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SPP WITH DISCONTINOUS FUNCTION AND SPECTRAL APPROXIMATION[1]

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Abstract. We shall consider the problem $-\varepsilon^2 y" + g(x)y = f(x)$, $y(0) = a$, $y(1) = b$, where the function $f(x)$ is not continuous at some point $d \in (a, b)$. The approximate solution will be constructed inside the layers using truncated Chebyshev series.

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1. Introduction

In this paper we shall consider a self-adjoint singularly perturbed reaction-diffusion boundary value problem in one dimension with a discontinuous source term, described by

$$-\varepsilon^2 u_\varepsilon'' + a(x)u_\varepsilon = f(x), \quad x \in [0.d) \cup (d, 1]$$

$$u_\varepsilon(0) = a, \quad u_\varepsilon(1) = b \tag{1}$$

$$f(d^-) \neq f(d^+), \quad a(x) \geq \alpha > 0, \quad x \in [0, 1]$$

It was shown in [1] that the problem (1) has a unique solution $u_\varepsilon \in C^1[0, 1] \cap C^2([0, d) \cup (d, 1])$ given by

$$u_\varepsilon(x) = \begin{cases} y_1(x) + (a - y_1(0))\phi_1(x) + A\phi_2(x), & x \in [0, d) \\ y_2(x) + B\phi_1(x) + (b - y_2(1))\phi_2(x), & x \in (d, 1] \end{cases}$$

where $y_1(x)$ and $y_2(x)$ are particular solutions of the differential equations

$$-\varepsilon^2 y_1'' + a(x)y_1 = f(x), \quad x \in [0, d)$$

$$-\varepsilon^2 y_2'' + a(x)y_2 = f(x), \quad x \in (d, 1],$$

functions $\phi_1(x)$ and $\phi_2(x)$ are the solutions of the boundary value problems

$$-\varepsilon^2 \phi_1'' + a(x)\phi_1 = 0, \quad x \in (0, 1), \quad \phi_1(0) = 1, \phi_1(1) = 0$$
$$-\varepsilon^2 \phi_2'' + a(x)\phi_2 = 0, \quad x \in (0, 1), \quad \phi_2(0) = 0, \phi_2(1) = 1$$

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and constants A and B are chosen in such a way that

$$u_\varepsilon(d^-) = u_\varepsilon(d^+), \quad u'_\varepsilon(d^-) = u'_\varepsilon(d^+),$$

which means that $u_\varepsilon(x) \in C^1[0,1]$.

The jump of function $f(x)$ at the point $x = d$ implies that the solution of the reduced problem will also have a discontinuity at $x = d$, so we shall represent it as

$$z_\varepsilon(x) = \begin{cases} z_l(x), & x \in [0,d) \\ z_r(x), & x \in (d,1] \end{cases}$$

where

$$\begin{aligned} a(x)z_l(x) &= f(x), \quad x \in [0,d) \\ a(x)z_r(x) &= f(x), \quad x \in (d,1]\ . \end{aligned}$$

We can see that

$$z_l(d^-) = \frac{f(d^-)}{a(d)} \quad \text{and} \quad z_r(d^+) = \frac{f(d^+)}{a(d)}\ ,$$

which means that the solution $u_\varepsilon(x)$ has

- boundary layers at $x = 0$ and $x = 1$ and
- interior layer at $x = d$.

It is well known that the layer length is of order $O(\varepsilon)$.

2. Approximation of the solution

In order to solve problem (1) we shall divide the interval $[0,1]$ into six subintervals using division points $x_0 = c_0\varepsilon$, $x_l = d - c_l\varepsilon$, $x = d$, $x_r = d + c_r\varepsilon$ and $x_1 = 1 - c_1\varepsilon$ which choice will be discussed in the next section. Upon subintervals $[x_0, x_l]$ and $[x_r, x_1]$ exact solution is approximated by the reduced solution $z_\varepsilon(x)$, and upon the other four subintervals the approximate solution will be represented as the sum of the reduced solution and appropriate layer solution. Thus, the exact solution is approximated by

$$u(x) = \begin{cases} z_l(x) + u_0(x) & x \in [0, c_0\varepsilon] \\ z_l(x) & x \in (c_0\varepsilon, d - c_l\varepsilon) \\ z_l(x) + u_l(x) & x \in [d - c_l\varepsilon, d) \\ z_r(x) + u_r(x) & x \in (d, d + c_r\varepsilon] \\ z_r(x) & x \in (d + c_r\varepsilon, 1 - c_1\varepsilon) \\ z_r(x) + u_1(x) & x \in [1 - c_1\varepsilon, 1] \end{cases} \tag{2}$$

where functions $u_0(x)$, $u_l(x)$, $u_r(x)$ and $u_1(x)$ represent layer solutions and satisfy

$$-\varepsilon^2 u_0''(x) + a(x)u_0(x) = \varepsilon^2 z_l''(x),\ x \in [0, c_0\varepsilon], \tag{3}$$

$$u_0(0) = a - z_l(0) = A, \quad u_0(c_0\varepsilon) = 0$$

(4) $$-\varepsilon^2 u_l''(x) + a(x)u_l(x) = \varepsilon^2 z_l''(x),\ x \in [d - c_l\varepsilon, d)$$

(5) $$-\varepsilon^2 u_r''(x) + a(x)u_r(x) = \varepsilon^2 z_r''(x),\ x \in (d, d + c_r\varepsilon]$$

(6) $$u_l(d - c_l\varepsilon) = 0, \quad u_r(d + c_r\varepsilon) = 0$$

(7) $$z_l(d) + u_l(d) = z_r(d) + u_r(d),$$

(8) $$z_l'(d) + u_l'(d) = z_r'(d) + u_r'(d),$$

(9) $$-\varepsilon^2 u_1''(x) + a(x)u_1(x) = \varepsilon^2 z_r''(x),\ x \in [1 - c_1\varepsilon, 1],$$

$$u_1(1 - c_1\varepsilon) = 0, \quad u_1(1) = b - z_r(1) = B.$$

In order to evaluate layer solutions we shall use standard spectral approximation which means that we shall represent them in the form of truncated Chebyshev series. The procedure for boundary layer functions $u_0(x)$ and $u_1(x)$, which approximate solutions of the problems (3) and (9), was constructed in some earlier authors' papers (see e.g. [2]). Using the same technique, we shall carry out the procedure for interior layer functions $u_l(x)$ and $u_r(x)$. In that purpose we introduce two stretching variables t and s given by:

(10) $$x = \varphi(t) = \frac{c_l\varepsilon}{2}(t - 1) + d,$$

which transforms the interior layer subinterval $[d - c_l\varepsilon, d]$ into $[-1, 1]$, and

(11) $$x = \psi(s) = \frac{c_r\varepsilon}{2}(s + 1) + d,$$

which transforms the interior layer subinterval $[d, d + c_r\varepsilon]$ into $[-1, 1]$. Now we can represent the layer solutions in the form of truncated Chebyshev series of degree n

(12) $$u_l(x) = u_l\left(\frac{c_l\varepsilon}{2}(t - 1) + d\right) = w_l(t) = {\sum_{k=0}^{n}}{}' \beta_k T_k(t)$$

and

(13) $$u_r(x) = u_r\left(\frac{c_r\varepsilon}{2}(s + 1) + d\right) = w_r(s) = {\sum_{k=0}^{n}}{}' \gamma_k T_k(s).$$

3. Division points

In some of their earlier papers concerning standard self-adjoint SPP (see e.g. [3]), the authors have shown that the accuracy of the spectral approximation vitally depends on the choice of the division points $x_0 = c_0\varepsilon$ and $x_1 = 1 - c_1\varepsilon$. The optimal choice was derived by the use of so-called resemblance function, evaluating numbers c_0 and c_1 in terms of degree n of the appropriate truncated Chebyshev series. It is necessary to perform the same procedure to evaluate the interior division points $x_l = d - c_l\varepsilon$ and $x_r = d + c_r\varepsilon$.

Definition 1. *The resemblance function for the point $x_r = d + c_r\varepsilon$ is a polynomial $q(x)$ of degree n such that*

a) $q(x_r) = 0$ *is the minimum for* $q(x)$ *if* $z_r(d) < z_l(d)$, *and maximum if* $z_r(d) > z_l(d)$

b) $q(x)$ *is concave if* $z_r(d) < z_l(d)$, *and convex if* $z_r(d) > z_l(d)$ *for all* $x \in (d, x_r)$

c) $q(d) = \frac{z_l(d) - z_r(d)}{2} = z^*$.

Verifying the conditions from Definition 1 it can be easily proved that the following lemma holds:

Lemma 1. *Polynomial*

$$q(x) = z^* \left(\frac{d + c_r\varepsilon - x}{c_r\varepsilon} \right)^n \tag{14}$$

is the resemblance function for the point $x_r = d + c_r\varepsilon$.

The division point is evaluated from the request that resemblance function has to satisfy the appropriate differential equation at the layer point.

Lemma 2. *The number c_r which determines division point $x_r = d + c_r\varepsilon$ is given by*

$$c_r = \sqrt{\frac{n(n-1)z^*}{a(d)z^* - \varepsilon^2 z_r''(d)}} \approx \sqrt{\frac{n(n-1)}{a(d)}}.$$

Proof: We introduce (14) into the differential equation (5) and ask that it is satisfied at the layer point $x = d$, which gives us

$$n(n-1) \cdot z^* - c_r^2 a(d) \cdot z^* = -c_r^2 \varepsilon^2 z_r''(d).$$

The positive solution of the above equation is

$$c_r = \sqrt{\frac{n(n-1)z^*}{a(d)z^* - \varepsilon^2 z_r''(d)}}.$$

If ε is sufficiently small, we can neglect the term $\varepsilon^2 z_r''(d)$, so we come to

$$c_r \approx \sqrt{\frac{n(n-1)}{a(d)}}.$$

Using the same procedure for the division point $x_l = d - c_l\varepsilon$ we obtain that

$$c_l = \sqrt{\frac{n(n-1)z^*}{a(d)z^* - \varepsilon^2 z_l''(d)}} \approx \sqrt{\frac{n(n-1)}{a(d)}}.$$

4. Spectral approximation of the layer solutions

Once the division points are determined, we can proceed to construct spectral approximation for the layer solutions. In that purpose we have to determine coefficients β_k and γ_k, $k = 0, \ldots, n$ in (12) and (13).

Theorem 1. *The coefficients β_k and γ_k, $k = 0, \ldots, n$, which determine spectral approximation (12) and (13) for the layer solutions $u_l(x)$ and $u_r(x)$ of the problem (4)–(8), represent the solution of the system*

$$(15)\quad \sum_{k=0}^{n}{}' \left(-\varepsilon^2 T_k''(t_i) + a\left(\varphi(t_i)\right) T_k(t_i)\right)\beta_k = \varepsilon^2 z_l''\left(\varphi(t_i)\right),\ i = 1, \ldots, n-1$$

$$(16)\quad \sum_{k=0}^{n}{}' \left(-\varepsilon^2 T_k''(t_i) + a\left(\psi(t_i)\right) T_k(t_i)\right)\gamma_k = \varepsilon^2 z_r''\left(\psi(t_i)\right),\ i = 1, \ldots, n-1$$

with $t_i = \cos\frac{i\pi}{n}$

$$(17)\quad \sum_{k=0}^{n}{}'(-1)^k\beta_k = 0\ , \quad \sum_{k=0}^{n}{}'\gamma_k = 0$$

$$(18)\quad \sum_{k=0}^{n}{}' \left(\beta_k - (-1)^k\gamma_k\right) = z_r(d) - z_l(d)$$

$$(19)\quad \sum_{k=0}^{n}{}' k^2 \left(\beta_k - (-1)^{k+1}\gamma_k\right) = z_r'(d) - z_l'(d)$$

Proof: We introduce truncated Chebyshev series (12) and (13) into (4) and (5), apply transformation of variables (10),(11) and collocate the obtained equalities at Gauss-Lobatto nodes $t_i = \cos\frac{i\pi}{n}$, $i = 1, \ldots, n-1$, which gives us the first $2n-2$ equations (15),(16). Equations (17)-(19) are obtained introducing (12) and (13) into (6)-(8) and using that

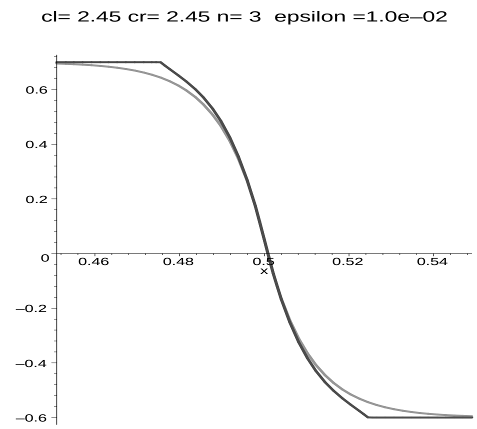
$$T_k(\pm 1) = (\pm 1)^k \quad \text{and} \quad T_k'(\pm 1) = (\pm 1)^k k^2, \quad k = 0, 1, \ldots.$$

5. Numerical example

We have tested numerical example given in [1]

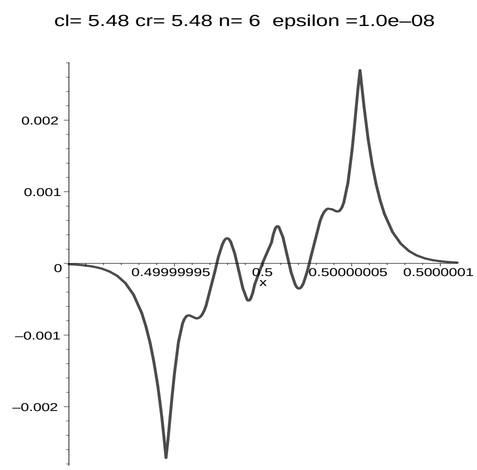
$$-\varepsilon^2 u'' + u = \begin{cases} 0.7 & x \in [0, 0.5) \\ -0.6 & x \in (0.5, 1] \end{cases}$$

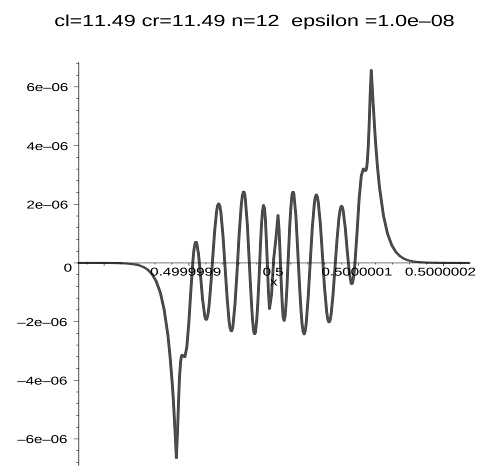
$$u(0) = 0\ , \quad u(1) = 0$$



The first picture represents the graph of the exact solution and the approximate solution constructed by the proposed procedure upon the interval $[d - 2c_l\varepsilon, d + 2c_r\varepsilon]$, which includes interior layer. Quite modest values $\varepsilon = 10^{-2}$ and $n = 3$ are chosen in purpose to distinguish the exact solution from the approximate one.

The second picture represents the error estimate (the difference between the exact solution and the approximate one) upon the same interval for $\varepsilon = 10^{-8}$ when $n = 6$ and $n = 12$. It shows a high accuracy of the presented method.





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