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CONVOLUTION EQUATIONS IN COLOMBEAU'S SPACES

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Abstract. The modified Colombeau's space $\mathcal{G}_{\mathbf{t}}$ is used as the frame for solving convolution equations via Fourier transformation and division.

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1. Introduction

The basic space in this paper is $\mathcal{G}_{\mathbf{t}}$ which is introduced in [5]. The reason why we use $\mathcal{G}_{\mathbf{t}}$ and corresponding $\mathbf{t}$-notions instead of Colombeau's $\mathcal{G}_\tau$ and τ-notions is that τ-convolution is not associative and commutative in a general case while $\mathbf{t}$-convolution has both properties (in $(g.t.d.)$ and $(G.t.d)$ sense). Further on, in $\mathcal{G}_{\mathbf{t}}$ the exchange formula holds, and this is not the case in Colombeau's space $\mathcal{G}_\tau$.

Using exchange formula we obtain sufficient conditions for solvability of a convolution equation in the associated sense in $\mathcal{G}_{\mathbf{t}}$.

In this paper we use the idea of division in $\mathcal{G}$, which is given in [6], and the main result, Corrolary 1, of this paper is a generalization of Theorem 2 in [6].

2. Notation and Basic Notions

We shall recall some facts from [1]. $\mathcal{A}_q$, $q \in \mathbf{N}$ are subsets of $\mathcal{D}$ with the following properties:

$$\mathrm{diam}(\mathrm{supp}(\phi)) = 1, \int x^\alpha \phi(x)dx = 0, \text{ and } \int \phi(x)dx = 1,$$

for every $\phi \in \mathcal{A}_q$, $q \in \mathbf{N}$, $\alpha \in \mathbf{N}_0^n$, $1 \leq |\alpha| \leq q$. $\mathcal{A}_0$ is a set of all $\phi \in \mathcal{D}$ such that $\int \phi(x)dx = 1$. Put $\phi_\varepsilon(\cdot) = \varepsilon^{-n}\phi(\cdot/\varepsilon)$. Obviously, $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots$;

$\mathcal{E}$ is defined as a set of all functions $F_{\phi,\varepsilon} : \mathcal{A}_0 \times (0,1) \times \mathbf{R}^n \to \mathbf{C}$, which are smooth on $\mathbf{R}^n$.

$\mathbf{C}_M$ is the set of all $A_{\phi,\varepsilon} : \mathcal{A}_0 \times (0,1) \to \mathbf{C}$ such that there exists $N \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$|A_{\phi,\varepsilon}| < C\varepsilon^{-N}, \ \varepsilon < \eta. \tag{1}$$

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$\mathcal{E}_M$ is the set of all $G_{\phi,\varepsilon} \in \mathcal{E}$ such that for every compact set K and every $\beta \in \mathbf{N}_0^n$ there exists $N \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta G_{\phi,\varepsilon}(x)| < C\varepsilon^{-N},\ \varepsilon < \eta,\ x \in K. \tag{2}$$

Denote by Γ the family of all increasing sequences which tend to infinity.

$\mathbf{C}_0$ is the set of all $A \in \mathbf{C}_M$ such that there exist $g \in \Gamma$ and $N \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_q$, $q \geq N$, there exist $C > 0$ and $\eta > 0$ such that

$$|A_{\phi,\varepsilon}| < C\varepsilon^{g(q)-N},\ \varepsilon < \eta. \tag{3}$$

$\mathcal{N}$ is the set of all $G \in \mathcal{E}_M$ such that for every $\beta \in \mathbf{N}_0^n$ and every compact set K there exist $N \in \mathbf{N}_0$ and $g \in \Gamma$ such that for every $\phi \in \mathcal{A}_q$, $q \geq N$, there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta G_{\phi,\varepsilon}(x)| < C\varepsilon^{g(q)-N},\ \varepsilon < \eta,\ x \in K. \tag{4}$$

The spaces of Colombeau's generalized complex numbers and generalized functions are defined by $\overline{\mathbf{C}} = \mathbf{C}_M/\mathbf{C}_0$ and $\mathcal{G} = \mathcal{E}_M/\mathcal{N}$.

If $g \in \mathcal{D}'$, then by

$$G_{\phi,\varepsilon}(x) =< g(\xi), \varepsilon^{-n}\phi((\xi - x)/\varepsilon) >,\ x \in \mathbf{R}^n$$

is denoted the representative of the corresponding element in $\mathcal{E}_M$. Its class is called Colombeau's regularization of g and denoted by $\mathrm{Cd}(g)$.

The inclusions $\mathcal{E} \subset \mathcal{D}' \subset \mathcal{G}$ are valid.

$\mathcal{E}_\mathbf{t}$ is the set of all elements $G \in \mathcal{E}$ with the following property: For every $\beta \in \mathbf{N}_0^n$ there exist $N \in \mathbf{N}_0$ and $\gamma > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta G_{\phi,\varepsilon}(x)| < C(1 + |x|)^\gamma \varepsilon^{-N},\ \varepsilon < \eta,\ x \in \mathbf{R}^n. \tag{5}$$

$\mathcal{N}_\mathbf{t}$ is the set of elements $G \in \mathcal{E}_\mathbf{t}$ with the following property: For every $\beta \in \mathbf{N}_0^n$ there exist $\gamma > 0$, $N \in \mathbf{N}_0$ and $g \in \Gamma$ such that for every $\phi \in \mathcal{A}_q$, $q \geq N$, there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta G_{\phi,\varepsilon}(x)| < C(1 + |x|)^\gamma \varepsilon^{g(q)-N},\ \varepsilon < \eta,\ x \in \mathbf{R}^n. \tag{6}$$

It is an ideal of $\mathcal{E}_\mathbf{t}$. The Colombeau's space of tempered generalized functions is defined by $\mathcal{G}_\mathbf{t} = \mathcal{E}_\mathbf{t}/\mathcal{N}_\mathbf{t}$. In [1] this space is denoted by $\mathcal{G}_\tau$. In [5] we have considered a class of spaces $\mathcal{G}_\mathbf{a}$ such that $\mathcal{G}_\mathbf{t}$ is a special space of this class. From now on we shall use notation and notions from [5].

A net of functions μ_ε, $\varepsilon > 0$ from $\mathcal{D}$ is called a unit net related to $\mathbf{t}$ if it satisfies the following properties:

1. $0 \leq \mu_\varepsilon(x) \leq 1,\ x \in \mathbf{R}^n,\ \varepsilon > 0.$

2. For some $b > 0$ and $r > 0$,

$$\mu_\varepsilon(x) = 1, |x| < b/\varepsilon,\ \mu_\varepsilon(x) = 0, |x| > b/\varepsilon + r,\ \varepsilon > 0.$$

3. For every $l \in \mathbf{N}_0^n$ there exists $c_l > 0$ such that $|\partial^l \mu_\varepsilon(x)| \leq c_l,\ x \in \mathbf{R}^n,\ \varepsilon > 0$.

Let μ_ε be a unit net related to $\mathbf{t}$, B a measurable subset of $\mathbf{R}^n$ and $G \in \mathcal{G}_{\mathbf{t}}$. Then we define

$$\int_B^{\mathbf{t},\mu} \mathbf{G}(x)dx \in \overline{\mathbf{C}} \text{ by its representative } \int_B G_{\phi,\varepsilon}(x)\mu_\varepsilon(x)dx \in \mathbf{C}_M.$$

If $B = \mathbf{R}^n$ then the symbol $\int^{\mathbf{t},\mu}$ is used. In [5] is proved that $G_{\phi,\varepsilon} \in \mathcal{N}_{\mathbf{t}}$ implies $\int_B G_{\phi,\varepsilon}(x)\mu_\varepsilon(x)dx \in \mathbf{C}_0$. (In this case we say that a definition is correct.)

Define $\mathcal{S}_G$ as the set of elements Ψ from $\mathcal{G}_{\mathbf{t}}$ for which there exists the representative $\Psi_{\phi,\varepsilon}$ such that for every $\beta \in \mathbf{N}_0^n$ there exists $N \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_N$ and $p \in \mathbf{N}$ there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta \Psi_{\phi,\varepsilon}(x)| < (1 + |x|)^{-p}\varepsilon^{-N},\ \varepsilon < \eta,\ x \in \mathbf{R}^n.$$

$\mathcal{S}_G$ is called the space of generalized rapidly decreasing functions. Clearly, $\mathcal{S} \subset \mathcal{S}_G$ and they are not equal. Let $\Psi \in \mathcal{S}_G$ and $G \in \mathcal{G}_{\mathbf{t}}$. Then we define

$$< G, \Psi >= \int G(x)\Psi(x)dx$$

given by the representative

$$\int G_{\phi,\varepsilon}(x)\Psi_{\phi,\varepsilon}(x)dx. \tag{7}$$

One can prove that this definition is correct. Moreover, for every $\mathbf{G} \in \mathcal{G}_{\mathbf{t}}, \Psi \in \mathcal{S}_G$, and a unit net μ_ε related to $\mathbf{t}$,

$$\int^{\mathbf{t},\mu} G(x)\Psi(x)dx = \int G(x)\Psi(x)dx.$$

It is said that $G \in \mathcal{G}$ ($G \in \mathcal{G}_{\mathbf{t}}$) is equal to $H \in \mathcal{G}$ ($H \in \mathcal{G}_{\mathbf{t}}$) in generalized distribution sense, $G = H(g.d.)$, (in generalized tempered distribution sense, $G = H(g.t.d.)$) if every $\psi \in \mathcal{D}$ ($\psi \in \mathcal{S}$)

$$< G - H, \psi >= 0.$$

If we use $\Psi \in \mathcal{S}_G$ instead of $\phi \in \mathcal{S}$ we obtain $(G.t.d.)$-equality instead of $(g.t.d.)$-equality.

$A \in \overline{\mathbf{C}}$ is associated to $c \in \mathbf{C}$ ($A \approx c$) if there exists $N \in \mathbf{N}_0$ such that $\lim_{\varepsilon \to 0} A_{\phi,\varepsilon} = c$ for every $\phi \in \mathcal{A}_q$.

$G \in \mathcal{G}$ is associated to $H \in \mathcal{G}$ $(G \approx H)$ if there exists $N \in \mathbf{N}_0$ such that for every $\psi \in \mathcal{D}$

$$\lim_{\varepsilon \to 0} < G_{\phi,\varepsilon} - H_{\phi,\varepsilon}, \psi >= 0$$

for every $\phi \in \mathcal{A}_N$.

If one takes $\psi \in \mathcal{S}$ ($\Psi \in \mathcal{S}_G$ instead $\phi \in \mathcal{D}$ then the definition of $\mathbf{t}$-association ($\mathbf{T}$-association) is obtained instead of association.

All defined associations and equalities are equivalence relations.

Now, we define a convolution in $\mathcal{G}_{\mathbf{t}}$. Let $G_1, G_2 \in \mathcal{G}_{\mathbf{t}}$, and let μ_ε be a unit net related to $\mathbf{t}$. Then we define $G_1 \,{}^{\mathbf{t}}\!\star^{\mu}\, G_2$ as an element of $\mathcal{G}_{\mathbf{t}}$ by

$$G_1 \,{}^{\mathbf{t}}\!\star^{\mu}\, G_2(x) = \int^{\mathbf{t},\mu} G_1(x-y)G_2(y)dy, \; x \in \mathbf{R}^n. \tag{8}$$

The correctness of this definition and that $G_1, G_2 \in \mathcal{G}_{\mathbf{t}}$ implies $G_1 \,{}^{\mathbf{t}}\!\star^{\mu}\, G_2 \in \mathcal{G}_{\mathbf{t}}$ are proved by standard methods in [5].

Let μ be a unit net related to $\mathbf{t}$. Then the $\mathbf{t}, \mu$- Fourier transformation $\mathcal{F}_{\mathbf{t},\mu}$ on $\mathcal{G}_{\mathbf{t}}$ is defined by

$$\mathcal{F}_{\mathbf{t},\mu}(G)(x) = \int^{\mathbf{t},\mu} G(y)e^{-ixy}dy, \; x \in \mathbf{R}^n. \tag{9}$$

It is an element of $\mathcal{G}_{\mathbf{t}}$.

The inverse $\mathbf{t}, \mu$-Fourier transformation is defined by

$$\mathcal{F}^{-1}_{\mathbf{t},\mu}(G) = (2\pi)^{-n/2} \int^{\mathbf{t},\mu} G(y)e^{ixy}dy, x \in \mathbf{R}^n. \tag{10}$$

In the same way as for $\mathcal{F}_{\mathbf{t},\mu}$, one can prove that the definition is correct.

Proposition 1. *([5]) Let G, G_1, G_2 be in $\mathcal{G}_{\mathbf{t}}$ and let μ_ε be a unit net related to* $\mathbf{t}$. *Then for every $\psi \in \mathcal{S}$*

1. $< \mathcal{F}_{\mathbf{t},\mu}(G), \psi >=< G, \mathcal{F}(\psi) >$.

   1. implies that the Fourier transformation in $\mathcal{G}_t$ does not depend on a unit net in the sense of $(g.t.d.)$ equality, so we shall omit the symbol μ in the symbol for the Fourier transformation.

2. $\mathcal{F}_{\mathbf{t}}(G_1 \,{}^{\mathbf{t}}\!\star^{\mu}\, G_2) = \mathcal{F}_{\mathbf{t}}(G_1)\mathcal{F}_{\mathbf{t}}(G_2)(g.t.d.)$.

3. $\mathcal{F}_{\mathbf{t}}(\partial^\alpha G) = (i\cdot)^\alpha \mathcal{F}_{\mathbf{t}}(G)(g.t.d.)$.

4. *If* $\mathcal{F}_{\mathbf{t}}(G_1) = \mathcal{F}_{\mathbf{t}}(G_2)(g.t.d.)$ *then* $G_1 = G_2(g.t.d.)$.

5. *The quoted assertions hold with the use of the inverse Fourier transformation.*

6. $\mathcal{F}_{\mathbf{t}}(\mathcal{F}_{\mathbf{t}}^{-1}(G)) = G(g.t.d.)$.

7. $G_1 \ {}^{\mathbf{t}}\star^{\mu}\ G_2 = G_2 \ {}^{\mathbf{t}}\star^{\mu}\ G_1(g.t.d.)$.

8. $(G_1 \ {}^{\mathbf{t}}\star^{\mu}\ G_2) \ {}^{\mathbf{t}}\star^{\mu}\ G_3 = G_1 \ {}^{\mathbf{t}}\star^{\mu}\ (G_2 \ {}^{\mathbf{t}}\star^{\mu}\ G_3)(g.t.d.)$.

9. $\partial^{\alpha}(G_1 \ {}^{\mathbf{t}}\star^{\mu}\ G_2) = \partial^{\alpha}G_1 \ {}^{\mathbf{t}}\star^{\mu}\ G_2(g.t.d.)$.

For the unit nets $\mu_{1,\varepsilon}, \mu_{2,\varepsilon}$ related to $\mathbf{t}$ and $\psi \in \mathcal{S}$

$$< G_1 \ {}^{\mathbf{t}}\star^{\mu_1}\ G_2, \psi > = < \mathcal{F}_{\mathbf{t}}(\mathcal{F}_{\mathbf{t}}^{-1}(G_1 \ {}^{\mathbf{t}}\star^{\mu_1}\ G_2)), \psi >$$

$$= < \mathcal{F}_{\mathbf{t}}(\mathcal{F}_{\mathbf{t}}^{-1}(G_1)\mathcal{F}_{\mathbf{t}}^{-1}(G_2)), \psi >$$

$$= < \mathcal{F}_{\mathbf{t}}(\mathcal{F}_{\mathbf{t}}^{-1}(G_1 \ {}^{\mathbf{t}}\star^{\mu_2}\ G_2)), \psi > = < G_1 \ {}^{\mathbf{t}}\star^{\mu_2}\ G_2, \psi > .$$

This implies that the **t**-convolution does not depend in $(g.t.d.)$ sense on the unit nets. So in the sequel for the **t**-convolution we use the symbol $\star$ and for the **t**-Fourier transformation the symbol $\mathcal{F}$.

Remark If we use $\Psi \in \mathcal{S}_G$ instead of $\psi \in \mathcal{S}$, all assertions are valid for $(G.t.d)$-equality because the **t**-Fourier transformation is bijection from $\mathcal{S}_G$ into $\mathcal{S}_G$, as one can prove by standard technique which is used to prove that Fourier transformation is bijection from $\mathcal{S}$ onto $\mathcal{S}$ in classical case.

3. Convolution Equations

Let $\psi_j,\ j \in \mathbf{N}$, be a locally finite partition of unity from $\mathcal{D}$ such that for every $\beta \in \mathbf{N}_0^n$ there is $D_\beta > 0$ such that

$$|\partial^{\beta}\psi_j(x)| \leq D_\beta,\ j \in \mathbf{N}. \tag{11}$$

Denote

$$K_j = \text{supp}\psi_j,\ K_{j,1} = \{x \in \mathbf{R}^n |\ d(x, K_j) \leq 1\},\ j \in \mathbf{N},$$

$k_x = \{j|\ x \in K_{j,1}\}$, and card$(k_x)$ is its cardinal number, $x \in \mathbf{R}^n$ ($d(x, K_j)$ is the distance between x and K_j).

We shall assume

$$\sup_{x \in \mathbf{R}^n} (\text{card } k_x) = r < \infty, \text{ and mes}(K_j) \leq 1.$$

It is easy to find such partition of unity.

Our aim is to find the solution to

$$F \cdot G \approx^T 1, \tag{12}$$

where $F \in \mathcal{G}$ satisfies the following assumptions.

(I) $\mathrm{mes}(V) = 0$, $V = \overline{\mathrm{compl}(U)}$, where U is the set of all $x \in \mathbf{R}^n$ such that for every K_j there exists $N_{j,F} \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_{N_{j,F}}$ there exist $C_{j,F} > 0$, $\gamma_j > 0$, and $\eta_{j,F} > 0$ such that

$$|F_{\phi,\varepsilon}(x)| \geq C_{j,F}(1+|x|)^{-\gamma_j}\varepsilon^{N_{j,F}},\ \varepsilon < \eta_{j,F},\ x \in K_j \cap U, \tag{13}$$

and

$$\gamma_{1,F} = \sup_{j\in\mathbf{N}} \gamma_j < \infty,\ C_{1,F} = \sup_{j\in\mathbf{N}} C_{j,F} < \infty,$$

$$N_{1,F} = \sup_{j\in\mathbf{N}} N_{j,F} < \infty,\ \eta_{1,F} = \inf_{j\in\mathbf{N}} \eta_{j,F} > 0.$$

(II) For every K_j there exists $N_j \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_{N_j}$ there exist $m_j > 1$, $C_j > 0$ and $\eta_j > 0$ such that

$$|F_{\phi,\varepsilon}(x)| \geq C_j \cdot d(x,V)^m \varepsilon^{N_j},\ \varepsilon < \eta_j,\ x \in K_j, \tag{14}$$

$$\sup_{j\in\mathbf{N}} m_j = m < \infty, \inf_{j\in\mathbf{N}} \eta_j = \eta > 0, \sup_{j\in\mathbf{N}} N_j = N_2 < \infty, \tag{15}$$

and

$$C_x = \max_{j\in k_x} 1/C_j \leq C_2(1+|x|)^{\gamma}. \tag{16}$$

(III) V can be decomposed in a finite union of subvarieties of dimension less or equal to $n-1$,

$$V = V_1 \cup ... \cup V_{r_V},\ \dim V_i = n_i \leq n-1,$$

such that every $x \in V_i$ is given by

$$x = (\kappa_1(x_1,...,x_{n_i}),...,\kappa_{n_i}(x_1,...,x_{n_i}),\kappa_{n_i+1}(x_1,...,x_{n_i}),...,\kappa_n(x_1,...,x_{n_i})),$$

where $\kappa_l(x_1,...,x_{n_i}) = x_l$, for $l \leq n_i$, and $x_l = \kappa_l(x_1,...,x_{n_i})$, $l > n_i$,

$(x_1,...,x_{n_i}) \in \mathbf{R}^{n_i}$, are of polynomial growth in infinity with respect to variables $x_1,...,x_{n_i}$.

The following result is obtained by adopting division procedure from the space $\mathcal{G}$ ([6]) to the space $\mathcal{G}_{\mathbf{t}}$.

Theorem 1. *Let $F \in \mathcal{G}_{\mathbf{t}}$ satisfies assumptions (I), (II), and (III). Then there exists $G \in \mathcal{G}_{\mathbf{t}}$ such that $F \cdot G \approx^t 1$.*

Proof. We can suppose that $\inf m_j = m_0 > 1$, $\eta_j \leq \eta_{j,F}$. Put $C_0 = C_{1,F} + C_2$, and $N = N_{1,F} + N_2$.

Let $\Psi \in \mathcal{S}_G$. Then there exists N_Ψ such that for every $\phi \in \mathcal{A}_{N_\Psi}$ there exist η_Ψ and C_Ψ such that for every $s > 0$

$$|\Psi_{\phi,\varepsilon}(x)| \leq C_\Psi(1+|x|)^{-s}\varepsilon^{-N_\Psi},\ \varepsilon < \eta_\Psi,\ x \in \mathbf{R}^n.$$

Let $j \in \mathbf{N}$ and $\phi \in \mathcal{A}_q$, where q will be chosen later. Put

$$G_{1,\phi,\varepsilon,j}(x) = \begin{cases} 1/F_{\phi,\varepsilon}(x), & d(x,V) > \varepsilon^{q/m_j}, x \in K_j, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

$$G_{\phi,\varepsilon,j}(x) = \psi_j(x)(G_{1,\phi,\varepsilon,j} \star \phi_{\varepsilon^{qm_j}})(x), G_{\phi,\varepsilon}(x) = \sum_{j\in\mathbf{N}} G_{\phi,\varepsilon,j}(x),\ x \in \mathbf{R}^n.$$

We shall use

$$F_{\phi,\varepsilon}(x - \varepsilon^{qm_j}y) = F_{\phi,\varepsilon}(x) + \sum_{|\alpha|=1} \int_0^1 \partial^\alpha F_{\phi,\varepsilon}(x - t\varepsilon^{qm_j}y)(-\varepsilon^{qm_j}y)^\alpha dt,$$

which implies that there exists $N_D \in \mathbf{N}_0$ and $\gamma_D > 0$, independent on q because $|t\varepsilon^{qm_j}y| \le 1$, such that for every $\phi \in \mathcal{A}_{N_D}$ there exist $C_D > 0$ and $\eta_D > 0$ such that

$$|F_{\phi,\varepsilon}(x) - F_{\phi,\varepsilon}(x - \varepsilon^{qm_j}y)| \le C_D(1 + |x|)^{\gamma_D}\varepsilon^{qm_j - N_D}, \tag{18}$$

for $0 < \varepsilon < \eta_D$, $x \in \mathbf{R}^n$, $x - \varepsilon^{qm_j}y \in K_j$, $|y| < 1$.

First, we will prove that $G_{\phi,\varepsilon} \in \mathcal{E}_{\mathbf{t}}$. Suppose $q \ge \max\{N, N_\Psi\}$ and $qm_0 - q - N - N_D - N_\Psi > 0$. Then, by the Leibnitz formula

$$|\partial^\alpha G_{\phi,\varepsilon}(x)| \le 2^{|\alpha|} \max_{\gamma+\beta=\alpha} A_{\gamma,\beta},\ x \in \mathbf{R}^n,$$

where

$$A_{\gamma,\beta} = |\sum_{j\in k_x} \partial^\gamma\psi_j(x)\partial^\beta \int_{d(y,V)>\varepsilon^{q/m_j}} G_{1,\phi,\varepsilon,j}(y)\varepsilon^{-nqm_j}\phi((x-y)/\varepsilon^{qm_j})dy|$$

$$\le \sum_{j\in k_x} \varepsilon^{-q|\beta|m_j}|\partial^\gamma\psi_j(x) \int_{d(\varepsilon^{qm_j}y,V)>\varepsilon^{q/m_j}} G_{1,\phi,\varepsilon,j}(\varepsilon^{qm_j}y)\partial^\beta\phi(x/\varepsilon^{qm_j} - y)dy|$$

$$\le D_\gamma \sum_{j\in k_x} \varepsilon^{-q|\beta|m_j}(\sup_{y\in A_j}|G_{1,\phi,\varepsilon,j}(\varepsilon^{qm_j}y)|)(\sup_{y\in A_j}|\partial^\beta\phi(x/\varepsilon^{qm_j} - y)|)\cdot \text{ mes } A_j,$$

where D_γ is from (11) and

$$A_j = \{y|\ \varepsilon^{qm_j}y \in K_j, d(\varepsilon^{qm_j}y, V) > \varepsilon^{q/m_j}, |y - x/\varepsilon^{qm_j}| < 1\}.$$

From (I) and (II) we have

$$|G_{1,\phi,\varepsilon,j}(\varepsilon^{qm_j}y)| = |1/F_{\phi,\varepsilon})(\varepsilon^{qm_j}y)| \le 1/(C_j(\varepsilon^{q/m_j})^{m_j - N})$$
$$= 1/(C_j\varepsilon^{q-N}),\ y \in A_j, \varepsilon < \eta.$$

If $|y - x/\varepsilon^{qm_j}| \le 1$ then $|y| \le |x|/\varepsilon^{qm_j} + 1$ and the ball with the radius $R = |x|/\varepsilon^{qm_j} + 1$ at the center 0 has the volume

$$V = \int dV = \int_{S^{n-1}}(\int_0^R r^{n-1}dr)d\omega = \max_{x\in K_j}(2\pi^{n/2}/\Gamma(n/2))(|x|/\varepsilon^{qm_j} + 1)^n/n.$$

This implies

$$\mathrm{mes}(A_j) \leq \max_{x\in K_j}(2\pi^{n/2}/\Gamma(n/2)) \cdot (|x|/\varepsilon^{qm_j}+1)^n/n.$$

Since $\sup_{t\in\mathbf{R}^n}|\partial^\beta\phi(t)| = \tilde{D}_\beta < \infty$, we have

$$|\partial^\alpha G_{\phi,\varepsilon}(x)| \leq 2^{|\alpha|}\sup_{\gamma+\beta=\alpha} D_\gamma \sum_{j\in k_x}(1/C_j)\varepsilon^{-q|\beta|m_j+q})$$

$$\cdot\tilde{D}_\beta(2\pi^{n/2}/\Gamma(n/2))(|x|+\varepsilon^{qm_j})^n/(n\varepsilon^{nqm_j}), x\in\mathbf{R}^n, \varepsilon<\eta.$$

This proves that $G_{\phi,\varepsilon}\in\mathcal{E}_{\mathbf{t}}$.

Let us prove that for every $\Psi_{\phi,\varepsilon}\in\mathcal{S}_G$, $< F_{\phi,\varepsilon}G_{\phi,\varepsilon}-1, \Psi_{\phi,\varepsilon} >\to 0$, when $\varepsilon\to 0$.

Put

$$\Lambda^j_+(x) = \{y\in\mathbf{R}^n|\ |y|\leq 1,\ d(x-\varepsilon^{qm_j}y, V) > \varepsilon^{q/m_j},\ x\in K_j,\ x-\varepsilon^{qm_j}y\in K_j\},$$

$$\Lambda^j_-(x) = \{y\in\mathbf{R}^n|\ |y|\leq 1\}\setminus\Lambda^j_+(x).$$

Since

$$\int_{\Lambda^j_-(x)}\phi(y)dy = 0 \text{ for } d(x,V) > 2\varepsilon^{q/m_j},\ x\in K_j,$$

we have

$$\int(F_{\phi,\varepsilon}(x)\sum_{j\in\mathbf{N}}\psi_j(x)\int G_{1,\phi,\varepsilon,j}(x-y)\phi_{\varepsilon^{qm_j}}(y)dy-1)\Psi_{\phi,\varepsilon}(x)dx$$

$$=\int\sum_{j\in k_x}|\int_{\Lambda^j_+(x)}\frac{F_{\phi,\varepsilon}(x)-F_{\phi,\varepsilon}(x-\varepsilon^{qm_j}y)}{F_{\phi,\varepsilon}(x-\varepsilon^{qm_j}y)}\psi_j(x)\phi(y)dy\Psi_{\phi,\varepsilon}(x)dx$$

$$+\int\sum_{j\in k_x}\int_{\Lambda^j_-(x)}dy\Psi_{\phi,\varepsilon}(x)dx = I_1+I_2.$$

We have

$$|I_1| \leq \int\sum_{j\in k_x}|\int_{\Lambda^j_+(x)}\frac{C_D\varepsilon^{qm_j-N_D}(1+|x|)^{\gamma_D}}{C_j(\varepsilon^{qm_j})^{1/m_j}}\tilde{\tilde{D}}_j\psi_j(x)\phi(y)||\Psi_{\phi,\varepsilon}(x)|dydx$$

$$\leq\int\sum_{j\in k_x}C_j^{-1}\tilde{\tilde{D}}_j\tilde{D}_0\varepsilon^{qm_j-q-N-N_D}(1+|x|)^{\gamma_j+\gamma_D}|\Psi_{\phi,\varepsilon}(x)|dx$$

$$< C_\Psi\int(1+|x|)^{\gamma_F+\gamma_D-s}dx\cdot\varepsilon^{qm-q-N-N_H-N_\Psi}.$$

On the other hand,

$$|I_2| \le (2\pi^{n/2}/\Gamma(n/2)) \int (\int_{\Lambda^j_-(x)} \psi(y)dy)|\Psi_{\phi,\varepsilon}(x)|dx$$

$$= (2\pi^{n/2}/\Gamma(n/2))(\int_{|x_i|\le 1/\varepsilon, d(x,V)>2\varepsilon^{q/m_j}} |\Psi_{\phi,\varepsilon}(x)|dx$$

$$+ \int_{|x_i|>1/\varepsilon, d(x,V)>2\varepsilon^{q/m_j}} |\Psi_{\phi,\varepsilon}(x)| = J_1 + J_2.$$

By standard arguments one can prove that

$$\int_{|x_i|>1/\varepsilon, d(x,V)>2\varepsilon^{q/m_j}} |\Psi_{\phi,\varepsilon}(x)|dx \in \mathbf{C}_0,$$

which implies that $J_2 \in \mathbf{C}_0$. Let us prove that $J_1 \to 0$ as $\varepsilon \to 0$. From assumption (III) it follows that the measure of V_i in $\mathbf{R}^{n_i}$ is bounded by $C_{V_i}\varepsilon^{-N_i}$ for some $C_{V_i} > 0$ and $N_i > 0$ if $|x_l| \le \varepsilon^{-1}$, $1 \le l \le n_i$ because

$$\mathrm{mes}(V_i) = \int_{|x_l|\le 1/\varepsilon, 1\le l\le n_i} (\det(a_{ij}))^{1/2} dx_1...dx_{n_i},$$

$$a_{ij} = (\frac{\partial\kappa_1}{\partial x_i}, ..., \frac{\partial\kappa_n}{\partial x_i}) \cdot (\frac{\partial\kappa_1}{\partial x_j}, ..., \frac{\partial\kappa_n}{\partial x_j}).$$

Let $\tilde{N} = \max_{1\le i\le r_V} N_i$. we can suppose that $q > (\tilde{N} + N_\Psi)/m$. Let $\phi \in \mathcal{A}_q$. Then

$$M_i = \mathrm{mes}\{x \in \mathbf{R}^n |\; |x_l| \le 1/\varepsilon,\ 1 \le l \le n_i,\ d(x, V_i) \le 2\varepsilon^{q/m}\}$$

$$\le (C_{V_i}\varepsilon^{-N_i} + 2\varepsilon^{q/m}) \cdot \mathrm{mes}\{x \in \mathbf{R}^n |\; |x| \le 2\varepsilon^{q/m}\}$$

which implies

$$\mathrm{mes}\{x \in \mathbf{R}^n |\; |x_l| \le 1/\varepsilon,\ 1 \le l \le n,\ d(x, V) \le 2\varepsilon^{q/m}\}$$

$$\le \sum_{i=1}^{r_V} M_i \le (2\pi^{n/2}/\Gamma(n/2))(\max_{1\le i\le r_V} C_{V_i}\varepsilon^{\tilde{N}} + 1) \cdot 2\varepsilon^{q/m}.$$

Since

$$(\max_{1\le i\le r_V} C_{V_i}\varepsilon^{-\tilde{N}} + 1) \cdot 2\varepsilon^{q/m - N_\Psi} C_\psi (1 + |x|)^{-s} \to 0 \text{ as } \varepsilon \to 0,$$

it follows that $J_1 \to 0$ as $\varepsilon \to 0$. This proves the theorem.

Now we shall give the main result of the paper.

Corollary 1. *Let $F \in \mathcal{G}_{\mathbf{t}}$. If $\mathcal{F}(F)$ satisfies the conditions of Theorem 1. Then for every $H \in \mathcal{G}_{\mathbf{t}}$ there exists $G \in \mathcal{G}_{\mathbf{t}}$ such that*

$$F \star G \approx^T H. \tag{19}$$

Proof. Since $H\Psi \in \mathcal{S}_G$ for $H \in \mathcal{G}_{\mathbf{t}}$ and $\Psi \in \mathcal{S}_G$, we have that $\int G\Psi dx \approx \int F\Psi dx$ implies $\int GH\Psi dx \approx \int FH\Psi dx$, i.e. that $G \approx^T F$ implies $GH \approx^T FH$ for every $F, G, H \in \mathcal{G}_{\mathbf{t}}$.

The bijectivity of the **t**-Fourier and inverse **t**-Fourier transformation from $\mathcal{S}_G$ onto $\mathcal{S}_G$ implies that from $F \approx^T G$ we obtain $\mathcal{F}(F) \approx^T \mathcal{F}(G)$ and $\mathcal{F}^{-1}(F) \approx^T \mathcal{F}^{-1}(G)$ because

$$\int \mathcal{F}(F)\Psi dx \approx^T \int F\mathcal{F}(\Psi)dx \approx^T \int G\mathcal{F}(\Psi)dx \approx^T \int \mathcal{F}(G)\Psi dx.$$

These statements enable us to prove that $F \approx^T G$ implies $F \star H \approx^T G \star H$, for every $F, G, H \in \mathcal{G}_{\mathbf{t}}$:

$$\mathcal{F}(F \star H) \approx^T \mathcal{F}(F)\mathcal{F}(H) \approx^T \mathcal{F}(G)\mathcal{F}(H) \approx^T \mathcal{F}(G \star H).$$

Because of that, the equation $F \star G \approx^T \delta$ by **t**-Fourier transformation becomes $\mathcal{F}(F)\mathcal{F}(G) \approx^T 1$, and by Theorem 1 this equation has the solution $\mathcal{F}(G)$. Then $G = \mathcal{F}^{-1}(\mathcal{F}(G))$ is a solution to $F \star G \approx^T \delta$, and $G_1 = G \star H$ is a solution to $F \star G_1 \approx^T H$, for every $H \in \mathcal{G}_{\mathbf{t}}$. This proves the corollary.

Remark Theorem 2 in [6] is special case of this corrolary.

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