Novi Sad J. Math.
Vol. 33, No. 2, 2003, 43–48

ON Φ–RECURRENT SASAKIAN MANIFOLDS

U. C. De[1], A. A. Shaikh[2], Sudipta Biswas[1]

Abstract. The objective of the present paper is to study ϕ–recurrent Sasakian manifolds.

AMS Mathematics Subject Classification (2000): 53C05, 53C20, 53C25

Key words and phrases: ϕ–recurrent Sasakian manifold, Einstein manifold, sectional curvature

1. Introduction

In 1977, T. Takahashi [2] introduced the notion of locally ϕ–symmetric Sasakian manifolds and studied their several interesting properties. In this paper we introduce the notion of ϕ–recurrent Sasakian manifolds which generalizes the notion of locally ϕ–symmetric Sasakian manifolds. After preliminaries, in Section 3, we study ϕ–recurrent Sasakian manifolds and show that such a manifold is always an Einstein manifold. Again, it is proved that in a ϕ–recurrent Sasakian manifold, the characteristic vector field ξ and the vector field ρ associated to the 1–form A are codirectional. Also we obtain some other interesting results of this manifold.

2. Preliminaries

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Sasakian manifold with the structure (ϕ,ξ,η,g). Then the following relations hold [1]:

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$a)\ \ \eta(\xi) = 1, \quad b)\ \ g(X,\xi) = \eta(X), \quad c)\ \ \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$R(\xi,X)Y = (\nabla_X\phi)(Y) = g(X,Y)\xi - \eta(Y)X, \tag{2.4}$$

$$a)\ \nabla_X\xi = -\phi X, \quad b)\ \ (\nabla_X\eta)(Y) = g(X,\phi Y), \tag{2.5}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.7}$$

[1]Department of Mathematics, University of Kalyani, Kalyani–741235, Nadia, West Bengal, India, e–mail: ucde@klyuniv.ernet.in (U. C. De)

[2]Department of Mathematics, University of North Bengal, P. O. NBU–734430, Darjeeling, West Bengal, India, e–mail: aask2003@yahoo.co.in, aask@epatra.com

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.8}$$

$$S(X,\xi) = 2n\ \ \eta(X), \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n\, \eta(X)\eta(Y), \tag{2.10}$$

for all vector fields X, Y, Z, where ∇ denotes the operator of covariant differentiation with respect to g, ϕ is a skew–symmetric tensor field of type (1,1), S is the Ricci tensor of type (0,2) and R is the Riemannian curvature tensor of the manifold.

Definition 2.1. [2] *A Sasakian manifold is said to be a locally ϕ–symmetric manifold if*

$$\phi^2\left((\nabla_W R)(X,Y)Z\right) = 0 \tag{2.11}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.2. *A Sasakian manifold is said to be a ϕ–recurrent manifold if there exists a non–zero 1–form A such that*

$$\phi^2\left((\nabla_W R)(X,Y)Z\right) = A(W)R(X,Y)Z \tag{2.12}$$

for arbitrary vector fields X, Y, Z, W.

If the 1–form A vanishes, then the manifold reduces to a ϕ–symmetric manifold.

3. ϕ–recurrent Sasakian manifolds

Let us consider a ϕ–recurrent Sasakian manifold. Then by virtue of (2.1) and (2.12) we have

$$-(\nabla_W R)(X,Y)Z + \eta\left((\nabla_W R)(X,Y)Z\right)\xi = A(W)R(X,Y)Z, \tag{3.1}$$

from which it follows that

$$\begin{aligned} -g\left((\nabla_W R)(X,Y)Z, U\right) + \eta\left((\nabla_W R)(X,Y)Z\right)\eta(U) \\ = A(W)g(R(X,Y)Z,U). \end{aligned} \tag{3.2}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, 2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.2) and taking summation over i, $1 \le i \le 2n+1$, we get

$$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{n} \eta\left((\nabla_W R)(e_i,Y)Z\right)\eta(e_i) = A(W)S(Y,Z). \tag{3.3}$$

The second term of (3.3) by putting $Z=\xi$ takes the form

$$g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi)$$

which is denoted by E. In this case E vanishes. Namely we have

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ &\quad - g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi) \end{aligned}$$

at $p\in M$. Since $\{e_i\}$ is an orthonormal basis, $\nabla_X e_i = 0$ at p. Using (2.2) and (2.4) we obtain

$$g(R(e_i,\nabla_W Y)\xi,\xi) = g(\nabla_W Y,\xi)g(e_i,\xi) - g(\xi,e_i)g(\nabla_W Y,\xi) = 0.$$

Thus we obtain

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W\xi,\xi).$$

In virtue of $g(R(e_i,Y)\xi,\xi) = g(R(\xi,\xi)Y,e_i) = 0$, we have

$$g(\nabla_W(R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W\xi) = 0,$$

which implies

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W\xi) - g(R(e_i,Y)\nabla_W\xi,\xi).$$

Using (2.5) and applying the skew–symmetry of R we get

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(R(\phi W,\xi)Y,e_i) + g(R(\xi,\phi W)Y,e_i).$$

Hence we reach

$$\begin{aligned} E &= \sum_{i=1}^{n}\{g(R(\phi W,\xi)Y,e_i)g(\xi,e_i) + g(R(\xi,\phi W)Y,e_i)g(\xi,e_i)\} \\ &= g(R(\phi W,\xi)Y,\xi) + g(R(\xi,\phi W)Y,\xi) = 0. \end{aligned}$$

Replacing Z by ξ in (3.3) and using (2.9) we have

$$-(\nabla_W S)(Y,\xi) = 2n\, A(W)\eta(Y). \tag{3.4}$$

Now we have $(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W\xi)$. Using (2.9) and (2.5) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = 2n\, g(W,\phi Y) + S(Y,\phi W). \tag{3.5}$$

In view of (3.4) and (3.5) we obtain

$$-[2n\, g(W,\phi Y) + S(Y,\phi W)] = 2n\, A(W)\eta(Y). \tag{3.6}$$

Replacing Y by ϕY in (3.6) and then using (2.1), (2.2) and (2.10) we get

$$S(Y,W) = 2n\, g(Y,W) \qquad \text{for all } \; Y,W. \tag{3.7}$$

This leads to the following:

Theorem 3.1. *A ϕ–recurrent Sasakian manifold (M^{2n+1}, g) is an Einstein manifold.*

Now from (3.1) we have

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z. \tag{3.8}$$

From (3.8) and the Bianchi identity we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0. \tag{3.9}$$

By virtue of (2.8) we obtain from (3.9)

$$\begin{aligned} &A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+ A(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] \\ &+ A(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)] = 0. \end{aligned} \tag{3.10}$$

Putting $Y = Z = e_i$ in (3.10) and taking summation over i, $1 \le i \le 2n+1$, we get

$$A(W)\eta(X) = A(X)\eta(W) \quad \text{for all vector fields } X, W. \tag{3.11}$$

Replacing X by ξ in (3.11), it follows that

$$A(W) = \eta(W)\eta(\rho) \quad \text{for any vector field } W, \tag{3.12}$$

where $A(\xi) = g(\xi,\rho) = \eta(\rho)$, ρ being the vector field associated to the 1–form A, i.e., $g(X,\rho) = A(X)$. From (3.11) and (3.12) we can state the following:

Theorem 3.2. *In a ϕ–recurrent Sasakian manifold (M^{2n+1}, g), $n > 1$, the characteristic vector field ξ and the vector field ρ associated to the 1–form A are co–directional and the 1–form A is given by (3.12).*

Next, in view of (2.5) and (2.6) it can be easily seen that in a Sasakian manifold the following relation holds:

$$(\nabla_W R)(X,Y)\xi = g(W,\phi Y)X - g(W,\phi X)Y + R(X,Y)\phi W. \tag{3.13}$$

By virtue of (2.8), it follows from (3.13) that

$$\eta((\nabla_W R)(X,Y)\xi) = 0. \tag{3.14}$$

Again from Tanno [3] we have

$$\begin{aligned} R(X,Y)\phi Z = {} & g(\phi X,Z)Y - g(Y,Z)\phi X - g(\phi Y,Z)X \\ & + g(X,Z)\phi Y + \phi R(X,Y)Z \end{aligned} \tag{3.15}$$

for any $X, Y, Z \in T_pM$. From (3.13) and (3.15), it follows that

$$(\nabla_W R)(X,Y)\xi = g(X,W)\phi Y - g(Y,W)\phi X + \phi R(X,Y)W. \tag{3.16}$$

In view of (3.16) and (3.14), we obtain from (3.1) that

$$g(X,W)\phi Y - g(Y,W)\phi X + \phi R(X,Y)W = -A(W)R(X,Y)\xi. \tag{3.17}$$

Using (2.6) and (3.12) in (3.17) we have

$$\begin{aligned} &g(X,W)\phi Y - g(Y,W)\phi X + \phi R(X,Y)W \\ &\qquad = -\eta(W)\eta(\rho)[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.18}$$

Hence if X and Y are orthogonal to ξ, then (3.18) reduces to

$$\phi R(X,Y)W = g(Y,W)\phi X - g(X,W)\phi Y. \tag{3.19}$$

Operating ϕ on both sides of (3.19) and using (2.1) we get

$$R(X,Y)W = g(Y,W)X - g(X,W)Y \qquad \text{for all} \ \ X, Y, W. \tag{3.20}$$

Hence we can state the following:

Theorem 3.3. *A ϕ–recurrent Sasakian manifold (M^{2n+1}, g), $n > 1$, is a space of constant curvature, provided that X and Y are orthogonal to ξ.*

We now suppose that a Sasakian manifold (M^{2n+1}, g), $n > 1$, is ϕ–recurrent. Then from (3.8) and (3.16), it follows that

$$\begin{aligned} (\nabla_W R)(X,Y)Z = \{&g(Y,W)g(\phi X, Z) - g(X,W)g(\phi Y, Z) \\ &- g(\phi R(X,Y)W, Z)\}\,\xi - A(W)R(X,Y)Z. \end{aligned} \tag{3.21}$$

This leads to the following:

Theorem 3.4. *If a Sasakian manifold (M^{2n+1}, g), $n > 1$, is ϕ–recurrent then the relation (3.21) holds.*

Let us now suppose that in a Sasakian manifold, the relation (3.21) holds. Then from (3.21) it follows that

$$\left\{ \begin{aligned} \phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z - A(W)\{&g(Y,Z)\eta(X) \\ &-g(X,Z)\eta(Y)\}\xi, \end{aligned} \right. \tag{3.22}$$

which yields

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z,$$

if X and Y are orthogonal to ξ. Hence we can state the following:

Theorem 3.5. *A Sasakian manifold (M^{2n+1}, g), $n > 1$, satisfying the relation (3.21) is ϕ–recurrent provided that X and Y are orthogonal to ξ.*

Next, we suppose that in a ϕ–recurrent Sasakian manifold, the sectional curvature of a plane $\pi \subset T_pM$ defined by

$$K_p(\pi) = g(R(X,Y)Y,X)$$

is a non-zero constant k, where $\{X,Y\}$ is any orthonormal basis of π. Then we have

$$g((\nabla_Z R)(X,Y)Y,X) = 0. \tag{3.23}$$

By virtue of (3.23) and (3.1) we obtain

$$g((\nabla_Z R)(X,Y)Y,\xi)\eta(X) = A(Z)g(R(X,Y)Y,X). \tag{3.24}$$

Since in a ϕ–recurrent Sasakian manifold, the relation (3.21) holds good, using (3.21) in (3.24) we get

$$\left\{ \begin{aligned} &\eta(X)[g(Y,Z)g(\phi X,Y) - g(X,Z)g(\phi Y,Y) - g(\phi R(X,Y)Z,Y)] \\ &\qquad -A(Z)[g(Y,Y)\eta(X) - g(X,Y)\eta(Y)] = kA(Z). \end{aligned} \right. \tag{3.25}$$

Putting $Z = \xi$ in (3.25) we obtain

$$\eta(\rho)[k + \{g(Y,Y)\eta(X) - g(X,Y)\eta(Y)\}] = 0,$$

which implies that

$$\eta(\rho) = 0.$$

Hence by (3.12) we obtain from (2.12) that

$$\phi^2((\nabla_W R)(X,Y)Z) = 0.$$

This leads to the following:

Theorem 3.6. *If a ϕ–recurrent Sasakian manifold (M^{2n+1}, g), $n > 1$, has a non–zero constant sectional curvature, then it reduces to a locally ϕ–symmetric manifold in the sense of Takahashi.*

References

[1] Blair, D.E., Contact manifolds in Riemannian geometry. Lecture Notes in Math. No. 509. Springer 1976.

[2] Takahashi, T., Sasakian ϕ–symmetric spaces. Tohoku Math. J. 29 (1977), 91–113.

[3] Tanno, S., Isometric Immersions of Sasakian manifold in spheres. Kodai Math. Sem. Rep. 21 (1969), 448–458.

Received by the editors September 25, 2003