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SOME PROPERTIES OF THE SUM OF LINEAR OPERATORS IN THE NON–DIFFERENTIAL CASE

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Abstract. In this paper we recall some results concerning an application of the sum of linear operators in the infinite matrix theory. Then, we give several extensions of these results in order to obtain new properties of infinite linear systems.

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1. Introduction

The theory of the sum of operators is well known, and has been studied by many authors such as Da Prato and Grisvard [1,2], Furman [3], R. Labbas and B. Terreni [7,8]. It can also be found in the work of de Malafosse [15], giving an application of the sum of operators in the infinite matrix theory in the commutative case. The aim is to study the equation

$$A.X + B.X - \lambda X = Y, \quad \lambda > 0 \tag{1}$$

in a Banach space E, where Y is given in E and A, B are two closed linear operators with domains $D(A)$, $D(B)$ included in E. This work extends the results obtained in the paper [6], entitled: "An application of the sum of linear operators in infinite matrix theory", where E is not reflexive. Here, equation (1) is regarded as an infinite linear system in the space $E = l^{\infty}$. $A + B$ is considered as the sum of two particular infinite matrices defined respectively on $D(A) = s_{(1/a_n)_n}$ and $D(B) = s_{1/\beta}$. In our case, it has been proved that the operators $(-A)$ and $(-B)$ are generators of analytic semigroups. The relative boundedness with respect A or B being not satisfied, the classical perturbation theory given by Kato [4] or Pazy [18], cannot be applied. The choice of the two infinite matrices is motived by the resolution of a class of non-symmetric linear infinite systems. Note that some results concerning the infinite matrices of operators are given in Maddox [9].

This work is organized as follows. In Section 2 are recalled the definitions and properties of some Banach spaces of infinite matrices used in R. Labbas and B. de Malafosse [5] and de Malafosse [11, 12, 14, 16]. We also give the main

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results of the sum-strategy as in Labbas-Terreni [7] and define the two infinite matrices A and B regarded as two unbounded linear operators on $E = l^{\infty}$ and study their sum. In Section 3 are used the regularity property of A in order to give results on the sum of operators in the non-differential case. Further, we apply the previous results to new matrices obtained from the matrix $A+B+\lambda I$. Then, we deal with the linear infinite system $(A+B+\lambda I).X = Y$ for $\lambda \geq 0$. Finally, we study the properties of ${}^{t}(A+B+\lambda I)$ when the trace of the kernull of the operator $A+B+\lambda I$ on $D(A) \bigcap D(B)$ is not reduced to $\{0\}$.

2. Recall of definitions and properties

2.1. Spaces S_c and s_c

For a sequence $c = (c_n)$, where $c_n > 0$ for every integer $n \geq 1$, we define the Banach algebra

$$S_c = \left\{ M = (a_{nm})_{n,m \geq 1} \mid \sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| \frac{c_m}{c_n} \right) < \infty \right\}, \tag{2}$$

normed by

$$\|M\|_{S_c} = \sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| \frac{c_m}{c_n} \right),$$

see [10, 12, 14, 16, 17]. We also define the Banach space s_c of one-row matrices by

$$s_c = \left\{ X = (x_n)_n \mid \sup_{n} \left(\frac{|x_n|}{c_n} \right) < \infty \right\}, \tag{3}$$

normed by

$$\|X\|_{s_c} = \sup_{n} \left(\frac{|x_n|}{c_n} \right). \tag{4}$$

If $c = (c_n)_n$, and $c' = (c'_n)_n$ are two sequences such that $0 < c_n \leq c'_n$ $\forall n$, then:

$$s_c \subset s_{c'}.$$

A very useful particular case is the one when $c_n = r^n$, $r > 0$. We denote then by S_r and s_r the spaces S_c and s_c. When $r = 1$ we obtain the space of the bounded sequences $l^{\infty} = s_1$.

If $\|I - M\|_{S_c} < 1$ we shall say that A satisfies the condition Γ_c. If $c = (r^n)$, Γ_c is replaced by Γ_r.

S_c being a unital algebra, we have the useful result:

if M satisfies the condition Γ_c, M is invertible in the space S_c and for every $B \in s_c$ the equation $MX = B$ admits one and only one solution in s_c given by

$$X = \sum_{n=0}^{\infty} (I - M)^n B. \tag{5}$$

Similarly, we define the Banach algebra S_r with unit element $I = (\delta_{nm})_{n,m\geq 1}$, (with $\delta_{nm} = 0$ if $n \neq m$, and $\delta_{nn} = 1$) by

$$S_r = \left\{ M = (a_{nm})_{n,m\geq 1} \mid \sup_n \left(\sum_m |a_{nm}| \, r^{m-n} \right) < \infty \right\} \tag{6}$$

normed by

$$\|M\|_{S_r} = \sup_n \left(\sum_m |a_{nm}| \, r^{m-n} \right), \tag{7}$$

see [10-13]. Let us recall that the product of two matrices of S_r belongs to this space and

$$\forall M \in S_r \quad \forall X \in s_r : MX \in s_r$$

with $\|MX\|_{s_r} \leq \|M\|_{S_r} \|X\|_{s_r}$.

2.2. Sum of linear operators

We recall here some results given in Da Prato-Grisvard [1] and Labbas-Terreni [7]. E being a Banach space, we consider two closed linear operators A and B, whose domains are $D(A)$ and $D(B)$ included in E. For every $X \in D(A) \bigcap D(B)$ we then define their sum $SX = AX + BX$.

The spectral properties of A and B are:

$$(H_1) \begin{cases} \exists C_A, C_B > 0, \ \varepsilon_A, \varepsilon_B \in]0, \pi[\text{ such that} \\ i) \ \rho(A) \supset \sum_A = \{ z \in C / |Arg(z)| < \pi - \varepsilon_A \} \\ \left\| (A - zI)^{-1} \right\|_{L(E)} \leq \dfrac{C_A}{|z|}, \forall z \in \sum_A - \{0\}, \\ ii) \ \rho(B) \supset \sum_B = \{ z \in C / |Arg(z)| < \pi - \varepsilon_B \} \\ \left\| (B - zI)^{-1} \right\|_{L(E)} \leq \dfrac{C_B}{|z|}, \forall z \in \sum_B - \{0\}, \\ iii) \ \varepsilon_A + \varepsilon_B < \pi \end{cases}$$

Here, we are not in the commutative case, that is:

$$(A - zI)^{-1} (B - z'I)^{-1} - (B - z'I)^{-1} (A - zI)^{-1} = \left[(A - zI)^{-1}, (B - z'I)^{-1} \right],$$

is not equal to zero for all $z \in \rho(A)$ and $z' \in \rho(B)$. Furthermore, the density of $D(A)$ and $D(B)$ being not true, we must assume that (see Labbas-Terreni [7], [8])

$$(H_2) \begin{cases} \exists C > 0, \ k \in N, \ (\xi_i)_{1\leq i\leq k}, \ (\eta_i)_{1\leq i\leq k} \text{ such that} \\ \forall i \geq 1 : 0 \leq 1 - \xi_i < \eta_i \leq 2 \text{ and} \\ \left\| \mu A (A - \lambda I)^{-1} . \left[A^{-1}; (B + \mu I)^{-1} \right] \right\|_{L(E)} \leq K(\varepsilon_A, \varepsilon_B) \sum_{i=1}^{k} |\lambda|^{-\xi_i} |\mu|^{-\eta_i} \\ \text{for } |\lambda|, \ |\mu| \to \infty; \ \lambda \in \rho(A), \ \mu \in \rho(B). \end{cases}$$

Now introduce the real interpolation space between $D(A)$ and E, defined for all $\sigma \in]0,1[$ by

$$D_A(\sigma,\infty) = \left\{ X \in E \,/ \sup_{Z\in\Gamma} \left\| z^\sigma A (A - zI)^{-1} X \right\|_E < \infty \right\}, \tag{8}$$

Γ being a simple infinite sectorial curve lying in $\rho(A - \lambda I) \bigcap \rho(-B)$. Here we have

$$D_A(\sigma,\infty) = \left\{ X \in E \,/ \sup_{t>0} \left\| t^\sigma A (A + tI)^{-1} X \right\|_E < \infty \right\}.$$

In the same way $D_B(\sigma,\infty)$ denotes the interpolation space between $D(B)$ and E . Now we can express the main result given in [8], where

$$\delta = \min_{1\le i\le k} (\xi_i + \eta_i - 1) > 0.$$

Theorem 1. *Suppose that* (H_1) *and* (H_2) *are satisfied. There exists* λ^* *such that* $\forall\lambda \ge \lambda^*$ *and* $\forall Y \in D_A(\sigma,\infty)$ *equation* $[A + B - \lambda I].X = Y$ *has a unique solution* X_0 *in the space* $D(A) \cap D(B)$ *such that*
i) $(A - \lambda I) X_0 \in D_A(\theta,\infty) \quad \forall\theta \in]0, \min(\sigma,\delta)[$,
ii) $BX_0 \in D_A(\theta,\infty) \quad \forall\theta \in]0, \min(\sigma,\delta)[$,
iii) $(A - \lambda I) X_0 \in D_B(\theta,\infty) \quad \forall\theta \in]0, \min(\sigma,\delta)[$.

2.3. Definition of operators A and B

As in [6], we apply the results of the previous subsections to particular matrices.

Let A be the infinite matrix:

$$\begin{bmatrix} a_1 & b_1 & & O \\ & \cdot & \cdot & \\ O & & a_n & b_n \\ & & & \cdot \end{bmatrix}, \tag{9}$$

where (a_n) and (b_n) satisfy

$$\begin{cases} \text{i) } a_n > 0 \ \forall n, \ (a_n) \text{ is strictly increasing, and } \lim_{n\to\infty} a_n = \infty \\ \text{ii) } \exists M_A > 0 \text{ such that: } |b_n| \le M_A \text{ for all } n. \end{cases} \tag{10}$$

In the same way we denote by B the lower triangular matrix

$$\begin{bmatrix} \beta_1 & & & O \\ \cdot & \cdot & & \\ & \gamma_n & \beta_n & \\ O & & & \cdot \end{bmatrix}, \tag{11}$$

where $(\beta_n)_n$ and $(\gamma_n)_n$ satisfy:

$$\text{(12)} \qquad \begin{cases} \text{i) } \beta_n > 0 \quad \forall n, \text{ and } \beta_{2n} \to L, \\ \text{ii) } \left(\beta_{2n+1}/a_{2n+1}\right)_n \to \infty \ , \\ \text{iii) } \exists M_B > 0 \text{ such that } |\gamma_n| \le M_B \text{ for all } n. \end{cases}$$

A is defined on $D(A) = s_{(1/a_n)_n}$ and B is defined on $D(B) = s_{(1/\beta_n)_n}$, these spaces being included in $E = l^\infty = s_1$. We deduce from (12) i), ii) that $D(A)$ is not included in $D(B)$ and $D(B)$ is not included in $D(A)$. In [6], the following results are proved:

Proposition 2. *In the Banach space E the two linear operators A and B are closed and satisfy*

i) $D(A) = s_{(1/a_n)_n} = \{X = (x_n) \,/\, a_n x_n = O(1) \ \ (n \to \infty)\}$,

ii) $D(B) = s_{(1/\beta_n)_n}$,

iii) $\overline{D(A)} \neq s_1$, $\overline{D(B)} \neq s_1$.

iv) There exist numbers ε_A, $\varepsilon_B > 0$ (with $\varepsilon_A + \varepsilon_B < \pi$) such that

$$\left\| (A - \lambda I)^{-1} \right\|_{L(s_1)} \le \frac{M}{|\lambda|}, \ \forall \lambda \neq 0 \text{ and } |Arg(\lambda)| \ge \varepsilon_A,$$

$$\left\| (B + \mu I)^{-1} \right\|_{L(s_1)} \le \frac{M}{|\mu|}, \ \forall \mu \neq 0 \text{ and } |Arg(\mu)| \le \pi - \varepsilon_B.$$

Now let us consider the following additional assumption on A

$$\text{(13)} \qquad \sup_{n \ge 1} \left(\frac{|b_{n-1}| \, \beta_n}{a_n} \right) < \infty.$$

Then we have [6]

Proposition 3. *Under (10), (12), and (13) there exists a constant $K(\varepsilon_A, \varepsilon_B) > 0$ such that*

$$\left\| \mu A (A - \lambda I)^{-1} . \left[A^{-1}; (B + \mu I)^{-1} \right] \right\|_{L(s_1)} \le K(\varepsilon_A, \varepsilon_B) \left[\frac{1}{|\lambda| \, |\mu|} + \frac{1}{|\mu|^2} \right],$$

$\forall \lambda \neq 0$ *such that* $|Arg(\lambda)| \ge \varepsilon_A$ *and* $\forall \mu \neq 0$ *such that* $|Arg(\mu)| \le \pi - \varepsilon_B$.

As we deal with the non-commutative case, we must use the interpolation space defined in (8). It has been proved in [6] that for all $\theta \in]0, 1[$

$$D_A(\theta, \infty) = s_{(1/a_n^\theta)} = \left\{ X = (x_n) \,/ \sup_{t > 0} \left\| t^\theta A (A + tI)^{-1} X \right\|_{s_1} < \infty \right\}.$$

Then we can assert the following result:

Theorem 4. *A and B satisfy the hypotheses (10), (12), (13). For any* $\theta \in]0,1[$ *there exists* λ^* *such that* $\forall \lambda \geq \lambda^*$ *and* $\forall Y \in s_{(1/a_n^\theta)}$*, the linear infinite system* $(-A-B-\lambda I).X = Y$ *has a unique solution* X_0 *in the space* $D(A) \cap D(B) = s_{(1/a_n)} \cap s_{(1/\beta_n)}$ *such that*

i) $(A+\lambda I) X_0 \in s_{(1/a_n^\theta)_n}$,

ii) $BX_0 \in s_{(1/a_n^\theta)_n}$.

Remark 1. *It is easy to see that* $s_{(1/a_n)_n} \cap s_{(1/\beta_n)_n} = s_d$*, where* $d = (d_n)_n$ *is defined by*

$$d_n = \begin{cases} \dfrac{1}{a_{2k}} \text{ if } & n = 2k, \\ \dfrac{1}{\beta_{2k+1}} \text{ otherwise.} \end{cases}$$

Corollary 5. *Conditions i) and ii) in Theorem 4 are equivalent to the condition* $AX_0 \in s_{(1/a_n^\theta)_n}$.

Proof. First we see that for every $\theta \in]0,1[$, $s_d \subset s_{(1/a_n^\theta)_n}$. In fact, take $X = (x_n)_n \in s_d$. Since we have (10) i), $x_{2n} = O(1/a_{2n})$ implies $x_{2n} = O\left(1/a_{2n}^\theta\right)$ as $n \to \infty$. And from (12) ii), $x_{2n+1} = O(1/\beta_{2n+1})$ implies $x_{2n+1} = O\left(1/a_{2n+1}^\theta\right)$ as $n \to \infty$. Then $X \in s_{(1/a_n^\theta)}$. We deduce that $Z_0 = (A+\lambda I) X_0 \in s_{(1/a_n^\theta)}$ is equivalent to

$$AX_0 = Z_0 - \lambda X_0 \in s_{(1/a_n^\theta)}.$$

Elsewhere i)⇔ii), since $BX_0 = (A+B+\lambda I).X_0 - Z_0 \in s_{(1/a_n^\theta)}$. □

3. New properties of the operator $A+B$, in the non–differential case

3.1. Consequence of the regularity property

More precisely, from i) in Theorem 4 we have:

Corollary 6. *The unique solution* $X_0 = \left(x_n^0\right)$ *satisfies the following property*

$$\forall \sigma \in]0,\theta[\qquad x_n^0 = \frac{o(1)}{a_n^{\sigma+1}} \qquad (n \to \infty).$$

Proof. From Corollary 5 we deduce that $AX_0 \in s_{(1/a_n^\theta)}$ implies that there exists a real $K > 0$ such that $\forall \sigma \in]0,\theta[$, $\forall n$

$$a_n^\sigma \left| a_n x_n^0 + b_n x_{n+1}^0 \right| \leq \frac{K}{a_n^{\theta-\sigma}}.$$

Then

$$a_n^\sigma \left(a_n x_n^0 + b_n x_{n+1}^0 \right) = o(1) \qquad (n \to \infty). \tag{14}$$

On the other hand, (10) i) implies that there exists $K > 0$ such that

$$\left|b_n a_n^{\sigma} x_{n+1}^0\right| \leq \begin{cases} \dfrac{K a_{2k}^{\sigma}}{\beta_{2k+1}} & \text{if } n = 2k, \\ \dfrac{K a_{2k+1}^{\sigma}}{a_{2k+2}} & \text{if } n = 2k+1. \end{cases} \tag{15}$$

Using (10) i) and (12) ii) we deduce that $b_n a_n^{\sigma} x_{n+1}^0 = o(1)$ $(n \to \infty)$, and from (15) we conclude that $a_n^{\sigma+1} x_n^0 = o(1)$ as n tends to infinity. □

3.1.1. Numerical application

Assume here that $a_n = \alpha^n$ with $\alpha > 1$ and consider the matrix $M_\lambda(t_1) = (A + B + \lambda I)(t_1)$ obtained from $A + B + \lambda I$, by adding the supplementary row $t_1 = \left(0, \alpha^{2\omega}, 0, \alpha^{4\omega}, ...\right)$ with $1 \leq \omega < 2$. $Y(u)$ is the matrix obtained from Y by adding the number u. From the regularity property we get

$$\forall \sigma \in]0, \theta[\qquad x_{2n}^0 = \frac{o(1)}{\alpha^{2n(\sigma+1)}} \quad (n \to \infty),$$

for any $\theta \in]0, 1[$. We deduce that $\exists C > 0$ such that $\alpha^{2\omega m}\left|x_{2m}^0\right| \leq C/\alpha^{2m(\sigma+1-\omega)}$. Thus, the series $\sum_m \alpha^{2\omega m}\left|x_{2m}^0\right|$ is convergent, since for a given $\omega \in [1, 2[$ one can associate $\theta \in]0, 1[$ and $\sigma \in]0, \theta[$, such that $\sigma + 1 - \omega > 0$. Then the product $M_\lambda(t_1) X_0$ exists and belongs to the space $s_{(1/\alpha^{n\theta})}$. The equation $M_\lambda(t_1) X = Y(u)$, where $Y \in s_{(1/\alpha^{n\theta})}$ admits a unique solution in s_d if and only if $u = \sum_m \alpha^{2\omega m} x_{2m}^0$. Notice that the property: $X_0 \in D(A) \cap D(B) = s_d$ is not sufficient to assure the convergence of the series $\sum_m \alpha^{2\omega m} x_{2m}^0$.

3.2. Expression of the solution in the Banach space s_d

In this section, we use only the hypotheses (10), (12) and the following supplementary condition

$$\left(\frac{a_{2n}}{a_{2n-1}}\right) \in s_1. \tag{16}$$

Then we obtain the expression of the solution X_0 in Theorem 4 for any second member $Y \in s_1$. So we shall see that there exists $Y \in s_1 - s_{(1/a_n^{\theta})}$ such that the equation $(A + B + \lambda I) X = Y$ admits a solution in the space $D(A) \cap D(B)$ satisfying the property $AX_0 \notin s_{(1/a_n^{\theta})}$. In the case we shall study, this means that $(A + B + \lambda I)$ is a bijection from $D(A) \cap D(B)$ into l^{∞}.

For the following results we shall put $\nu_n = a_n + \beta_n$ for short.

Proposition 7. *There exists $\lambda^* > 0$ such that $\lambda \geq \lambda^*$ implies that for every $Y \in s_1$ equation*

$$(A + B + \lambda I) X = Y \tag{17}$$

admits in $D(A) \cap D(B) = s_d$ *a unique solution which can be written in the form*

$$X_0 = \sum_{n=0}^{\infty} \left[I - D_\lambda (A + B + \lambda I)\right]^n D_\lambda Y, \tag{18}$$

where $D_\lambda = \left(\frac{\delta_{nm}}{\nu_n + \lambda}\right)_{n,m \geq 1}$.

Proof. (17) is equivalent to

$$D_\lambda \left[(A + B + \lambda I) X\right] = D_\lambda Y$$

itself equivalent to

$$\left[D_\lambda (A + B + \lambda I)\right] X = D_\lambda Y. \tag{19}$$

We see that

$$D_\lambda (A + B + \lambda I) = \begin{pmatrix} 1 & \frac{b_1}{\nu_1+\lambda} & & & \\ \frac{\gamma_2}{\nu_2+\lambda} & 1 & \frac{b_2}{\nu_2+\lambda} & & O \\ & . & . & . & \\ O & & \frac{\gamma_n}{\nu_n+\lambda} & 1 & \frac{b_n}{\nu_n+\lambda} \\ & & & . & . \end{pmatrix}.$$

We get

$$\left\|D_\lambda (A + B + \lambda I) - I\right\|_{S_\delta} = \sup\left(\sup_{n\geq 1} (\mu_n), \sup_{n \geq 1} (\mu'_n)\right),$$

where

$$\mu_n = \left|\frac{\gamma_{2n}}{\nu_{2n} + \lambda}\right| \frac{a_{2n}}{\beta_{2n-1}} + \left|\frac{b_{2n}}{\nu_{2n} + \lambda}\right| \frac{a_{2n}}{\beta_{2n+1}}, \tag{20}$$

and

$$\mu'_n = \left|\frac{\gamma_{2n+1}}{\nu_{2n+1} + \lambda}\right| \frac{\beta_{2n+1}}{a_{2n}} + \left|\frac{b_{2n+1}}{\nu_{2n+1} + \lambda}\right| \frac{\beta_{2n+1}}{a_{2n+2}}. \tag{21}$$

We see that the sequence defined by

$$\rho_n = |\gamma_{2n}| \frac{a_{2n}}{\beta_{2n-1}} + |b_{2n}| \frac{a_{2n}}{\beta_{2n+1}},$$

is bounded. Indeed, from (12) and (16) ii) the sequence defined by

$$\frac{a_{2n}}{\beta_{2n-1}} = \frac{a_{2n}}{a_{2n-1}} \frac{a_{2n-1}}{\beta_{2n-1}}$$

is bounded. It is the same for the sequence $\left(\frac{a_{2n}}{\beta_{2n+1}}\right)$, since (a_n) is strictly increasing. Taking $\lambda > \xi = \sup_{n\geq 1} (\rho_n)$, we have $\sup_{n \geq 1} (\mu_n) < 1$. Further, we deduce from (10) and (12) that there exists an integer N such that

$$\sup_{n \geq N+1} \left(\frac{|\gamma_{2n+1}|}{a_{2n}} + \frac{|b_{2n+1}|}{a_{2n+2}}\right) < 1,$$

which implies that $\sup_{n\geq N+1}(\mu'_n) < 1$. Let now

$$\xi'_N = M_B \sup_{n\leq N}\left(\frac{\beta_{2n+1}}{a_{2n}}\right) + M_A \sup_{n\leq N}\left(\frac{\beta_{2n+1}}{a_{2n+2}}\right).$$

If $\lambda \geq \xi'_N$ we have $\sup_{n\leq N}(\mu'_n) < 1$ and

$$\sup_n (\mu'_n) = \sup\left(\sup_{n\leq N}(\mu'_n)\,,\ \sup_{n\geq N+1}(\mu'_n)\right) < 1.$$

Put now $\lambda^* = \sup(\xi, \xi'_N)$. If $\lambda > \lambda^*$

$$\sup\left(\sup_{n\geq 1}(\mu_n)\,, \sup_{n\geq 1}(\mu'_n)\right) < 1$$

and the matrix $D_\lambda (A + B + \lambda I)$ satisfies the condition Γ_d. So, (17) admits a unique solution X_0 in $D(A) \cap D(B) = s_d$ for every Y such that

$$D_\lambda Y = \left(\frac{y_n}{\nu_n + \lambda}\right) \in s_d.$$

The previous property is also satisfieded for all $Y \in s_1$, since

$$a_{2n}\left|\frac{y_{2n}}{\nu_{2n+1} + \lambda}\right| = O(1)\,, \quad \beta_{2n+1}\left|\frac{y_{2n+1}}{\nu_{2n+1} + \lambda}\right| = O(1)\,,$$

as n tends to infinity. Then we can write X_0 in the form (18). □

3.3. Resolution of systems obtained from the preceding

In this subsection we suppose that A and B satisfied (10), (12) and (13). We are going to generalize the results of Theorem 4, in which $A + B + \lambda I$ was an infinite tridiagonal matrix. So, we shall use an infinite upper triangular matrix $P = (p_{nm})_{n,m\geq 1} \in S_1$ such that $p_{nn} = 1\ \forall n$, and consider the infinite matrix $C = (c_{nm})_{n,m\geq 1}$ defined by

$$c_{nm} = \begin{cases} \nu_1 + \lambda + p_{12}\gamma_2 & \text{if } m = n = 1\,, \\ p_{1m-1}b_{m-1} + p_{1m}(\nu_m + \lambda) + p_{1m+1}\gamma_{m+1} & \text{if } n = 1\,, m \geq 2\,, \\ p_{nn-1}\gamma_{n-1} & \text{if } n \geq 3\,, m = n-2\,, \\ p_{nn-1}(\nu_{n-1} + \lambda) + \gamma_n & \text{if } n \geq 2\,, m = n-1\,, \\ p_{nn-1}b_{n-1} + \nu_n + \lambda + p_{nn+1}\gamma_{n+1} & \text{if } n \geq 2\,, m = n\,, \\ p_{nm}b_m + p_{nm+1}(\nu_{m+1} + \lambda) + p_{nm+2}\gamma_{m+2} & \text{if } n \geq 2\,, m \geq n+1\,, \end{cases}$$

the other elements corresponding to $1 \leq m \leq n-2$ with $n \geq 3$ being equal to zero.

Proposition 8. *Let $\theta \in]0,1[$, and assume that the sequence $(p_{nm})_{n,m\geq 1}$ satisfies*

$$\sup_n \left[\sum_{m=n+1}^{\infty} |p_{nm}| \left(\frac{a_n}{a_m}\right)^{\theta} \right] < 1. \tag{22}$$

There exists $\lambda^* > 0$ such that for every $\lambda \geq \lambda^*$ and $Y \in s_{(1/a_n^\theta)_n}$ the system defined by

$$\sum_{m\geq 1} c_{nm} x_m = y_n \quad (n = 1, 2, ...)$$

admits a unique solution X_0 in s_d such that $AX_0 \in s_{(1/a_n^\theta)_n}$.

Proof. We deduce from (22) that P satisfies the condition $\Gamma_{(1/a_n^\theta)}$, i.e.

$$\|I - P\|_{S_{\left(1/a_n^\theta\right)}} < 1. \tag{23}$$

We have $C = P(A + B + \lambda I)$. And since $P \in S_1$ the series of general terms $p_{nm}\gamma_{m-1}x_{m-1}$, $p_{nm}(a_m + \beta_m)x_m$ and $p_{nm}b_{m+1}x_{m+1}$, are absolutely convergent. We deduce that the matrix equation $CX = Y$ is equivalent to

$$P[(A + B + \lambda I)X] = Y$$

and to

$$(A + B + \lambda I)X = P^{-1}Y.$$

Using (23), we have $P^{-1} \in S_{(1/a_n^\theta)_n}$, then $P^{-1}Y \in s_{(1/a_n^\theta)_n}$ and Theorem 4 can be applied. □

This method allows us to consider systems having a zero on the main diagonal. In this case, there exists no sequence $c = (c_n)$, $(c_n > 0, \forall n)$, such that the matrix of the coefficients satisfies the condition Γ_c. Taking $\lambda_0 > \lambda^*$, we deduce from the preceding, the following result:

Corollary 9. *Let $(\alpha_n)_{n\geq 2}$ be any sequence and $\theta \in]0,1[$. Consider the system*

$$(S_1) \begin{cases} (b_1\gamma_2 - \nu_1'\nu_2')x_2 - \nu_1' b_2 x_3 = y_1, \\ \gamma_n x_{n-1} + (\nu_n' + \alpha_n\gamma_{n+1})x_n + \left(b_n + \alpha_n\nu_{n+1}'\right)x_{n+1} + \alpha_n b_{n+1}x_{n+2} = y_n, \\ \hfill n \geq 2, \end{cases}$$

where $\nu_n' = \nu_n + \lambda_0$, and $\gamma_2 \neq 0$. Assume that

$$\frac{\nu_1'}{|\gamma_2|}\left(\frac{a_1}{a_2}\right)^{\theta} < 1 \quad \text{and} \quad \sup_{n\geq 2}\left(|\alpha_n|\left(\frac{a_n}{a_{n+1}}\right)^{\theta}\right) < 1. \tag{24}$$

Then for all $Y \in s_{(1/a_n^\theta)_n}$, the system (S_1) admits a unique solution $X_0 = \left(x_n^0\right)_n$ in s_d such that

$$\sup_{n\geq 1}\left(a_n^\theta \left|a_n x_n^0 + b_n x_{n+1}^0\right|\right) < \infty.$$

Proof. Define here the infinite matrix $P = (p_{nm})_{n,m}$ by $p_{nm} = 0$ for every n, m such that $m \neq n, n+1$; $p_{nn} = 1$ for all $n \geq 1$; $p_{12} = -\frac{\nu_1'}{\gamma_2}$, and $p_{nn+1} = \alpha_n$ $\forall n \geq 2$. (24) implies that $\|I - P\|_{S_{\left(1/\alpha_n^\theta\right)}} < 1$. Doing the product $P(A + B + \lambda_0 I)$ we get the matrix:

$$\begin{pmatrix} 0 & b_1 - \frac{\nu_1'\nu_2'}{\gamma_2} & -\frac{\nu_1'}{\gamma_2}b_2 & 0 & & \\ \gamma_2 & \nu_2' + \alpha_2\gamma_3 & b_2 + \alpha_2\nu_3' & \alpha_2 b_3 & O & . \\ . & . & . & . & . & . \\ & & \gamma_n & \nu_n' + \alpha_n\gamma_{n+1} & b_n + \alpha_n\nu_{n+1}' & \alpha_n b_{n+1} \\ & & & . & . & . \\ & O & & . & . & . \end{pmatrix}.$$

We conclude reasoning as above. □

3.4. Study of equation $(A + B + \lambda I)X = Y$, for $\lambda \geq 0$

In this subsection we suppose that (10), (12) and (16) hold. Let κ be an integer and denote here $t_1 = \left(1, \dfrac{b_1}{\nu_1 + \lambda}, 0, ...\right)$ and

$$t_n = \left(0, ..., \frac{\gamma_n}{\nu_n + \lambda}, 1, \frac{b_n}{\nu_n + \lambda}, 0, ...\right),$$

where $2 \leq n \leq \kappa$. t_1,...,t_κ are the κ first rows of $D_\lambda(A + B + \lambda I)$. Let Q_κ be the matrix obtained from $D_\lambda(A + B + \lambda I)$ by replacing its κ first rows t_1, t_2,...t_κ, by e_1, e_2,...e_κ, where $e_n = (...0, 1, 0, ...)$, (1 being in the nth position). We have $Q_\kappa = (q_{nm})_{n,m\geq 1}$ with $q_{nn} = 1$ for all n and for every $n > \kappa$ $q_{nn-1} = \dfrac{\gamma_n}{\nu_n + \lambda}$ and $q_{nn+1} = \dfrac{b_n}{\nu_n + \lambda}$, the other terms being equal to zero. Consider now the determinant

$$\Delta(\lambda) = \begin{vmatrix} 1 & \frac{b_1}{\nu_1 + \lambda} & 0 & . & . & t_1 X_\kappa \\ \frac{\gamma_2}{\nu_2 + \lambda} & 1 & \frac{b_2}{\nu_2 + \lambda} & 0 & . & t_2 X_\kappa \\ 0 & \frac{\gamma_3}{\nu_3 + \lambda} & 1 & \frac{b_3}{\nu_3 + \lambda} & 0 & . \\ & & . & . & & . \\ & & & . & 1 & t_{\kappa-1} X_\kappa \\ & O & & & \frac{\gamma_\kappa}{\nu_\kappa + \lambda} & t_\kappa X_\kappa \end{vmatrix},$$

and recall some definitions and results given in [16, 12, 13]. Let M be an infinite matrix. $M(t_1, t_2, ...t_k)$, where k is an integer, is the infinite matrix obtained from M by addition of the following rows

$$t_1 = (t_{1,m})_{m\geq 1}, \ t_2 = (t_{2,m})_{m\geq 1}, ... \ t_k = (t_{k,m})_{m\geq 1}, \ t_{ii} \neq 0 \quad (i = 1, 2, ...k),$$

where t_{ij} is any scalar. In the same way, set

$$^{t}Y(u_1, u_2, ...u_k) = (u_1, ...u_k, b_1, b_2, ...),$$

and let $D(k)$ be the diagonal matrix whose elements are the inverses of the diagonal elements of $M(t_1, t_2, ...t_k) = (a'_{nm})$, that is, $D(k) = (a'^{-1}_{nn}\delta_{nm})$. Then we have the following result:

Proposition 10. *Let $c = (c_n)$ with $c_n > 0$ for all n be a sequence such that*

$$\| I - D(k)M(t_1, t_2, ...t_k) \|_{S_c} < 1, \tag{25}$$

and

$$D(k)Y(u_1, u_2, ..., u_k) \in s_c, \tag{26}$$

then

i) solutions of $MX = Y$ in the space s_c are

$$X = [D(k)M(t_1, t_2, ...t_k)]^{-1}D(k)Y(u_1, u_2, ...u_k) \quad u_1, u_2, ...u_k \in C.$$

ii) The linear space $KerM \cap s_c$ of the solutions of $MX = 0$ in the space s_c is of dimension k and is given by

$$(KerM) \cap s_c = span(X_1, X_2, ...X_k)$$

where

$$X_i = [M(t_1, t_2, ...t_k)]^{-1}.^{t}e_i, \; i = 1, 2, ...k.$$

Remark 2. *The solutions given in i) can be also written as $X = X_0 + \sum_{i=1}^{k} u_i X_i$ where $X_0 = [D(k)M(t_1, t_2, ...t_k)]^{-1}D(k)Y(0, 0, .., 0)$ is a particular solution of $MX = Y$.*

From the preceding we can deduce the following result:

Proposition 11. *i) For all $\lambda \geq 0$ such that $\Delta(\lambda) \neq 0$,(17) admits a unique solution in s_d for all $Y \in s_1$.*

ii) If $\Delta(\lambda) = 0$, equation (17) where $Y \in s_1$, either does not admit any solution in s_d, or admits infinitely many solutions in s_d.

Proof. From (20) and (21) we see that μ_n tends to 0 as n tends to infinity and it is the same for μ'_n, since

$$0 \leq \mu'_n \leq \frac{M_B}{a_{2n}} + \frac{M_A}{a_{2n+2}}.$$

We deduce that there exists κ such that $\|I - Q_\kappa\|_{S_d} < 1$. Denote now by $P^*_{\kappa,\lambda}$ the matrix obtained from Q_κ by deleting its κ first rows and by $Y'_\kappa \in s_1$ the one column matrix ${}^tY'_\kappa = \left(y'_{\kappa+1}, y'_{\kappa+2}, ...\right)$ with $y'_n = y_n/(\nu_n + \lambda)$. Applying Proposition 10 we see that equation $P^*_{\kappa,\lambda}X = Y'_\kappa$ admits infinitely many solutions defined for all scalars u_1, u_2,...,u_κ by

(27) $$X = (Q_\kappa)^{-1} .^t \left(u_1, u_2, ...u_\kappa, y'_{\kappa+1}, y'_{\kappa+2}, ...\right).$$

Let now

(28) $$X_0 = (Q_\kappa)^{-1} .^t \left(0, 0, ...0, y'_{\kappa+1}, y'_{\kappa+2}, ...\right).$$

It is easy to see that the $\kappa - 1$ first rows of $(Q_\kappa)^{-1}$ are e_1, e_2,...$e_{\kappa-1}$, and if we denote by X_κ its $\kappa-$th column we deduce, using (27) and (28), that

(29) $$X = X_0 + \sum_{i=1}^{\kappa-1} u_i e_i + u_\kappa X_\kappa.$$

Replacing now these solutions in the κ first equations of the system

$$D_\lambda \left[(A + B + \lambda I) X\right] = D_\lambda Y,$$

(D_λ defined in Proposition 7), we obtain the finite linear system $t_n X = y'_n$, $n = 1$, 2, ...κ. This one is equivalent to

$$(S) \qquad \sum_{i=1}^{\kappa-1} u_i t_n e_i + u_\kappa t_n X_\nu = y'_n - t_n X_0 \qquad n = 1, 2, ..., \kappa,$$

where u_1, u_2,...u_κ are the unknowns. Doing the calculations of $t_n e_i$, ($1 \leq n \leq \kappa$, $1 \leq i \leq \kappa - 1$) we deduce that $\Delta(\lambda)$ is the determinant of the coefficients of (S). One can apply the well-known results on finite linear systems and conclude, considering the cases where $\Delta(\lambda)$ is equal to 0 or not. This completes the proof. □

In the case when $\Delta(\lambda) = 0$, we have the following property: if $b_n \neq 0$, $\forall n$, the rank of the system (S) is equal to $\kappa - 1$. In fact, it suffices to adapt Proposition 10 and apply this one to the matrix triangle obtained from $D_\lambda (A + B + \lambda I)$, by adding the row e_1. Thus,

$$\dim\left[KerD_\lambda (A + B + \lambda I)\right] \bigcap s_d = 1.$$

Then we can suppose for instance that the determinant $\Delta'_\kappa(\lambda)$, obtained from $\Delta(\lambda)$ by striking out the κth row and the κth column, is not equal to 0. Considering then the determinant $\Delta'_Y(\lambda)$ obtained from $\Delta(\lambda)$ by replacing the κth column by y''_1,...y''_κ with $y''_n = y'_n - t_n X_0$, $(1 \leq n \leq \kappa)$ we have

Corollary 12. *Assume that* $\Delta(\lambda) = 0$.

i) If $Y \in s_1$ *satisfies* $\Delta'_Y(\lambda) \neq 0$, *equation (17) does not admit any solution.*

ii) If $\Delta'_Y(\lambda) = 0$, *there exist scalars* $\alpha_1, \alpha_2, ..., \alpha_{\kappa-1}, \mu_1, \mu_2, ... \mu_{\kappa-1}, u_\kappa$ *and a vector* $X_\kappa \in s_d$ *such that* $\forall Y \in s_1$ *equation (17) admits infinitely many solutions in* s_d *which can be written*

$$X = X_0 + \left(X_\kappa + \sum_{i=1}^{\kappa-1} \alpha_i e_i \right) u_\kappa + \sum_{i=1}^{\kappa-1} \mu_i e_i. \tag{30}$$

Proof. i) is obvious. Assertion ii). (17) being equivalent to (S), u_κ is the variable and there exist $\alpha_1, \alpha_2, ..., \alpha_{\kappa-1}, \mu_1, \mu_2, ... \mu_{\kappa-1}$ such that

$$u_i = \alpha_i u_\kappa + \mu_i \qquad i = 1, 2, ..., \kappa - 1.$$

Then using (29), the solutions can be written as

$$X = X_0 + \sum_{i=1}^{\kappa-1} (\alpha_i u_\kappa + \mu_i) e_i + u_\kappa X_\kappa,$$

which permits us to make the conclusion. □

Remark 3. *Let* ${}^t Y'_\kappa = \left(y'_{\kappa+1}, y'_{\kappa+2}, ... \right)$. *Then we see that for every* $Y'_\kappa \in s_1$, *one can associate a unique* $X_0 \in s_d$ *and a subspace* V *of* C^κ, *such that if* $(y'_1, ... y'_\kappa) \in V$ *equation (17) admits in* s_d *the solutions* $X = X_0 + xW_1 + W_2$, *where* $W_1 = X_N + \sum_{i=1}^{\kappa-1} \alpha_i e_i$, $W_2 = \sum_{i=1}^{\kappa-1} \mu_i e_i$ *for all scalars* x.

Remark 4. *If we assume that (10), (12) and (13) hold, then Proposition 11 and Corollary 12 remain true if we replace the condition* $Y \in s_1$ *by* $Y \in s_{(1/a_n^\theta)_n}$. *Note that we do not have necessarily the property of regularity.*

3.5. Property of the operator ${}^t(A + B + \lambda I)$, for $\lambda \geq 0$

In this part A and B satisfy (10) and (12). Denote $A + B + \lambda I = (a_{nm})_{n,m \geq 1}$ for short and recall that $l^1 = \{X = (x_n) / \sum_n |x_n| < \infty\}$, then we have

Proposition 12. *Assume that for a real* $\lambda \geq 0$

$$[Ker(A + B + \lambda I)] \bigcap D(A) \bigcap D(B) \neq \{0\}, \tag{31}$$

there exists a non-empty set $I \subset N^*$ *such that* $\forall b \neq 0$, *the equation*

$$^t(A + B + \lambda I) X = b\, {}^t e_{n_0} \qquad n_0 \in I, \tag{32}$$

does not admit any solution in l^1.

Proof. Let $Z = (z_n)$ be a non-zero element of $[Ker(A+B+\lambda I)] \bigcap D(A) \bigcap D(B)$ and denote $I = \{n \in N^* / \ z_n \neq 0\}$. Then for all $\chi = (\chi_n) \in l^1$ we have

$$\sum_{n=1}^{\infty} \chi_n \left(\sum_{m=1}^{\infty} a_{nm} z_m \right) = 0,$$

and using the fact that $Z \in s_d$ we deduce that there exists $K > 0$ such that for every $n \geq 2$

$$\sum_{m=1}^{\infty} |a_{nm}| \, |z_m| \, |\chi_n| \leq K \left(|\gamma_n| \, d_{n-1} + \nu_n d_n + |b_n| \, d_{n+1} \right) |\chi_n| \, ;$$

and, since the series $\sum\limits_{n=2}^{\infty} |\gamma_n| \, d_{n-1} \, |\chi_n|$, $\sum\limits_{n=2}^{\infty} \nu_n d_n \, |\chi_n|$, $\sum\limits_{n=2}^{\infty} |b_n| \, d_{n+1} \, |\chi_n|$ are convergent we deduce that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{nm}| \, |z_m| \, |\chi_n| < \infty.$$

Thus

$$\sum_{n=1}^{\infty} \chi_n \left(\sum_{m=1}^{\infty} a_{nm} z_m \right) = \sum_{m=1}^{\infty} z_m \left(\sum_{n=1}^{\infty} a_{nm} \chi_n \right) = 0.$$

Now let (τ_n) be a sequence such that for an integer $n_0 \in I$, $\tau_{n_0} \neq 0$, the other terms being equal to 0. If the system

$$\sum_{n=1}^{\infty} a_{nm} \chi_n = \tau_m \quad m = 1, 2, ...$$

would admit a solution $\chi = (\chi_n)$ in the space l^1 we should have

$$\sum_{m=1}^{\infty} z_m \left(\sum_{n=1}^{\infty} a_{nm} \chi_n \right) = z_{n_0} \tau_{n_0} \neq 0,$$

which is contradictory. This completes the proof. □

Remark 5. *We see applying the proposition that if equation*

$$^t \left(A + B + \lambda I \right) X = Y, \ \ (\forall Y \in s_1),$$

admits a solution in l^1 *it has not a solution any more when the* $n - th$ *term of* Y, $n \in I$, *is modified. Indeed, let* $n_0 \in I$ *and denote by* Y' *the matrix obtained from* Y *by replacing the* $n_0 - th$ *coefficient by another one. If the equation* $^t \left(A + B + \lambda I \right) X = Y'$ *admitted a solution in* l^1 *the equation* $^t \left(A + B + \lambda I \right) X = Y$ *would not admit any solution, since* $Y' - Y$ *is of the form* $b^t e_{n_0}$, $b \neq 0$.

Remark 6. *An important case is the one when* $A + B + \lambda I$ *is symmetric (see [17]).*

Remark 7. *Let us give an example of matrices* A *and* B *satisfying (31). Put* $a_n = 2^n$ *and* $\beta_{2n} = 1$ *for all* $n \geq 1$*;* $\beta_{2n+1} = (2n+1)!$ *for all* $n \geq 0$*;* $b_1 = 3$*,* $\gamma_2 = 5$*,* $b_2 = 0$ *and* $b_n = \gamma_n = 0$ *for all* $n \geq 3$*. Then we see that for* $\lambda = 0$*,* $\Delta(\lambda) = 0$ *and* $[Ker(A+B)] \bigcap D(A) \bigcap D(B)$ *is the space of all sequences defined by* ${}^tX = (x, -x, 0, ...)$ *for all scalars* x*.*

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