{m}[\alpha_i, x_i]$. Then G/ρ is an additive commutative semigroup. Now G/ρ forms a semiring with the multiplication defined by $(\sum_{i=1}^{m}[\alpha_i, x_i])(\sum_{j=1}^{n}[\beta_j, y_j]) = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$. We denote this semiring by R and call it the *right operator semiring* of the Γ-semiring S. Dually we define the left operator semiring L of the Γ-semiring S where $L = \{\sum_{i=1}^{m}[x_i, \alpha_i] : \alpha_i \in \Gamma,\ x_i \in S,\ i = 1, 2, \ldots, m;\ m \in Z^+\}$ and the multiplication on L is defined as $(\sum_{i=1}^{m}[x_i, \alpha_i])(\sum_{j=1}^{n}[y_j, \beta_j]) = \sum_{i,j}[x_i\alpha_i y_j, \beta_j]$. For $N \subseteq S$ and $\Delta \subseteq \Gamma$ we denote by $[N, \Delta]$ the set of all finite sums $\sum_{i=1}^{m}[x_i, \alpha_i]$ in L, where $x_i \in N$ and $\alpha_i \in \Delta$. Thus in particular $[S, \Gamma] = L$. Similarly, we denote by $[\Delta, N]$ the set of all finite sums $\sum_{j=1}^{n}[\beta_j, y_j]$ in R where $y_j \in N$, $\beta_j \in \Delta$ and in particular $[\Gamma, S] = R$. For simplicity $[\{x\}, \Gamma]$ is written as $[x, \Gamma]$ and $[\Gamma, \{x\}]$ is written as $[\Gamma, x]$. We also have $[x, \Gamma] \subseteq P([\Gamma, x] \subseteq P)$ if and only if $[x, \alpha] \in P$ (respectively $[\alpha, x] \in P$) for all $\alpha \in \Gamma$, where P is a subset of L (respectively R) and $x \in S$. For $P \subseteq L(P \subseteq R)$ we define $P^+ = \{a \in S\ :\ [a, I] \subseteq P\}$ (respectively $P^* = \{a \in S\ :\ [\Gamma, a] \subseteq P\}$). For $Q \subseteq S$ we define $Q^{+'} = \{\sum_{i=1}^{m}[x_i, \alpha_i] \in L\ :\ (\sum_{i=1}^{m}[x_i, \alpha_i])S \subseteq Q\}$ where $(\sum_{i=1}^{m}[x_i, \alpha_i])S$ denotes the set of all finite sums $\sum_{i,k} x_i\alpha_i s_k$, $s_k \in S$ and $Q^{*'} = \{\sum_{i=1}^{m}[\alpha_i, x_i] \in R\ :\ S(\sum_{i=1}^{m}[\alpha_i, x_i]) \subseteq Q\ \}$ where $S(\sum_{i=1}^{m}[\alpha_i, x_i])$) is the set of all finite sums $\sum_{k,i} s_k\alpha_i x_i$, $s_k \in S$. Here we note that $S(\sum_{i=1}^{m}[\alpha_i, x_i]) \subseteq Q(\sum_{i=1}^{m}[x_i, \alpha_i])\ S$

$\subseteq Q$) if and only if $\sum_{i=1}^{m} s\alpha_i x_i \in Q$ (respectively $\sum_{i=1}^{m} x_i\alpha_i s \in Q$) for all $s \in S$. If P is a $(k-, h-)$ ideal of $L(R)$, then $P^+(P^*)$ is a $(k-, h-)$ ideal of S. If Q is a $(k-, h-)$ideal of S then so is $Q^{+^{\text{`}}}(Q^{*^{\text{`}}})$ in $L(R)$. For a Γ-semiring S if there exists an element $0 \in S$ such that $0 + x = x$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in S$ and $\alpha \in \Gamma$ then 0 is called the *zero* of the Γ-semiring S and in that case we say that the Γ-semiring S is with zero. In such a case $[0, \alpha]$ is the zero of L and $[\alpha, 0]$ is the zero of R for any $\alpha \in \Gamma$. Again, if there exists an element $\sum_{i=1}^{m}[e_i, \delta_i] \in L$ $(\sum_{j=1}^{n}[\gamma_j, f_j] \in R)$ such that $\sum_{i=1}^{m} e_i\delta_i a = a$ $(\sum_{j=1}^{n} a\gamma_j f_j = a)$ for all $a \in S$ then S is said to have the *left unity* $\sum_{i=1}^{m}[e_i, \delta_i]$ (respectively the *right unity* $\sum_{j=1}^{n}[\gamma_j, f_j]$). The left (right) unity of the Γ-semiring S, if it exists, is the identity of the left operator semiring L (respectively the right operator semiring R) of S. An equivalence relation ρ, defined on a Γ-semiring S satisfying the condition that if $r\rho r'$ and $s\rho s'$ in S then $(r + s)\rho(r' + s')$ and $(r\alpha s)\rho(r'\alpha s')$ for all $\alpha \in \Gamma$, is called a Γ-*congruence* on the Γ-semiring S. For a proper ideal A of a Γ-semiring S the Γ-congruence on S, denoted by ρ_A, defined as $s\rho_A s'$ if and only if $s + a_1 = s' + a_2$ for some $a_1, a_2 \in A$, is called the *Bourne* Γ-*congruence* on S defined by the ideal A. We denote the Bourne Γ-congruence (ρ_A) class of an element r of S by r/ρ_A or simply by r/A and denote the set of all such Γ-congruence classes of the Γ-semiring S by S/ρ_A or by S/A. It should be noted here that for any proper ideal A of S and for any $s \in S$, s/A is not necessarily equal to $s + A = \{s + a \ : \ a \in A\}$ but surely contains it. For any proper ideal A of a Γ-semiring S, if the Bourne Γ-congruence ρ_A, defined by A, is proper i.e. $0/A \neq S$ then S/A is a Γ-semiring with the following operations: $s/A + s'/A = (s + s')/A$ and $(s/A)\alpha(s'/A) = (s\alpha s')/A$ for all $\alpha \in \Gamma$. We call this Γ-semiring the *Bourne factor* Γ-*semiring* or simply the *factor* Γ-*semiring* of S by A.

For preliminaries of semirings, Γ-semirings, operator semirings of a Γ-semiring and Γ-rings we refer to [4], [1], [2], [7].

Throughout this paper the Γ-semiring S is assumed to be with zero, left unity and right unity.

3. Irreducible, semi-irreducible, faithful Γ-semimodules

Definition 3.1. *Let S be a Γ-semiring. An additive commutative monoid M is said to be a right Γ-semiring S-semimodule or simply a ΓS-semimodule, if there exists a mapping $M \times \Gamma \times S \to M$ (images to be denoted by $a\alpha S$ for $a \in M$, $\alpha \in \Gamma$, $s \in S$) satisfying the following conditions:*

(i) $(a+b)\alpha s = a\alpha s + b\alpha s$,
(ii) $a\alpha(s+t) = a\alpha s + a\alpha t$,
(iii) $a(\alpha+\beta)s = a\alpha s + a\beta s$,
(iv) $a\alpha(s\beta t) = (a\alpha s)\beta t$ *and*
(v) $0_M \alpha s = 0_M = a\alpha 0_S$ *for all* $a, b \in M$, *for all* $s, t \in S$ *and for all* $\alpha, \beta \in \Gamma$.

If in addition to the above conditions $\sum\limits_j a\gamma_j f_j = a$ holds for all $a \in M$, where $\sum\limits_{j=1}^{n} [\gamma_j, f_j]$ is the right unity of the Γ-semiring S, then M is said to be a unitary ΓS-semimodule.

Left Γ-semimodule of S can be defined in a similar manner and it is called $S\Gamma$-semimodule.

Example 3.2. Let S be a Γ-semiring, where S is the additive commutative semigroup of all 2×3 matrices over the set of all nonnegative rational numbers Q_0^+ and Γ is the additive commutative semigroup of all 3×2 matrices over the same set and $a\alpha b$ denotes the usual matrix product of a, α, b where $a, b \in S$ and $\alpha \in \Gamma$. Let M be the additive commutative monoid of all 3×3 matrices over Q_0^+. Then M is a unitary ΓS-semimodule, where $m\alpha a$ denotes the usual matrix product of m, α, a with $m \in M$, $a \in S$ and $\alpha \in \Gamma$. Here the right unity of S is $\sum\limits_{i=1}^{3} [\gamma_i, f_i]$ where

$$\gamma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \gamma_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \gamma_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$f_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}, \ f_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \quad \text{and} \quad f_3 = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A nonempty subset N of a ΓS-semimodule M is said to be a ΓS-subsemimodule of M if i) $a+b \in N$, ii) $a\alpha s \in N$ for all $a, b \in N$, for all $s \in S$ and for all $\alpha \in \Gamma$. N contains the zero of M.

A ΓS-subsemimodule N of a ΓS-semimodule M is said to be a *$k\Gamma S$-subsemimodule* of M if $a+b$, $b \in N$, $a \in M$ imply that $a \in N$. Let N be a ΓS-subsemimodule of a ΓS-semimodule M. Then *k-closure* of N, denoted by $\bar{N}$, is defined by $\bar{N} = \{a \in M : a+b = c \text{ for some } b, c \in N\}$. A ΓS-subsemimodule N of a ΓS-semimodule M is said to be an *$h\Gamma S$-subsemimodule* of M if $x+n_1+z = n_2+z$, $n_1, n_2 \in N$, $x, z \in M$ imply that $x \in N$. Let N be a ΓS-subsemimodule of a ΓS-semimodule M. Then *h-closure* of N, denoted by $\hat{N}$, is defined by $\hat{N} = \{a \in M : a+n_1+z = n_2+z \text{ for some } n_1, n_2 \in N \text{ and for some } z \in M\}$.

Proposition 3.3. *Let N be a ΓS-subsemimodule of a ΓS-semimodule M. Then*

N is a $k\Gamma S$*-(*$h\Gamma S$*-)subsemimodule if and only if* $\bar{N} = N$ $(\hat{N} = N)$.

Proof. The proof is a matter of routine verification. □

A ΓS-semimodule M is said to be *cancellative* if $a + b = a + c$, $a, b, c \in M$ implies that $b = c$.

Throughout the rest of the paper a ΓS-semimodule is assumed to be cancellative.

Definition 3.4. *A* ΓS*-semimodule* $M \neq \{0\}$ *is said to be irreducible if for any arbitrary fixed pair* $u, v \in M$ *with* $u \neq v$ *and for any* $x \in M$ *there exist* $x_i, y_j \in S$, $\alpha_i, \beta_j \in \Gamma$ *(*$i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$, m, n *are positive integers) such that* $x + \sum_i u\alpha_i x_i + \sum_j v\beta_j y_j = \sum_j u\beta_j y_j + \sum_i v\alpha_i x_i$. *A* ΓS*-semimodule* M *is said to be semi-irreducible if* $M\Gamma S \neq \{0\}$ *and* M *does not have any* $k\Gamma S$*-subsemimodule other than 0 and* M.

The notions of both irreduciblity and semi-irreducibility coincide with the notion of irreducibility in a Γ-ring ([7], [8], [9]) S or in a ring R when R or S is treated as a Γ-semiring, where $\Gamma = R$ in case of R.

Proposition 3.5. *Let* P *be an ideal of a* Γ*-semiring* S *and* M *be a* ΓS*-semimodule with* $M\Gamma P \neq \{0\}$. *Then the following statements are true.*

(1) If M *is semi-irreducible and* m *is an element of* M *then* $m = 0$ *if and only if* $m\alpha p = 0$ *for all* $\alpha \in \Gamma$ *and for all* $p \in P$ *i.e.* $m = 0$ *if and only if* $m\Gamma P = \{0\}$.

(2) If M *is irreducible and u,v are elements of* M *then* $u = v$ *if and only if* $\sum_{i=1}^{m} u\alpha_i x_i = \sum_{i=1}^{m} v\alpha_i x_i$, *for all* $\alpha_i \in \Gamma$, *for all* $x_i \in S$, $i = 1, 2, \ldots, p$; p *is any positive integer.*

Proof. (1) Let M be a semi-irreducible ΓS-semimodule and $m\alpha p = 0$ for all $p \in P$ and for all $\alpha \in \Gamma$. Let $M_0 = \{y \in M \ : \ y\Gamma P = \{0\}\}$. Then $m \in M_0$. Let $x, y \in M_0$. Then $(x + y)\Gamma P \subseteq x\Gamma P + y\Gamma P = \{0\}$. Thus $x + y \in M_0$. Let $\alpha \in \Gamma$ and $p \in P$. Then $(x\alpha p)\Gamma P = 0\Gamma P = \{0\}$. So $x\alpha p \in M_0$. Thus M_0 is a ΓS-subsemimodule of M. Let $x+y$, $y \in M_0$ and $x \in M$. Then $(x+y)\alpha p = 0$ and $y\alpha p = 0$ for all $\alpha \in \Gamma$ and for all $p \in P$. This implies that $x\alpha p = x\alpha p + y\alpha p = (x + y)\alpha p = 0$ for all $\alpha \in \Gamma$ and for all $p \in P$ whence $x\Gamma P = \{0\}$. Hence $x \in M_0$ proving that M_0 is a $k\Gamma S$-subsemimodule of M. Since $M\Gamma P \neq \{0\}$, $M_0 \neq M$. Since S is semi-irreducible so $M_0 = \{0\}$. So $m = 0$. Conversely, if $m = 0$ then $m\alpha p = 0$ for all $\alpha \in \Gamma$ and for all $p \in P$.
(2) Let M be irreducible and $u, v \in M$ be such that $u \neq v$. Since $M\Gamma P \neq \{0\}$ so there exist $m \in M$, $\alpha \in \Gamma$, $p \in P$ such that $m\alpha p \neq \{0\}$. For this $m \in M$, there exist $x_i, y_j \in S$, $\alpha_i, \beta_j \in \Gamma$ $(1 \leq i \leq p,\ 1 \leq j \leq q$; p, q are

positive integers) such that $m+\sum_{i=1}^{p}u\alpha_i x_i+\sum_{j=1}^{q}v\beta_j y_j=\sum_{j=1}^{q}u\beta_j y_j+\sum_{i=1}^{p}v\alpha_i x_i$. Hence $m\alpha p+\sum_{i=1}^{p}u\alpha_i x_i\alpha p+\sum_{j=1}^{q}v\beta_j y_j\alpha p=\sum_{j=1}^{q}u\beta_j y_j\alpha p+\sum_{i=1}^{p}v\alpha_i x_i\alpha p$ i.e., $m\alpha p+\sum_{i=1}^{p}u\alpha_i x_i'+\sum_{j=1}^{q}v\beta_j y_j'=\sum_{j=1}^{q}u\beta_j y_j'+\sum_{i=1}^{p}v\alpha_i x_i'$ where $x_i'=x_i\alpha p$ and $y_j'=y_j\alpha p$ for all $i=1,2,\ldots,p$ and $j=1,2,\ldots,q$. Since M is cancellative and $m\alpha p\neq 0$, so at least one of $\sum_{i=1}^{p}u\alpha_i x_i'\neq\sum_{i=1}^{p}v\alpha_i x_i'$ and $\sum_{j=1}^{q}u\beta_j y_j'\neq\sum_{j=1}^{q}v\beta_j y_j'$ holds. Converse follows easily. □

Proposition 3.6. *Let M be a ΓS-semimodule and $M\neq\{0\}$. Then M is semi-irreducible if and only if for every non-zero $m\in M$ $\overline{m\Gamma S}=M$ i.e. for any $x\in M$ there exist $x_i,y_j\in S$, $\alpha_i,\beta_j\in\Gamma$ $(i=1,2,\ldots,p$ and $j=1,2,\ldots,q$, p,q are positive integers) such that $x+\sum_i m\alpha_i x_i=\sum_j m\beta_j y_j$.*

Proof. Let $M\neq 0$ be semi-irreducible. Then $M\Gamma S\neq\{0\}$. Let $m\in M$ such that $m\neq 0$. Hence by Proposition 3.5, $m\Gamma S\neq\{0\}$; so $\overline{m\Gamma S}\neq\{0\}$. Since $\overline{m\Gamma S}$ is a $k\Gamma S$-subsemimodule of M, $\overline{m\Gamma S}=M$. Hence for any $x\in M$ there exist $x_i,y_j\in S$, $\alpha_i,\beta_j\in\Gamma$ $(i=1,2,\ldots,p$ and $j=1,2,\ldots,q$; p,q are positive integers) such that $x+\sum_i m\alpha_i x_i=\sum_j m\beta_j y_j$.

Conversely, suppose for any nonzero $m\in M$, $\overline{m\Gamma S}=M$. Let $N\neq\{0\}$ be a $k\Gamma S$-subsemimodule of M. Then there exists $n\in N$ such that $n\neq 0$. So, by the given condition $\overline{m\Gamma S}=M$. Hence for any $x\in M$ there exist $x_i,y_j\in S$, $\alpha_i,\beta_j\in\Gamma$ $(i=1,2,\ldots,p$ and $j=1,2,\ldots,q$; p,q are positive integers) such that $x+\sum_i n\alpha_i x_i=\sum_j n\beta_j y_j$. Since N is $k\Gamma S$-subsemimodule of M and $\sum_i n\alpha_i x_i$, $\sum_j n\beta_j y_j\in N$, $x\in N$. Hence $N=M$. Now if $M\Gamma S=\{0\}$ then $m\Gamma S=\{0\}$ for all $m\in M$. In particular, $m\Gamma S=\{0\}$ for any nonzero $m\in M$. Hence $\overline{m\Gamma S}=\{0\}$ for any nonzero $m\in M$. This implies that $M=0$- a contradiction. Hence M is semi-irreducible. □

Corollary 3.7. *If a ΓS-semimodule M is irreducible, then it is semi-irreducible and $\overline{m\Gamma S}=M$.*

Proof. Let M be an irreducible ΓS-semimodule. Then $M\neq\{0\}$. So, there exists $m(\neq 0)\in M$. Thus for any $x\in M$ there exist $x_i,y_j\in S$, $\alpha_i,\beta_j\in\Gamma$ $(i=1,2,\ldots,p$ and $j=1,2,\ldots,q$; p,q are positive integers) such that

$x+\sum_i m\alpha_i x_i=\sum_j m\beta_j y_j$. Hence by Proposition 3.6, M is a semi-irreducible ΓS-semimodule. Then $M\Gamma S \neq \{0\}$ which implies that $\overline{m\Gamma S}=\{0\}$. Since $\overline{m\Gamma S}$ is a $k\Gamma S$-subsemimodule of M, $\overline{m\Gamma S}=M$. □

Proposition 3.8. *Let S be a Γ-semiring and R be its right operator semiring. Then M is an irreducible ΓS-semimodule if and only if M is an irreducible R-semimodule.*

Proof. Let M be an irreducible ΓS-semimodule. Now we define R-action on M as follows: for $a \in M$, $\sum_i[\alpha_i,x_i]\in R$, $a\sum_i[\alpha_i,x_i]=\sum_i a\alpha_i x_i$. If $\sum_i[\alpha_i,x_i]=\sum_j[\beta_j,y_j]$ in R then $\sum_i s\alpha_i x_i=\sum_i s\beta_j y_j$ for all $s\in S$. Since M is an irreducible ΓS-semimodule, $\overline{m\Gamma S}=M$. (Corollary 3.7). Then for $m\in M$, $m+\sum_k a_k\gamma_k s_k=\sum_t b_t\delta_t v_t$ where $a_k,b_t\in M$, $\gamma_k,\delta_t\in\Gamma$, $s_k,v_t\in S$ ($k=1,2,\dots,p$; $t=1,2,\dots,q$; p,q are positive integers). So, $\sum_i m\alpha_i x_i+\sum_{k,i}a_k\gamma_k s_k x_i\alpha_i$ $=\sum_{t,i}b_t\delta_t x_i\alpha_i$, implying that

$$\sum_i m\alpha_i x_i+\sum_{k,j}a_k\gamma_k s_k\beta_j y_j=\sum_{i,j}b_t\delta_t v_t\beta_j y_j. \tag{1}$$

Again

$$\sum_j m\beta_j y_j+\sum_{k,j}a_k\gamma_k s_k\beta_j y_j=\sum_{t,j}b_t\delta_t v_t\beta_j y_j \tag{2}$$

Since M is cancellative so we have from (1) and (2) $\sum_i m\alpha_i x_i=\sum_j m\beta_j y_j$. Thus the R-action defined above on M is well defined. Now it can be easily verified that M with the above action is an R-semimodule. Next, let $u,v\in M$ with $u\neq v$. Then for any $x\in M$ there exist $x_i,y_j\in S$, $\alpha_i,\beta_j\in\Gamma$ such that $x+\sum_i u\alpha_i x_i+\sum_j v\beta_j y_j=\sum_j u\beta_j y_j+\sum_i v\alpha_i x_i$ (using irreducibility of M as a ΓS-semimodule). This implies that $x+u\sum_i[\alpha_i,x_i]+v\sum_j[\beta_j,y_j]+v\sum_i[\alpha_i,x_i]$ where $\sum_i[\alpha_i,x_i]$, $\sum_j[\beta_j,y_j]\in R$. Hence M is an irreducible R-semimodule ([6]). Conversely, suppose M is an irreducible R-semimodule. We define Γ-action of S on M as follows: for $a\in M$, $\alpha\in\Gamma$ and $s\in S$, $a\alpha S=a[\alpha,s]$. Then, with this composition M is a ΓS-semimodule. Let $u,v\in M$ with $u\neq v$ and let $x\in M$.

Then there exist $\sum_i[\alpha_i, x_i],\ \sum_j[\beta_j, y_j] \in R$ such that

$$x + u\sum_i[\alpha_i, x_i] + v\sum_j[\beta_j, y_j] = u\sum_j[\beta_j, y_j] + v\sum_i[\alpha_i, x_i].$$

So $x + \sum_i u\alpha_i x_i + \sum_j v\beta_j y_j = \sum_j u\beta_j y_j + \sum_i v\alpha_i x_i$. Hence by definition M is an irreducible ΓS-semimodule. This completes the proof. □

Let S be a Γ-semiring. The *zeroid* of S, denoted by $Z(S)$, is defined as $Z(S) = \{x \in S\ :\ x + z = z$ for some $z \in S\}$. Clearly, 0 is a member of $Z(S)$ of a Γ-semiring S with zero element 0. The zeroid $Z(S)$ of a Γ-semiring S is an h-ideal of S. Let M be a ΓS-semimodule. We put $(0 : M) = \{x \in S\ :\ M\Gamma x = \{0\}\}$ where $M\Gamma x = \{\sum_{i=1}^{k} m_i\alpha_i x\ :\ m_i \in M, \alpha \in \Gamma,\ k$ is a positive integer$\}$. We call $(0 : M)$ the *annihilator* of M in S. We also denote it by $A_S(M)$. A ΓS-semimodule M is said to be *faithful* if $Z(S) = A_S(M)$.

Proposition 3.9. *Let M be a ΓS-semimodule. Then $A_S(M)$ is an h-ideal of S. Moreover, M is a faithful $\Gamma(S/A_S(M))$-semimodule.*

Proof. Clearly $A_S(M)$ is an additive subsemigroup of S. Now let $x \in A_S(M)$, $\alpha \in \Gamma$, $s \in S$. Then $M\Gamma(x\alpha S) = (M\Gamma x)\alpha S = \{0\}$. Hence, $x\alpha S \in A_S(M)$ proving that it is a right ideal of S. To prove that $A_S(M)$ is also a left ideal of S we see that $M\Gamma(S\Gamma A_S(M)) = (M\Gamma S)\Gamma A_S(M) \subseteq M\Gamma A_S(M) = \{0\}$ which means $S\Gamma A_S(M) \subseteq A_S(M)$. Thus $A_S(M)$ is a two-sided ideal of S. Next, let $x + a + z = b + z$ where $x, z \in S$, $a, b \in A_S(M)$. Then for all $\alpha \in \Gamma$, for all $m \in M$, $m\alpha a = 0$ and so $m\alpha x + m\alpha z = m\alpha b + m\alpha z$. Since M is cancellative we have $m\alpha x = m\alpha b = 0$ for all $m \in M$, for all $\alpha \in \Gamma$. Hence $x \in A_S(M)$. Thus $A_S(M)$ is an h-ideal of S. Now let us define a Γ-action of $S/A_S(M)$ on M as follows: $m\alpha(s/A_S(M)) = m\alpha S$ for $m \in M$, $\alpha \in \Gamma$, $s/A_S(M) \in S/A_S(M)$. If $s/A_s(M) = t/A_s(M)$ then $s + p_1 = t + p_2$ for some $p_1, p_2 \in A_S(M)$. Then $m\alpha S + m\alpha p_1 = m\alpha t + m\alpha p_2$ for all $m \in M$, for all $\alpha \in \Gamma$ i.e. $m\alpha s = m\alpha t$ for all $m \in M$, for all $\alpha \in \Gamma$. Hence the Γ-action of $S/A_S(M)$ on M is well-defined. Now it is easy to see that M is a $\Gamma(S/A_S(M))$-semimodule. It remains to show that $A_{S/A_S(M)}(M) = Z(S/A_S(M))$. Clearly $Z(S/A_S(M) \subseteq A_{S/A_S(M)}$. Now let $x/A_S(M) \in A_{S/A_S(M)}(M)$. Then $m\alpha(x/A_S(M)) = 0$ for all $m \in M$, for all $\alpha \in \Gamma$ i.e. $m\alpha x = 0$ for all $m \in M$, for all $\alpha \in \Gamma$. Hence $x \in A_S(M)$. This implies that $x/A_S(M) = 0/A_S(M)$. Hence $x/A_S(M) \in Z(S/A_S(M))$. Thus $A_{S/A_S(M))}(M) \subseteq Z(S/A_S(M))$. Hence $A_{S/A_S(M)} = Z(S/A_S(M))$ (whence M is a faithfuul $\Gamma(S/A_S(M))$-semimodule. □

Proposition 3.10. *Let S be a Γ-semiring and R be its right operator semiring. Then*

(i) $A_S(M)^{*'} = A_R(M)$ *and* $A_R(M)^* = A_S(M)$; *where* M *is an irreducible* ΓS*-semimodule (and hence an irreducible* R*-semimodule)*

(ii) $Z(S)^{*'} = Z(R)$ *and* $Z(R)^* = Z(S)$.

Proof. (i)

$$\begin{aligned} A_S(M)^{*'} &= \{\sum_i [\alpha_i, x_i] \in R \ : \ S(\sum_i [\alpha_i, x_i]) \subseteq A_S(M)\} \\ &= \{\sum_i [\alpha_i, x_i] \in R \ : \ M\Gamma S(\sum_i [\alpha_i, x_i]) = \{0\}\} \\ &= \{\sum_i [\alpha_i, x_i] \in R \ : \ M(\sum_i [\alpha_i, x_i]) = \{0\}\} \\ &= A_R(M). \end{aligned}$$

$$\begin{aligned} A_R(M)^* &= \{x \in S \ : \ [\Gamma, x] \subseteq A_R(m)\} \\ &= \{x \in S \ : \ M[\Gamma, x] = \{0\}\} \\ &= \{x \in S \ : \ M\Gamma x = \{0\}\} \\ &= A_S(M). \end{aligned}$$

(ii) By Propositions 6.14 ([1]) and since zeroid is an h-ideal, $(Z(S)^*)^* = Z(S)$ and $(Z(R)^*)^{*'} = Z(R)$. So it is sufficient to prove one of the two relations. Let $x \in Z(R)^*$. Then $[\Gamma, x] \subseteq Z(R)$. So $S\Gamma x \subseteq SZ(R) \subseteq Z(S)$. Since S has the left unity, $x \in Z(S)$. Thus $Z(R)^* \subseteq Z(S)$. Now let $\sum_{i=1}^m [\alpha_i, x_i] \in [\Gamma, Z(S)]$ where $x_i \in Z(S)$ for all $i = 1, 2, 3, \ldots, m$. Then $x_i + z_i = z_i$ for some $z_i \in S$ for all $i = 1, 2, \ldots, m$. Then $[\alpha_i, x_i] + [\alpha_i, z_i] = [\alpha_i, z_i]$ for all $i = 1, 2, 3, \ldots, m$. This implies that $\sum_{i=1}^{m} [\alpha_i, x_i] + \sum_{i=1}^{m} [\alpha_i, z_i] = \sum_{i=1}^{m} [\alpha_i, z_i]$ where $\sum_{i=1}^{m} [\alpha_i, z_i] \in R$. Hence $\sum_{i=1}^{m} [\alpha_i, x_i] \in Z(R)$ and so $[\Gamma, Z(S)] \subseteq Z(R)$. Thus $Z(S) \subseteq Z(R)^*$. Hence $Z(R)^* = Z(S)$. □

Proposition 3.11. *Let* S *be a* Γ*-semiring and* R *be its right operator semiring. Then* M *is a faithful irreducible* ΓS*-semimodule if and only if* M *is a faithful irreducible* R*-semimodule.*

Proof. Let M be a faithful irreducible ΓS-semimodule. Then by Proposition 3.8, M is an irreducible ΓS-semimodule. Again, $A_S(M) = Z(S)$. So $A_S(M)^{*'} = Z(S)^{*'}$. This implies by Proposition 3.10, $A_R(M) = Z(R)$. Hence M is a faithful irreducible ΓS-semimodule. Converse follows by reversing the above argument. □

Definitions 3.12. *A* Γ*-semiring* S *is said to be primitive if it has a faithful irreducible* ΓS*-semimodule.*

An ideal P of S is said to be primitive if the Bourne factor Γ-semiring S/P

is primitive. Hence a Γ-semiring S is primitive if $\{0\}$ is a primitive ideal of S.

Lemma 3.13. *Let S be a Γ-semiring and R be its right operator semiring and Q be a proper ideal of S. Then $R(S/Q)$ and $R/Q^{*'}$ are isomorphic, where $R(S/Q)$ is the right operator semiring of the Bourne factor Γ-semiring S/Q.*

Proof. We define a mapping $\phi : R(S/Q) \to R/Q^{*'}$ as follows: $\phi(\sum_{i=1}^{m}[\alpha_i, x_i/Q]) = \sum_{i=1}^{m}[\alpha_i, x_i]/Q^{*'}$. Now let $\sum_{i=1}^{m}[\alpha_i, x_i/Q] = \sum_{j=1}^{n}[\beta_j, y_j/Q]$ in $R(S/Q)$. Then $\sum_{i=1}^{m}(s/Q)\alpha_i(x_i/Q) = \sum_{j=1}^{n}(s/Q)\beta_j(y_j/Q)$ for all $s/Q \in S/Q$ i.e. $\sum_{i=1}^{m}(s\alpha_i x_i)/Q = \sum_{j=1}^{n}(s\beta_j y_j)/Q$ for all $s \in S$, which means that $\sum_{i=1}^{m} s\alpha_i x_i + q = \sum_{j=1}^{n} s\beta_j y_j + q'$ for some $q, q' \in Q$ and for all $s \in S$. This implies that $\sum_{i=1}^{m} f_k\alpha_i x_i + a_k = \sum_{j=1}^{n} f_k\beta_j y_j$ for some $a_k, b_k \in Q$, for all k, $1 \leq k \leq p$, where $\sum_{k=1}^{p}[\gamma_k, f_k]$ is the right unity of S. This implies that $\sum_{k,i} s\gamma_k f_k\alpha_i x_i + \sum_{k} s\gamma_k a_k = \sum_{k,j} s\gamma_k f_k\beta_j y_j + \sum_{k} s\gamma_k b_k$ for all $s \in S$ and for all $a_k, b_k \in Q, 1 \leq k \leq p$. This implies that $(\sum_{k=1}^{p}[\gamma_k, f_k])(\sum_{i=1}^{m}[[\alpha_i, x_i]) + \sum_{k=1}^{p}[\gamma_k, a_k] = (\sum_{k=1}^{p}[\gamma_k, f_k])(\sum_{j=1}^{n}[\beta_j, y_j]) + \sum_{k=1}^{p}[\gamma_k, b_k]$, where $\sum_{k=1}^{p}[\gamma_k, a_k], \sum_{k=1}^{p}[\gamma_k, b_k] \in Q^{*'}$ (Proposition 3.5 [3]) i.e., $\sum_{i=1}^{m}[\alpha_i, x_i] + \sum_{k=1}^{p}[\gamma_k, a_k] = \sum_{j=1}^{n}[\beta_j, y_j] + \sum_{k=1}^{p}[\gamma_k, b_k]$, where $\sum_{k=1}^{p}[\gamma_k, a_k], \sum_{k=1}^{p}[\gamma_k, b_k] \in Q^{*'}$. This implies that $\sum_{i=1}^{m}[[\alpha_i, x_i]]/Q^{*'} = \sum_{j=1}^{n}[\beta_j, y_j]/Q^{*'}$ i.e. $\phi(\sum_{i=1}^{m}[\alpha_i, x_i/Q]) = \phi(\sum_{j=1}^{n}[\beta_j, y_j/Q])$. Thus ϕ is well-defined.

Clearly, ϕ is surjective. Next, let $\phi(\sum_{i=1}^{m}[\alpha_i, x_i/Q]) = \phi(\sum_{j=1}^{n}[\beta_j, y_j/Q])$. Then $\sum_{i=1}^{m}[\alpha_i, x_i]/Q^* = \sum_{j=1}^{n}[\beta_j, y_j]/Q^*$. So $\sum_{i=1}^{m}[\alpha_i, x_i] + \sum_{k=1}^{p}[\gamma_k, a_k] = \sum_{j=1}^{n}[\beta_j, y_j] + \sum_{k=1}^{p}[\gamma_k, b_k]$, where $\sum_{k=1}^{p}[\gamma_k, a_k], \sum_{k=1}^{p} [\gamma_k, b_k] \in Q^{*'}$ (Proposition 3.5 [3]). This implies that $\sum_{i=1}^{m} s\alpha_i x_i + \sum_k s\gamma_k a_k = \sum_{j=1}^{n} s\beta_j y_j + \sum_k s\gamma_k b_k$ for all $s \in S$, where $\sum_k s\gamma_k a_k, \sum_k s\gamma_k b_k \in Q$ for all $s \in S$. This implies that $\sum_{i=1}^{m} s\alpha_i x_i/Q = \sum_{j=1}^{n} s\beta_j y_j/Q$ for all $s \in S$ i.e. $\sum_{i=1}^{m}(s/Q)\alpha_i(x_i/Q) = \sum_{j=1}^{n}(s/Q)\beta_j(y_j/Q)$ for all $s/Q \in S/Q$. This implies that $\sum_{i=1}^{m}[\alpha_i,(x_i/Q)] = \sum_{j=1}^{n}[\beta_j,(y_j/Q)]$. Hence ϕ is injective. Clearly, ϕ is a semiring homomorphism. Therefore ϕ is a semiring isomorphism, whence $R(S/Q)$ and R/Q^* are isomorphic. □

Proposition 3.14. *Let S be a Γ-semiring and R be its right operator semiring. If P is a primitive ideal of S then $P^{*'}$ is a primitive ideal of R.*

Proof. Let P be a primitive ideal of S. Then S/P is a primitive Γ-semiring. So there exists an irreducible faithful $\Gamma(S/P)$-semimodule M. Then by Proposition 3.11, M is a faithful irreducible $R(S/P)$-semimodule where $R(S/P)$ is the right operator semiring of S/P. Since $R(S/P)$ and $R/P^{*'}$ are isomorphic (Lemma 3.13), M is a faithful irreducible $R/P^{*'}$-semimodule. Consequently, $R/P^{*'}$ is a primitive semiring ([6]), i.e. $P^{*'}$ is a primitive ideal of R. □

Proposition 3.15. *Let S be a Γ-semiring and R be its right operator semiring. If Q is a primitive ideal of R then $Q^{*'}$ is a primitive ideal of S.*

Proof. Suppose that Q is a primitive ideal of R. Then R/Q is a primitive semiring. So, there exists a faithful irreducible R/Q-semimodule M. Then by Proposition 3.11, M is a faithful irreducible $\Gamma(S/Q^*)$-semimodule (noting the fact that $R(S/Q^*)$ and $R/(Q^*)^{*'}$, i.e. R/Q are isomorphic). So, S/Q^* is a primitive Γ-semiring, whence Q^* is a primitive ideal of the Γ-semiring S. □

From the above two propositions and Theorem 6.6 ([1]) the following theorem follows easily:

Theorem 3.16. *Let S be a Γ-semiring and R be its right operator semiring. Then there exists an inclusion preserving bijection between the set of all primitive ideals of S and the set of all primitive ideals of R via the mapping $P \to P^{*'}$, where P is an ideal of S.*

Theorem 3.17. *A Γ-semiring S is primitive if and only if its right operator semiring R is primitive.*

Proof. Let S be a primitive Γ-semiring. Then there is a faithful irreducible ΓS-semimodule M(say). Then, by Proposition 3.11, M is a faithful irreducible R-semimodule. So, R is a primitive semiring ([6]). Converse follows by reversing the above argument. □

Lastly, we have the following characterization of primitive h-ideal of a Γ-semiring which is analogous to that of a primitive ideal of a ring.

Theorem 3.18. *An h-ideal P of a Γ-semiring S is primitive if and only if $P = A_S(M)$ for some irreducible ΓS-semimodule M.*

Proof. Let the h-ideal P of the Γ-semiring S be primitive. Then by Proposition 6.11 ([1]) and Proposition 3.14, $P^{*'}$ is a primitive h-ideal of R. Hence $P^{*'} = A_R(M)$ ([6]), where M is an irreducible R-semimodule (Proposition 3.8). Then $(P^{*'})^* = A_R(M)^*$, which implies that $P = A_S(M)$ (Proposition 3.10). Converse follows by reversing the above argument. □

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References

[1] Dutta, T. K., Sardar, S. K., On the Operator semirings of a Γ-semiring. Southeast Asian Bull. Math., Springer Verlag, 26(2002), 203–213.

[2] Dutta, T. K., Sardar, S. K., Semiprime ideals and irreducible ideals of Γ-semirings. Novi Sad J. Math. 30(1)(2000), 97–108.

[3] Dutta, T. K., Sardar, S. K., Study of Noetherian Γ-semirings via operator semirings of a Γ-semiring. Southeast Asian Bull.Math., Springer Verlag 25(2002), 599–608.

[4] Golan, J. S., The theory of semirings with applications in Mathematics and theoretical computer science. Pitman Monographs and Surveys in Pure and Applied Mathematics, 54, Longman Sci. Tech. Harlow, 1992.

[5] Hebisch, U., Weinert, H. J., Semiring Theory, Algebraic Theory and Applications in Computer Science. World Scientific, 1998.

[6] Izuka, K., On the Jacobson radical of a semiring. Tohoku Math. J. 11(2)(1959), 409–421.

[7] Luh, J., On primitive Γ-rings with minimal one-sided ideals. Osaka J. Math. 5(1968), 165–173.

[8] Luh, J., On the theory of simple Γ-rings. Michigan Math. J. 16(1969), 65–75.

[9] Kyuno, S., On a Nobusawa's Gamma Ring $\begin{bmatrix} R & \Gamma \\ M & L \end{bmatrix}$. Math. Japonica 26(1981), 55–64.

[10] Rao, M.M.K., Γ-semirings-1. Southeast Asian Bull. Math. 19(1995), 49–54.

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