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SIMPLE SCORE SEQUENCES IN ORIENTED GRAPHS

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Abstract. We characterize irreducible score sequences of oriented graphs and give a condition for a score sequence to belong to exactly one oriented graph.

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1. Introduction

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Let D be an oriented graph with the vertex set $V = v_1, v_2, .., v_n$, and let *odv* and *idv* denote the outdegree and indegree, respectively, of a vertex v. Avery [1] defined $s_v = n - 1 +$ $odv - idv$, $0 \leq s_v \leq 2n - 2$, as the score of vertex v and $S = (s_1, s_2, .., s_n)$ in nondecreasing order is the score sequence of D. An arc from the vertex u to the vertex v is denoted by $u \to v$ and $u \sim v$ or $v \sim u$ means neither $u \to v$ nor $v \to u$. Avery [1] has characterized the score sequence of oriented graphs.

Theorem 1.1. [1] *A nondecreasing sequence of non-negative integers* $S = (s_1, s_2, \ldots, s_n)$ *is the score sequence of an oriented graph if and only if for* $k = 1, 2, \ldots, n$

$$\sum_{i=1}^{k} s_i \geq K(k-1)$$

and equality holds for $k = n$.

A triple in an oriented graph is an induced subdigraph with three vertices. The triples of the form $u \leftarrow v$, $v \leftarrow w$, $w \leftarrow u$ or $u \leftarrow v$, $u \leftarrow w$, $u \sim w$ are called intransitive triples and the triples $u \sim v$, $v \sim w$, $w \sim u$ or $u \sim v$, $v \sim w$, $u \leftarrow w$ are called transitive triples.

Avery [1] gave the following results.

Theorem 1.2. [1] *Let D and D be two oriented graphs with the same score sequence. Then D can be transformed to D by successively transforming appropriate triples in one of the following ways:*

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either (a) by changing a cyclic triple $u \leftarrow v$, $v \leftarrow w$, $w \leftarrow u$ *to a transitive triple* $u \sim v$, $v \sim w$, $w \sim u$, *which has the same score sequence, or vice versa;*

or (b) by changing an intransitive triple $u \leftarrow v$, $v \leftarrow w$, $u \sim w$ *to a transitive triple* $u \sim v$, $v \sim w$, $u \leftarrow w$, *which has the same score sequence, or vice versa.*

The next result provides a useful recursive test of whether a given sequence S of nonnegative integers is the score sequence or an oriented graph and if S is a score sequence, an oriented graph $\triangle(s)$ with score sequence S is constructed.

Theorem 1.3. [1] *Suppose* S *is a sequence of* n *integers between* 0 *and* $2n-2$ *inclusive. Let* S *be obtained from* S *by deleting the greatest entry,* $2n-2-r$ *say, and reducing each of the greatest* r *remaining entries in* S *by* 1. *Then* S *is a score sequence if and only if* S *is.*

2. Irreducible score sequences

An oriented graph D is reducible if it is possible to partition its vertices in to two nonempty sets V_1 and V_2 in such a way that every vertex of V_2 is adjacent to all vertices of V_1. Let D_1 and D_2 be induced digraphs having vertex sets V_1 and V_2 respectively. Then D consists of D_1 and D_2 and every vertex of D_2 is adjacent to all vertices of D_1. We write $D = [D_1, D_2]$. If this is not possible, then the oriented graph D is irreducible. Let $D_1, D_2, \ldots, D_k$ be irreducible oriented graphs with disjoint vertex sets. Now $D = [D_1, D_2, \ldots, D_k]$ denotes the oriented graph having all arcs of $D_i \leq i \leq k$, and every vertex of D_j is adjacent to all vertices of D_i with $1 \leq i \leq j \leq k$. $D_1, D_2, \ldots, D_k$ are called as irreducible components of D. Such a decomposition is known as irreducible component decomposition of D, which is unique.

A score sequence S is said to be irreducible if all the oriented graphs D_1 with score sequence S are irreducible.

In case of ordinary tournaments, the score sequence $S = (s_1, s_2, \ldots, s_n)$ is used to decide whether a tournament T having score sequence S, is strong or not [3]. This is not true in the case of oriented graphs. For example, the oriented graphs D_1 and D_2 in Figure 1, both have score sequence (2,2,2) but D_1 is strong and D_2 is not.

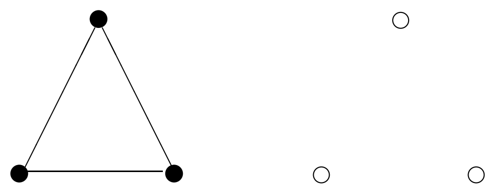

Figure 1

The following result charaterises irreducible oriented graphs.

Theorem 2.1. *Let D be an oriented graph having score sequence $(s_1, s_2, \ldots, s_n)$. Then D is irreducible if and only if, for $k = 1, 2, \ldots, n-1$*

$$\sum_{i=1}^{k} s_i > k(k-1) \tag{2.1}$$

and

$$\sum_{i=1}^{k} s_i = n(n-1) \tag{2.2}$$

Proof. Suppose D is an irreducible oriented graph having score $(s_1, s_2, \ldots, s_n)$. Condition (2) holds since Theorem 1.1 has already established it for any oriented graph. To verify inequalities (1), we observe that for any integer $k < n$, the subdigraph induced by any set of k vertices has a sum of scores $k(k-1)$. Since D is irreducible, there must be an arc from at least one of these vertices to one of the other $n-k$ vertices. Thus for $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} s_i > k(k-1).$$

For the converse, suppose conditions (1) and (2) hold, we know by Theorem 1.1 that there exists an oriented graph D with these scores. Assume that such an oriented graph D is reducible. Let $D = [D_1, D_2, \ldots, D_k]$ be the irreducible component decomposition of D. If m is the number of vertices in D_1, then $m < n$, and the equation

$$\sum_{i=1}^{m} s_i = m(m-1)$$

holds, which is a contradiction. This proves the converse part. □

The following result is an extension of Theorem 2, [2]. The proof is obvious.

Theorem 2.2. *Let D be an oriented graph with score sequence $S = (s_1, s_2, \ldots, s_n)$. Suppose that*

$$\sum_{i=1}^{p} s_i = p(p-1),$$

$$\sum_{i=1}^{q} s_i = q(q-1)$$

and

$$\sum i = 1^k s_i > k(k-1) \qquad \text{for } p+1 \leq k \leq q-1,$$

where $0 \leq p < q \leq n$.

Then the subdigraph induced by the vertices $v_{p+1}, v_{p+2}, \ldots, v_q$ is an irreducible component of D with score sequence $(s_{p+1} - 2p, \ldots, s_q - 2p)$.

Now S is irreducible if D is irreducible and the irreducible components of S are the score sequences of the irreducible components of D. Theorem 2.2 shows that the irreducible components of S are determined by the successive values of k for which

$$\sum_{i=1}^{k} s_i = k(k-1), 1 \leq k \leq n. \tag{2.3}$$

We illustrate it with the following example.

Let $S = (1, 2, 3, 8, 8, 8, 13, 13)$. Equation (3) is satisfied for $k = 3, 6, 8$. Thus the irreducible components of S are $(1, 2, 3)$, $(2, 2, 2)$ and $(1, 1)$ in ascending order.

3. Simple score sequences

A score sequence is simple if it belongs to exactly one oriented graph. Avery [2] has characterised simple score sequences in ordinary tournaments. Here we characterise simple score sequences of oriented graphs. First we have the following observation.

Theorem 3.1. *The score sequence S of an oriented graph is simple if and only if every irreducible component of S is simple.*

The following result determines which irreducible score sequences are simple.

Theorem 3.2. *Let S be an irreducible score sequence. Then S is simple if and only if it is one of* (0), *or* $(1, 1)$.

Proof. Suppose S is an irreducible score sequence and let D be an oriented graph having score sequence S. We have three cases to consider. (1) D has $n \geq 3$ vertices. (2) D has two vertices. (3) D has one vertex.

Case 1. D has $n \geq 3$ vertices. Since S is irreducible, therefore there exist vertices u, v and w such that D has a cyclic triple $u \leftarrow v$, $v \leftarrow w$, $w \leftarrow u$; or an intransitive triple $u \leftarrow v$, $v \leftarrow w$, $w \sim u$; or a transitive triple $u \sim v$, $v \sim w$, $u \leftarrow w$; or a transitive triple $u \sim v$, $v \sim w$, $w \sim u$. Now, if D contains the cyclic triple $u \leftarrow v$, $v \leftarrow w$, $w \leftarrow u$, it can be changed to the transitive triple $u \sim v$, $v \sim w$, $w \sim u$ to form an oriented graph D with the same score sequence, or vice versa. So the number of arcs in D and D is different. If D contains the intransitive triple $u \leftarrow v$, $v \leftarrow w$, $w \sim u$, we can transform it to the transitive triple $u \sim v$, $v \sim w$, $u \leftarrow w$ to form an oriented graph D having the same score sequence, or vice versa. Here, the number of arcs in D and D is also different. Since in every case the number of arcs in D and D is not same, therefore D is not isomorphic to D. Thus S is not simple.

Case 2. D has two vertices. Then $S = (1, 1)$ is the only irreducible score sequence and it belongs to exactly one oriented graph, namely $u \sim v$.

Case 3. D has just one vertex. Then $S = (0)$ which is obviously simple. Hence (0) and $(1, 1)$ are the only irreducible score sequences that are simple. □

Corollary 1. *The score sequence S is simple if and only if every irreducible component of S is one of (0), or (1,1).*

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