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THE β–DUALS OF SOME MATRIX DOMAINS IN FK SPACES AND MATRIX TRANSFORMATIONS[1]

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Abstract. We prove general results that reduce the determination of the β–duals of matrix domains X_T of triangles in certain FK spaces X to that of the β–dual of X, and the characterization of matrix transformations on X_T to that of matrix transformations on X.

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1. Introduction

By ω we denote the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ℓ_∞, c, c_0 and ϕ be the sets of all bounded, convergent, null and finite sequences, cs and bs be the sets of all convergent and bounded series, and $\ell_p = \{x \in \omega : \sum_{k=0}^{\infty}|x_k|^p < \infty\}$ for $1 \le p < \infty$.

By e and $e^{(n)}$ $(n = 0, 1, \ldots)$, we denote the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

An *FK space* X is a complete linear metric sequence space with continuous *coordinates* $P_k : X \to \mathbb{C}$ where $P_k(x) = x_k$ for all $x \in X$ and $k = 0, 1, \ldots$; a *BK space* is a normed FK space. We say that an FK space $X \supset \phi$ has AD if ϕ is dense in X; we say that X has AK if $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x$ $(n \to \infty)$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$.

If X and Y are subsets of ω, and z is a sequence, we write $z^{-1} * Y = \{x \in \omega : xz = (x_k z_k)_{k=0}^{\infty} \in Y\}$ and $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{z \in \omega : zx \in Y \text{ for all } x \in X\}$ for the *multiplier of* X *and* Y. In the special case when $Y = cs$, we write $z^\beta = z^{-1} * cs$, and the set $X^\beta = M(X, cs)$ is called the *β–dual of* X.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers, x be a sequence and X be a subset of ω. Then we write $A_n = (a_{nk})_{k=0}^{\infty}$ and $A^k =$

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$(a_{nk})_{n=0}^{\infty}$ for the sequences in the n–th row and the k–th column of A, $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ $(n = 0, 1, \ldots)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$, provided $A_n \in x^{\beta}$ for all n. The set $X_A = \{z \in \omega : A(z) \in X\}$ is called the *matrix domain of* A *in* X. Given any subsets X and Y of ω, then (X, Y) denotes the class of all matrices A that map X into Y, that is for which $A_n \in X^{\beta}$ for all n and $A(x) \in Y$ for all $x \in X$, or equivalently $A \in (X, Y)$ if and only if $X \subset Y_A$.

A matrix T is said to be a *triangle* if $t_{nk} = 0$ for all $k > n$ and $t_{nn} \neq 0$ $(n = 0, 1, \ldots)$. A subset X of ω is said to be *normal* if $x \in X$ and $|y_k| \leq |x_k|$ $(k = 0, 1, \ldots)$ for some sequence y simultaneously imply $y \in X$.

In this paper we prove general results that reduce the determination of the β–duals of the matrix domains X_T of triangles in certain FK spaces X to that of the β–dual of X, and the characterization of matrix transformations on X_T to that of matrix transformations on X. Furthermore, we give some applications of our general results.

2. The β– duals of matrix domains of triangles

In this section we reduce the determination of $(X_T)^{\beta}$ to that of X^{β}.

Throughout, let $T = (t_{nk})_{n,k=0}^{\infty}$ be a triangle and $Z = X_T$. It is well known (cf. [11, 1.4.8, p. 9] and [1, Remark 22 (a), p. 22]) that every triangle T has a unique inverse $U = (u_{nk})_{n,k=0}^{\infty}$ which also is a triangle, and $x = T(U(x)) = U(T(x))$ for all $x \in \omega$. Without explicitly mentioning it each time, we will frequently use the trivial facts that $x \in X$ if and only if $z = U(x) \in Z$, and $z \in Z$ if and only if $x = T(z) \in X$.

Proposition 2.1 *Let* T *be a triangle,* X *and* Y *be subsets of* ω *and* $a \in \omega$. *We define the matrix* $B = B(a, T)$ *by* $B^k = aU^k$, *that is* $b_{nk} = a_n u_{nk}$ *for* $0 \leq k \leq n$ *and* $b_{nk} = 0$ *for* $k > n$ $(n = 0, 1, \ldots)$. *Then* $a \in M(X_T, Y)$ *if and only if* $B \in (X, Y)$.

Proof. This is trivial since $B(x) = aU(x) = az$. □

If (X, d) is a linear metric sequence space and $a \in X$, we write

$$S_{\delta} = S_{X,\delta} = \{x \in \omega : d(x, 0) \leq \delta\} \ \ (\delta > 0) \text{ and } S = S_X = S_1 \text{ for short,}$$

and

$$\|a\|_D^* = \|a\|_{X,D}^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : x \in S_{1/D}\right\} \ \ (D > 0),$$

provided the expression on the right exists and is finite, which is the case whenever X is an FK space and $a \in X^{\beta}$ (cf. [11, Theorem 7.2.9, p. 107]). If X is a BK space, we write

$$\|a\|^* = \|a\|_X^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : \|x\| \leq 1\right\}.$$

It follows from [11, Theorem 4.3.12, p. 63] that if X is a BK space so is Z with $\|z\|_Z = \|T(z)\|$ $(z \in Z)$, and so $S_Z = S_X$.

If A is a matrix, we write A^t for its transpose, that is $a^t_{nk} = a_{kn}$ for all $n, k = 0, 1, \ldots$. We define the matrix $\Sigma = (\sigma_{nk})_{n,k=0}^{\infty}$ by $\sigma_{nk} = 1$ for $0 \le k \le n$ and $\sigma_{nk} = 0$ for $k > n$ $(n = 0, 1, \ldots)$.

To be able to determine $(X_T)^\beta$ in the case when X is an FK space, we need

Lemma 2.2 *Let X be a normal FK space with AK. We put $R = U^t$. Then $(X_T)^\beta \subset (X^\beta)_R$.*

Proof. We assume $a \in Z^\beta$ and write $C = \Sigma B$ where B is the matrix defined in Proposition 2.1. Then $B \in (X, cs)$ by Proposition 2.1, and this is the case if and only if $C \in (X, c)$ by [7, Theorem 3.8, p. 180]. Since A is an FK space with AK it follows from [7, Theorem 1.23, p. 155] and [11, 8.3.6, p. 123] that

$$R_k(a) = \lim_{n\to\infty} c_{nk} = \sum_{j=k}^{\infty} a_j u_{jk} \text{ exists for each } k, \tag{2.1}$$

and $\sup_n \|C_n\|^*_{X,D} < \infty$ for some $D > 0$, that is there is a constant K such that

$$|C_n(x)| = \left|\sum_{k=0}^{n} c_{nk} x_k\right| \le K \text{ for all } n \text{ and for all } x \in S_{X,1/D}. \tag{2.2}$$

Let $x \in X$ be given and $\delta = 1/(2D)$. We define the sequence $\tilde{x}$ by $\tilde{x}_k = x_k \text{sgn} R_k(a)$ $(k = 0, 1, \ldots)$. Then $\tilde{x} \in X$, since X is normal. Furthermore, since $S_{X,\delta}$ is absorbing (cf. [10, Chapter 4.1, Fact (ix), p. 53]) and X has AK, there are a real $\lambda > 0$ and a non–negative integer m_0 such that $\tilde{y}^{[m]} = \lambda^{-1}\tilde{x}^{[m]} \in S_{X,\delta}$ for all $m \ge m_0$. Let $m \ge m_0$ be given. Then for all $n \ge m$ by (2)

$$\left|\sum_{k=0}^{m} c_{nk} x_k \text{sgn} R_k(a)\right| = \lambda \left|\sum_{k=0}^{m} c_{nk} \tilde{y}_k^{[m]}\right| = \lambda \left|C_n(\tilde{y}^{[m]})\right| \le \lambda K,$$

and so by (1)

$$\sum_{k=0}^{m} |R_k(a) x_k| = \lambda \lim_{n\to\infty} |C_n(\tilde{y}^{[m]})| \le \lambda K.$$

Since $m \ge m_0$ was arbitrary, we conclude $R(a) \in x^\beta$, and since $x \in X$ was arbitrary, $R(a) \in \bigcap_{x \in X} x^\beta = X^\beta$, that is $a \in (X^\beta)_R$. □

Theorem 2.3 *Let X be a normal FK space with AK and $R = U^t$. Then $a \in (X_T)^\beta$ if and only if*

$$a \in (X^\beta)_R \text{ and } W \in (X, c_0) \tag{2.3}$$

where the matrix W is defined by

$$w_{mk} = \begin{cases} \sum_{j=m}^{\infty} a_j u_{jk} & (0 \le k \le m) \\ 0 & (k > m) \end{cases} \qquad (m = 0, 1, \ldots).$$

Furthermore, if $a \in (X_T)^\beta$ then

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} R_k(a) T_k(z) \text{ for all } z \in Z = X_T. \tag{2.4}$$

Proof. First we assume $a \in Z^\beta$. Then $R(a) \in X^\beta$ by Lemma 2.2, and w_{mk} converges for all m and k, thus the matrix W is defined. Furthermore

$$\sum_{k=0}^{m-1} a_k z_k = \sum_{k=0}^{m} R_k(a) T_k(z) - \sum_{k=0}^{m} w_{mk} T_k(z) \text{ for all } m \text{ and all } z. \tag{2.5}$$

Let $x \in X$ be given. Then $z = U(x) \in Z$ and so $a \in z^\beta$ and $a \in (x^\beta)_R$. Thus $W(x) \in c$ by (5). Since $x \in X$ was arbitrary, we have $W \in (X, c) \subset (X, \ell_\infty)$. Furthermore, since $R_k(a) = \sum_{j=k}^{\infty} a_j u_{jk}$ exists for each k, we have

$$\lim_{m\to\infty} w_{mk} = \lim_{m\to\infty} \sum_{j=m}^{\infty} a_j u_{jk} = 0, \tag{2.6}$$

and by [7, Theorem 1.23, p. 115] and [11, 8.3.6, p. 123] this and $W \in (X, \ell_\infty)$ together imply $W \in (X, c_0)$. Now if $a \in Z^\beta$ then the conditions in (3) hold by what we have just shown, and (4) follows from (5).
Conversely, we assume that the conditions in (3) are satisfied. Then $x = T(z) \in X$ and so $az \in cs$ for all $z \in Z$ by (5), that is $a \in Z^\beta$. □

Using a different proof, we may drop the assumption that X is normal in the case when X is a BK space.

Theorem 2.4 *Let X be a BK space with AK and $R = U^t$. Then $a \in (X_T)^\beta$ if and only if the conditions in (3) hold. Furthermore, if $a \in (X_T)^\beta$ then (4) holds.*

Proof. First we assume $a \in Z^\beta$. Then, as in the proof of Lemma 2.2, we conclude that condition (1) holds and

$$C = \Sigma B \in (X, \ell_\infty). \tag{2.7}$$

From (1), we obtain that the matrix W is defined and again (6) holds. Furthermore, since X is a BK space with AK, condition (7) implies $C^t \in (\ell_1, X^\beta)$ by [11, Theorem 8.3.9, p. 124]. Now X^β is a BK space with

$$\|y\|_\beta = \sup_m \left\{ \left| \sum_{k=0}^{m} y_k x_k \right| : x \in S_X \right\} = \|y^{[m]}\|_X^* \text{ for all } y \in X^\beta$$

by [11, Example 4.3.16, p. 65]. Therefore, by [11, Example 8.4.1, p. 126], the columns of the matrix C^t, that is the rows of C are a bounded set in X^β. Thus there is a constant K_1 such that

$$\left|\sum_{k=0}^{m} c_{nk}x_k\right| \leq K_1 \text{ for all } m \text{ and } n \text{ and all } x \in S_X. \tag{2.8}$$

Now (1) implies $|\sum_{k=0}^{m} R_k(a)x_k| \leq K_1$ for all m and all $x \in S_X$. Since $x \in S_X$ if and only if $z \in S_Z$, it follows from (5) and (8) that

$$|W_m(x)| \leq K_1 + \left|\sum_{k=0}^{m-1} a_k z_k\right| \text{ for all } x \in S_X,\ z \in S_Z \text{ and all } m. \tag{2.9}$$

We define the linear functionals f_m $(m = 0, 1, \ldots)$ on Z by $f_m(z) = \sum_{k=0}^{m-1} a_k z_k$. Since Z is a BK space, we have $f_m \in Z^*$ for all m, and $a \in Z^\beta$ implies $\lim_{m\to\infty} f_m \in Z^*$ by [11, Theorem 7.2.9, p. 107]. We are going to show that the sequence $(f_m)_{m=0}^\infty$ is pointwise bounded. Then it is norm bounded by the uniform boundedness principle, that is there is a constant K_2 such that

$$|f_m(z)| = \left|\sum_{k=0}^{m-1} a_k z_k\right| \leq K_2 \text{ for all } m \text{ and all } z \in S_Z. \tag{2.10}$$

Let $z \in Z \setminus \{0\}$ be given. Then $|f_m(x)| \leq \|f_m\|\,\|z\|$ or all m. Since $f(z) = \lim_{m\to\infty} f_m(z)$, there is a non–negative integer $m_0 = m_0(z)$ such that $|f_m(z) - f(z)| \leq \|z\|$ for all $m \geq m_0$, and so $|f_m(z)| \leq (\|f\| + 1)\|z\|$. We put $K_2(z) = (\max\{\|f\| + 1, \max_{0\leq m\leq m_0} \|f_m\|\})\|z\|$. Then $|f_m(z)| \leq K_2(z)$ for all m. Thus the sequence $(f_m)_{m=0}^\infty$ is pointwise bounded.

Now it follows from (9) and (10) that $|W_m(x)| \leq K_1 + K_2$ for all m and all $x \in S_X$, hence $\sup_m \|W_m\|_X^* < \infty$. It follows from this and (6) that $W \in (X, c_0)$ by [7, Theorem 1.23, p. 155] and [11, 8.3.6, p. 123]. Finally, from (5) we obtain $R(a) \in X^\beta$, that is $a \in (X^\beta)_R$.

The converse part of the proof is exactly the same as in the proof of Theorem 2.3. □

Remark 2.5 *Since $(c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$, the proof of Theorem 2.4 shows that Theorem 2.4 also holds for $x = c$ and $X = \ell_\infty$.*

Remark 2.6 *It seems that the condition that X is normal is needed in the proof of Lemma 2.2, hence in the hypotheses of Theorem 2.3. On the other hand, however, some arguments in the proof of Theorem 2.4 fail in the case of FK spaces, for instance the β–dual of an FK space need not be an FK space. Thus the proof of Theorem 2.4 does not extend to FK spaces in general.*

3. Matrix transformations

In this section we shall prove a result which reduces the characterization of the classes (X_T, Y) to that of (X, Y).

Theorem 3.1 *Let Y be an arbitrary subset of ω, X be a normal FK space (or a BK space) with AK and $R = U^t$. Then $A \in (X_T, Y)$ if and only if*

$$R^A \in (X, Y) \tag{3.1}$$

and

$$W^{A_n} \in (X, c_0) \text{ for all } n = 0, 1, \ldots \tag{3.2}$$

where $R^A = AU$ and the matrices W^{A_n} $(n = 0, 1, \ldots)$ are defined by

$$w_{mk}^{A_n} = \begin{cases} \sum_{j=m}^{\infty} a_{nj} u_{jk} & (0 \le k \le m) \\ 0 & (k > m) \end{cases} \qquad (m = 0, 1, \ldots).$$

Proof. First we assume $A \in (Z, Y)$. Then $A_n \in Z^\beta$ for all n, hence the condition (12) holds and

$$R(A_n) = (R^A)_n \in X^\beta \text{ for all } n \tag{3.3}$$

by Theorem 2.3 (or 2.4). Let $x \in X$ be given. Then $A_n \in Z$ implies

$$(R^A)_n(x) = A_n(z) \text{ for all } n \tag{3.4}$$

by (4) and $A(z) \in Y$ yields $R^A(x) \in Y$. Thus the condition (11) also holds. Conversely, we assume that the conditions (11) and (12) are satisfied. Then (13) holds, and this and (12) jointly imply $A_n \in Z^\beta$ for all n by Theorem 2.3 (or 2.4). Now (14) again holds and we conclude that $A \in (Z, Y)$. □

4. Some applications

In this section we apply our results.

First we give an application of Proposition 2.1.

Example 4.1 *Let $p = (p_k)_{k=0}^\infty \in \ell_\infty$, $p_k > 1$ and $q_k = p_k/(p_k - 1)$ for $k = 0, 1, \ldots$. We write $\ell(p) = \{x \in \omega : \sum_{k=0}^\infty |x_k|^{p_k} < \infty\}$, $bv(p) = (\ell(p))_\Delta$ and, for all $N \in I\!N \setminus \{1\}$,*

$$S(N) = \left\{ a \in \omega : \sup_n \sum_{k=0}^n |a_n|^{q_k} N^{-q_k} < \infty \right\}.$$

Then

$$M(bv(p), c_0) = c_0 \cap \bigcup_{N \in I\!N \setminus \{1\}} S(N); \tag{4.1}$$

if $(p_k)_{k=0}^\infty$ *is a constant sequence, that is* $p_k = p > 1$ *for all* k*, we write* $bv^p = bv(p)$ *and*

$$M(bv^p, c_0) = c_0 \cap \left((n+1)^{1/q})_{n=0}^\infty\right)^{-1} * \ell_\infty \tag{4.2}$$

Proof. Now $T = \Delta$ and $U = \Sigma$, hence $B_n = a_n e^{[n]}$ for the rows of the matrix B. By Proposition 2.1, $a \in M(bv(p), c_0)$ if and only if $B \in (\ell(p), c_0)$ and this is the case by [2, Theorem 1 (i)] and [11, 8.3.6, p. 123] if and only if

$$\sup_n \sum_{k=0}^\infty |b_{nk}|^{q_k} N^{-q_k} = \sup_n \sum_{k=0}^n |a_n|^{q_k} N^{-q_k} < \infty \text{ for some } N \in \mathbb{N} \setminus \{1\} \tag{4.3}$$

and $\lim_{n\to\infty} b_{nk} = \lim_{n\to\infty} a_n = 0$ for each k. Thus we have shown (15). Furthermore, the special case in (16) is an immediate consequence of (15). □

Now we give an application of Theorem 2.4.

Example 4.2 *Let* δ *be a positive real and* T *be the Cesàro matrix of order* δ*, that is*

$$t_{nk} = \begin{cases} \frac{A_{n-k}^{\delta-1}}{A_n^\delta} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \qquad (n = 0, 1, \ldots),$$

where $A_n^\delta = \binom{n+\delta}{n}$ *denotes the* n^{th} *Cesàro coefficient of order* δ*. We write* $C_\delta = c_T$*. Then* $a \in (C_\delta)^\beta$ *if and only if*

$$\sum_{k=0}^\infty A_k^\delta \left|\sum_{j=k}^\infty A_{j-k}^{-\delta-1} a_j\right| < \infty, \tag{4.4}$$

$$\sup_n \sum_{k=0}^n A_k^\delta \left|\sum_{j=n}^\infty A_{j-k}^{-\delta-1} a_j\right| < \infty \tag{4.5}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^n A_k^\delta \sum_{j=n}^\infty A_{j-k}^{-\delta-1} a_j = 0. \tag{4.6}$$

Proof. Now the matrices U and W are given by $u_{nk} = A_{n-k}^{-\delta-1} A_k^\delta$, $w_{nk} = A_k^\delta \sum_{j=n}^\infty A_{j-k}^{-\delta-1} a_j$ for $0 \le k \le n$ and $u_{nk} = w_{nk} = 0$ for $k > n$ $(n = 0, 1, \ldots)$. Furthermore, $R_k(a) = A_k^\delta \sum_{j=k}^\infty A_{j-k}^{-\delta-1} a_j$. First, $R(a) \in c^\beta = \ell_1$ if and only if condition the (18) holds. Furthermore, by [11, Theorem 1.3.6, p. 6], $W \in (c, c_0)$ if and only if $\sup_n \sum_{k=0}^\infty |w_{nk}| < \infty$ which is condition (19), $\lim_{n\to\infty} w_{nk} = 0$ which is redundant (cf. (6)), and $\lim_{n\to\infty} \sum_{k=0}^\infty w_{nk} = 0$, which is condition (20). Thus the statement holds by Theorem 2.4 and Remark 2.5. □

In the special case were $T = \Delta$, Theorem 2.3 yields [6, Theorem 2.5].

Corollary 4.3 *Let X be a normal FK space (or a BK space) with AK. We write E for the matrix with $e_{nk} = 1$ for $k \geq n$ and $e_{nk} = 0$ for $k < n$ ($n = 0, 1, \ldots$). Then $a \in (X_\Delta)^\beta$ if and only if $a \in (X^\beta)_E$ and $W \in (X, c_0)$ where $W_n = (E_n)^{[n]}(a)$ for the rows of the matrix W. This can be written in the form*

$$(X_\Delta)^\beta = \left(X^\beta \cap M(X_\Delta, c_0)\right)_E \quad \text{(cf. [6, Theorem 2.5])}. \tag{4.7}$$

Proof. If $T = \Delta$ then $R = E$ and $w_{nk} = \sum_{j=n}^{\infty} a_j$ for $0 \leq k \leq n$ and $w_{nk} = 0$ for $k > n$ ($n = 0, 1, \ldots$), that is $W_n = (E_n)^{[n]}(a)$ for $n = 0, 1, \ldots$. Thus the first statement is an immediate consequence of Theorem 2.3 (or 2.4). Applying Proposition 2.1 with X_Δ and $E(a)$ instead of X and a, we obtain $E(a) \in M(X_\Delta, c_0)$ if and only if $B \in (X, c_0)$ where $b_{nk} = E_n(a)$ for $0 \leq k \leq n$ and $b_{nk} = 0$ for $k > n$ ($n = 0, 1, \ldots$), that is $B = W$. □

Example 4.4 *We use the notations of Example 4.1 and Corollary 4.3 and write*

$$\tilde{S}(N) = \left\{ a \in \omega : \sum_{k=0}^{\infty} |a_k|^{q_k} N^{-q_k} < \infty \right\}$$

for all $N \in I\!N \setminus \{1\}$. Then

$$(bv(p))^\beta = \left(\bigcup_{N \in I\!N \setminus \{1\}} (\tilde{S}(N) \cap S(N)) \right)_E, \tag{4.8}$$

that is $a \in (bv(p))^\beta$ if and only there is an integer $N > 1$ such that

$$\sum_{k=0}^{\infty} |E_k(a)|^{q_k} N^{-q_k} < \infty \text{ and } \sup_n \sum_{k=0}^{n} |E_n(a)|^{q_k} N^{-q_k} < \infty, \tag{4.9}$$

where $E_k(a) = \sum_{j=k}^{\infty} a_j$ for $k = 0, 1, \ldots$. In the special case where $p_k = p$ for all k, we have

$$(bv^p)^\beta = \left(\ell_q \cap \left(((n+1)^{1/q})_{n=0}^{\infty} \right)^{-1} * \ell_\infty \right)_E. \tag{4.10}$$

Proof. By [3, Theorem 1], $a \in (\ell(p))^\beta$ if and only if $a \in \tilde{S}(N)$ for some $N \in \mathbb{N} \setminus \{1\}$, and as in Example 4.1, $W \in (\ell(p), c_0)$ if and only if the condition in (17) holds with a replaced by $E(a)$, that is $E(a) \in S(N)$ for some $N \in \mathbb{N} \setminus \{1\}$. This shows (22). □

Next we apply Theorem 2.4.

Example 4.5 *Let $bv = bv(e)$ and $bv_0 = bv \cap c_0$. Then $((bv_0)_\Delta)^\beta = (bs \cap ((n+1)_{n=0}^{\infty})^{-1} * \ell_\infty)_E$.*

Proof. The space bv_0 is a BK space with AK by [11, Theorem 7.3.5 (i), p. 110] which is not normal, since for the sequences x and y with $x_k = 1/(k+1)$ and $y_k = (-1)^k x_k$ we have $|y_k| \le |x_k|$ for all k and $x \in bv_0$, but $y \notin bv$. Therefore we apply Theorem 2.4 and Corollary 4.3 to obtain $a \in ((bv_0)_\Delta)^\beta$ if and only if $E(a) \in (bv_0)^\beta = bs$ (cf. [11, Theorem 7.3.5 (ii), p. 110]) and $W \in (bv_0, c_0)$. Now, by [11, Example 8.4.2A, p. 127], $W \in (bv_0, c_0)$ if and only if

$$\sup_{n,m} \left| \sum_{k=0}^{m} w_{nk} \right| = \sup_n (n+1) E_n(a) < \infty.$$

□

Next we apply Theorem 3.1 to generalize [9, Theorem 2, p. 59].

Corollary 4.6 *Let* $p = (p_k)_{k=0}^\infty, s = (s_k)_{k=0}^\infty \in \ell_\infty$, $p_k > 1$, $q_k = p_k/(p_k - 1)$ *for* $k = 0, 1, \ldots$ *and* $\ell_\infty(s) = \{x \in \omega : \sup_k |x_k|^{s_k} < \infty\}$. *Then* $A \in (bv(p), \ell_\infty(s))$ *if and only if*

(4.11) *for each* $n = 0, 1, \ldots$ *there is* $N_n > 1$ *such that*
$$\sup_m \sum_{k=0}^{m} \left| \sum_{j=m}^{\infty} a_{nj} \right|^{q_k} N_n^{-q_k} < \infty$$

and

(4.12)
$$\sup_n \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} a_{nj} N^{-1/s_n} \right|^{q_k} < \infty \text{ for some } N > 1.$$

Proof. By Theorem 3.1, $A \in (bv(p), \ell_\infty(s))$ if and only if $W^{A_n} \in (\ell(p), c_0)$ for all n which is (25), and $R^A \in (\ell(p), \ell(s))$ which is (26) by [4, Theorem 7]. □

Corollary 4.7 (cf. [9, Theorem 2, p. 59]) *Let* $s \in \ell_\infty$. *Then* $A \in (bv, \ell(s))$ *if and only if*

(4.13)
$$\sup_n \left(\sup_k \left| \sum_{j=k}^{\infty} a_{nj} \right| N^{-1} \right)^{s_n} < \infty \text{ for some } N > 1.$$

Proof. If $p = e$ in Corollary 4.6 then $bv^\beta = cs$ by [11, Theorem 7.3.5 (iii), p. 110], and so (25) becomes redundant in Corollary 4.6. Thus $A \in (bv, \ell(s))$ if and only if $R^A \in (\ell_1, \ell_\infty(s))$ by Theorem 3.1, and by [4, Theorem 5 (i)] this is (27). □

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