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ON AN INTEGRAL EQUATION

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Abstract. In the recent paper [2], the authors obtained new proofs on the existence and uniqueness of the solution of the Volterra linear equation. Applying their results, in this paper we express the exact and approximate solution of the equation in the field of Mikusiński operators, $\mathcal{F}$, which corresponds to an integro–differential equation.

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1. Introduction

In this paper we consider the Volterra linear equation

$$x(t) + \int_0^t k(t-\tau)x(\tau)\,d\tau = f(t), \tag{1}$$

the integro–differential–difference equation of the form

$$x'(t) + \lambda x(t) + \int_0^t k(t-\tau)x(\tau)\,d\tau = f(t), \quad \text{with } \ x(0) = x_0, \tag{2}$$

where k and f are continuous functions, and a general equation of the form

$$x^{(r)}(t) + \sum_{i=0}^{r-1} A_i x^{(i)} + \sum_{i=0}^{r_1} \int_0^t k_i(t-\tau)x^{(i)}(\tau)\,d\tau = f(t), \tag{3}$$

with the appropriate conditions

$$x^{(i)}(0) = x_i, \qquad i = 0, 1, \ldots, r_m - 1, \qquad r_m = \max(r_1, r), \tag{4}$$

where A_i, $i = 1, 2, \ldots, r-1$, and x_i, $i = 1, 2, \ldots, r_m - 1$, are complex numbers, while k_i, $i = 1, \ldots, r_1$, and f are continuous functions. In this paper we shall suppose that $r > r_1$, and therefore we have $r_m = r$.

In [2], the authors gave a proof that the space of continuous, complex–valued functions defined on $[0, \infty)$, denoted by $\mathcal{M}$, is a Jacobson radical algebra. This implies that for $f \in \mathcal{M}$ there exists an $\tilde{f} \in \mathcal{M}$ such that

$$f + \tilde{f} + f * \tilde{f} = 0, \tag{5}$$

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where $*$ stands for the convolution. The authors in [2] applied their theory to obtain new proofs on the existence and uniqueness of the solution of the Volterra linear equation (1) and of equation (2).

In this paper we continue the application of the results from [2], by expressing the exact solutions and also giving the approximate solutions of these equations in the field of Mikusiński operators. Also, we apply the results of [2] to equation (3) and construct the exact and the approximate solutions.

We analyze the character of the obtained solution if $k, f \in \mathcal{M}$ or f is a delta distribution.

The elements of the Mikusiński operator field, $\mathcal{F}$, are called *operators*. They are quotients of the form

$$\frac{f}{g}, \quad f \in \mathcal{C}_+,\ 0 \not\equiv g \in \mathcal{C}_+,$$

where the last division is observed in the sense of convolution.

Also, among the most important operators are the inverse operator to l, the differential operator s, while I is the identity operator. This means that it holds that $ls = I$. Also, the following relation is very important:

$$\{x^{(n)}(t)\} = s^n x - s^{n-1}x(0) - \cdots - x^{(n-1)}(0)I.$$

By $\mathcal{F}_I$ we denote the subset of $\mathcal{F}$ consisting of the elements of the form αI, for some numerical constant α, while by $\mathcal{F}_c$ we denote the subset of $\mathcal{F}$ consisting of the operators representing continuous functions.

The operators can be compared only if they are from $\mathcal{F}_c$. So, for the two operators $a = \{a(t)\}$ and $b = \{b(t)\}$ from $\mathcal{F}_c$ we define

$$a \le b \quad \text{iff} \quad a(t) \le b(t), \quad \text{for each} \ \ t \ge 0$$

(see [3], p. 237). Clearly, $a = b$, $a, b \in \mathcal{F}_c$ iff $a(t) = b(t)$, $t \ge 0$.

The absolute value of the operators from $\mathcal{F}_c$ are only considered. If $a \in \mathcal{F}_c$, then the absolute value of the operator is $|a| = \{|a(t)|\}$.

If $a, b \in \mathcal{F}_c$, then it holds

$$|ab| \le |a||b|, \quad |ab| \le_T MNl,$$

where $M = \max_{0 \le t \le T}|a(t)|$ and $N = \max_{0 \le t \le T}|b(t)|$.

If the operator a is not from $\mathcal{F}_c$, but there exists an operator k such that $ak \in \mathcal{F}_c$, then we consider the absolute value of the operator ka, i.e., $|ka|$.

1.1. The Voltera type equations

In the field of Mikusiński operators the equation

$$x + x \cdot k = f, \tag{6}$$

corresponds to the Volterra equation (1). In (6), x is the unknown operator, while k and f are operators representing continuous functions $k = k(t)$ and $f = f(t)$.

The field of Mikusiński operators has very good algebraic properties, which also means that the usual addition and multiplication with operators can be treated in the same way as with real numbers. Let us remember that multiplication in $\mathcal{F}$ corresponds to the convolution of continuous functions.

The solution of equation (6) in $\mathcal{F}$ has the form

$$x = \frac{f}{I+k} = f\sum_{i=0}^{\infty}(-1)^i k^i = f + f\cdot\sum_{i=1}^{\infty}(-1)^i k^i. \tag{7}$$

If the operators f and k are operators representing continuous functions, then the solution of equation (6) given by (7) represents a continuous function.

From the results of [2] it follows that if k represents a continuous function, then it has a quasi–inverse $\tilde{k}$ in $\mathcal{F}_c$ (see (5)) such that we have

$$x = f + f * \tilde{k}.$$

From

$$x = f + f\cdot\sum_{i=1}^{\infty}(-1)^i k^i,$$

it follows that the operator $\tilde{k}$ (the quasi–inverse to k) in the field of Mikusiński operators has the following form

$$\tilde{k} = \sum_{i=1}^{\infty}(-1)^i k^i,$$

and it represents a continuous function.

If f is a delta distribution in the Volterra equation (1), then in the operator equation (6) it represents an identical operator I and from (7) it follows that the solution of the operator equation (6) has the form

$$x = I + \sum_{i=1}^{\infty}(-1)^i k^i.$$

It does not represent a continuous function, but is a sum of an operator from $\mathcal{F}_I$ and a continuous function.

If, however, in the operator equation (6) f is neither an operator from $\mathcal{F}_c$, nor from $\mathcal{F}_I$, then the solution of equation (6) does not represent a continuous function. For example, if in equation (6) $f = s^p$, $p \in \mathbf{N}$, s is the differential operator, then the solution has the form

$$\begin{aligned} x &= \frac{s^p}{I+k} = s^p + s^p \sum_{i=1}^{\infty} (-1)^i k^i \\ &= s^p - s^p k + s^p k^2 + \cdots + s^p k^{m-1} + s^p \sum_{i=m}^{\infty} (-1)^i k^i, \end{aligned}$$

where the integer m is chosen such that the operator $s^p k^{m-1}$ does not represent a continuous function and the operator $s^p k^m$ represents a continuous function. Since the series $\sum_{i=r}^{\infty} (-1)^i k^i$ converges, the operator $s^p \sum_{i=r}^{\infty} (-1)^i k^i$ represents a continuous function.

1.2. The solution of integro–differential equations

In the field of Mikusiński operators the equation

$$sx + \lambda x + x \cdot k = f + x_0 I \tag{8}$$

corresponds to equation (2) with the appropriate condition. In (8), x is an unknown operator, s is the differential operator, I is the identical operator, while k and f are operators representing the functions k and f. Note that (8) is an algebraic equation in $\mathcal{F}$, which can be written as

$$x(s + \lambda I + k) = f + x_0 I. \tag{9}$$

Proposition 1. *If the operators f and k are operators representing continuous functions, then the solution of equation (9) exists, it is unique and it represents a continuous function.*

Proof. The solution of equation (9) has the form

$$x = \frac{f + x_0 I}{s + \lambda I + k} = \frac{lf + lx_0}{I + (\lambda l + lk)} = (lf + lx_0) \sum_{i=0}^{\infty} (-1)^i (l\lambda + kl)^i, \tag{10}$$

where l is the integral operator. The infinite series mentioned above converges in the field $\mathcal{F}$, because λl is the operator representing the constant function λ, and the operator $lk = \{\int_0^t k(\tau)\, d\tau\}$ also represents a continuous function. □

In view of paper [2], we shall explicitly express the operator $\tilde{K}$. If we denote by $K = \lambda l + kl$ and $F = x_0 l + fl$, then the operator $\tilde{K}$ satisfying

$$x = F + F\tilde{K}$$

has the form

$$\tilde{K} = \sum_{i=1}^{\infty} (-1)^i (\lambda l + lk)^i.$$

The operator $\tilde{K}$ represents a continuous function.

The problem (3), (4), in the field of Mikusiński operators, corresponds to the equation

$$s^r x + x\sum_{i=0}^{r-1} A_i s^i + x\sum_{i=1}^{r_1} k_i \cdot s^i = f + \sum_{i=0}^{r-1} x_i s^{r-1-i} + \sum_{j=1}^{r-1} A_j \left(\sum_{i=0}^{j-1} x_i s^{j-1-i}\right) \sum_{j=1}^{r_1} k_j \left(\sum_{i=0}^{j-1} x_i s^{j-1-i}\right), \tag{11}$$

where x_i, $i = 1, \ldots, r_m = \max(r_1, r)$.

The equation (11) can be written as

$$x\left(s^r + \sum_{i=0}^{r-1} A_i s^i + \sum_{i=0}^{r_1} k_i s^i\right) = f + \sum_{i=0}^{r-1} x_i s^{r-1-i} + \sum_{j=1}^{r-1} A_j \left(\sum_{i=0}^{j-1} x_i s^{j-1-i}\right) + \sum_{j=1}^{r_1} k_j \left(\sum_{i=0}^{j-1} x_i s^{j-1-i}\right).$$

Then the solution of equation (11) is of the form

$$x = \frac{f + \sum_{i=0}^{r-1} x_i s^{r-1-i} + \sum_{j=1}^{r-1} A_j\left(\sum_{i=0}^{j-1} x_i s^{j-i-1}\right) + \sum_{j=1}^{r_1} k_j\left(\sum_{i=0}^{j-1} x_i s^{j-1-i}\right)}{s^r + \left(\sum_{i=0}^{r-1} A_i s^i + \sum_{i=0}^{r_1} k_i s^i\right)} \tag{12}$$

$$= \frac{l^r f + \sum_{i=0}^{r-1} x_i l^{1+i} + \sum_{j=1}^{r-1} A_j\left(\sum_{i=0}^{j-1} x_i l^{r-j+1+i}\right) + \sum_{j=1}^{r_1} k_j\left(\sum_{i=0}^{j-1} x_i l^{r-j+1+i}\right)}{I + \sum_{i=0}^{r-1} A_i l^{r-i} + \sum_{i=0}^{r_1} k_i l^{r-i}}$$

$$= \left(l^r f + \sum_{i=0}^{r-1} x_i l^{1+i} + \sum_{j=1}^{r-1} A_j\left(\sum_{i=0}^{j-1} x_i l^{r-j+1+i}\right) + \sum_{j=1}^{r_1} k_j\left(\sum_{i=0}^{j-1} x_i l^{r-j+1+i}\right)\right) \cdot \sum_{m=0}^{\infty} (-1)^m \left(\sum_{i=0}^{r-1} A_i l^{r-i} + \sum_{i=0}^{r_1} k_i l^{r-i}\right)^m.$$

In view of [2], let us denote the solution of equation (11), given by (12), as

$$x = L + L\tilde{M}, \tag{13}$$

where

$$\begin{aligned} L &= l^r f + \sum_{i=0}^{r-1} x_i l^{1+i} + \sum_{j=1}^{r-1} A_j \left(\sum_{i=0}^{j-1} x_i l^{r-j+1+i} \right) + \sum_{j=1}^{r_1} k_j \left(\sum_{i=0}^{j-1} x_i l^{r-j+1+i} \right) \\ \tilde{M} &= \sum_{m=1}^{\infty} (-1)^m \left(\sum_{i=0}^{r-1} A_i l^{r-i} + \sum_{i=0}^{r_1} k_i l^{r-i} \right)^m . \end{aligned}$$

From the previous expressions we have the following

Proposition 2. *If in (13) it holds that $r > r_1$, and the operators k_i, $i = 0, \ldots, r_1$, represent continuous functions, then the operator $\tilde{M}$ represents a continuous function.*

Proof. The operators l^{r-i}, $i = 0, \ldots, r-1$, l^{r-i}, $i = 0, \ldots, r_1$, appearing in $\tilde{M}$, represent continuous functions, because $r > r_1$, and k_i, $i = 0, \ldots, r_1$, also represent continuous functions. Therefore the series $\tilde{M}$ converges in $\mathcal{F}$ to an operator representing a continuous function. □

Corollary 1. *The solution of equation (11) represents a continuous function if $r > r_1$, and f and k_i, $i = 1, \ldots, r_1$, represent continuous functions.*

Proof. Each operator appearing in L represents a continuous function and therefore the solution $x = L + L\tilde{M}$ also represents a continuous function. □

Corollary 2. *If in equation (11) $f = I$, meaning that in (3) f represents a δ distribution and $r > r_1$, k_i, $i = 1, \ldots, r_1$, represent continuous functions, then the solution also represents a continuous function.*

2. The approximate solution

Let us remark that in the exact solutions of equations (6), (8) and (11), given by (7), (10) and (12), respectively, in the field of Mikusiński operators the infinite sum of the operators of the form k^i, $i = 1, 2, \ldots$, appeared. The operator k^i represents the $i-$times applied convolution. It turns out that the obtained exact solution, given as an infinite series, is inconvenient for computer calculation. Thus, in this paper we give the approximate solutions of these equations, such that instead of the infinite sum we use a finite sum. In particular, the approximate solution of equation (9) has the form:

$$x_n = (lf + lx_0) \sum_{i=0}^{n} (-1)^i (l\lambda + kl)^i. \tag{14}$$

If f and k represent continuous functions, then the approximate solution of equation (9), given by equation (14), represents a continuous function.

In the field $\mathcal{F}$ the approximate solutions make the sequence of approximate solutions $(x_n)_{n\in\mathbf{N}}$. This sequence converges to the exact solution x in the field of Mikusiński operators in the type I convergence (see [3], p. 155). However, since we deal with continuous functions, we can say that the functional sequence $(x_n(t))_{n\in\mathbf{N}}$ (corresponding to the operators $(x_n)_{n\in\mathbf{N}}$ in $\mathcal{F}$), given by (14), converges uniformly to the function $x(t)$ (corresponding to the operator x, i.e., to the exact solution given by (10)).

The approximate solution of equation (11) has the form:

$$\begin{aligned} x_n \;=\; & \left(l^r f + \sum_{i=0}^{r-1} x_i l^{1+i} + \sum_{j=1}^{r-1} A_j \left(\sum_{i=0}^{j-1} x_i l^{r-j+1+i}\right)\right. \\ & \left. + \sum_{j=1}^{r_1} k_j \left(\sum_{i=0}^{j-1} x_i l^{r-j+1+i}\right)\right) \cdot \sum_{m=0}^{n} (-1)^m \left(\sum_{i=0}^{r-1} A_i l^{r-i} + \sum_{i=0}^{r_1} k_i l^{r-i}\right)^m . \end{aligned}$$

2.1. The error of approximation

In order to estimate the error of approximation, let us consider the absolute value of the difference between the exact solution (8) and the approximate one (10). Note that they are both from $\mathcal{F}_c$.

$$\begin{aligned} |x - x_n| &= |(lf + lx_0)\sum_{i=0}^{\infty}(-1)^i k^i - (lf + lx_0)\sum_{i=0}^{n}(-1)^i k^i| \\ &= |(lf + lx_0)\cdot(-1)^n k^n \sum_{i=0}^{\infty}(-1)^{i+1}k^{i+1}|. \end{aligned}$$

If $0 \le t \le T$, and $f_M = \max\limits_{0\le t\le T}|F(t)|$, where $\{F(t)\} = lf + lx_0$, and $k_M = \max\limits_{0\le t\le T}|k(t)|$, then we have

$$\begin{aligned} |x - x_n| &\le_T \quad f_M k_M^n l^{n+1} \sum_{i=0}^{\infty} k_M^{i+1} l^{i+1} \\ &\le_T \quad f_M k_M^{n+1} \frac{T^n}{n!} \sum_{i=0}^{\infty} k_M^i \frac{T^i}{i!} l = f_M k_M^{n+1} \frac{T^n}{n!} e^{k_M T} l. \end{aligned}$$

Let us remark that for small values of T the error of approximation is small, and with increasing T the error of approximation is also increasing.

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