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A UNIFORMLY ACCURATE COLLOCATION METHOD FOR A SINGULARLY PERTURBED PROBLEM [1]

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Abstract. A semilinear singularly perturbed reaction-diffusion problem is considered and the approximate solution is given in the form of a quadratic polynomial spline. Using the collocation method on a simple piecewise equidistant mesh, an approximation almost second order uniformly accurate in small parameter is obtained. Numerical results are presented in support of this result.

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1. Introduction

We consider the semilinear problem

$$\begin{cases} Ly = -\varepsilon^2 y'' + f(x,y) = 0 \qquad x \in I = [0,1], \\ \\ y(0) = 0, \qquad y(1) = 0. \end{cases} \tag{1}$$

Here ε is a positive parameter, $f(x,y) \in C^2(I \times \boldsymbol{R})$, $f(x,y)$ has bounded partial derivatives and $f_y(x,y) > \beta^2 > 0$ for all $(x,y) \in I \times \boldsymbol{R}$.

Differential equations with a small parameter ε multiplying the highest-order derivative terms are said to be singularly perturbed. They occur frequently in engineering applications and in environmental sciences for example, in fluid flow at high Reynolds number, advection-dominated heat and mass transfer, semiconductor device models, theory of plates, shellsand chemical kinetics. Small values of ε correspond to large values of characteristic numbers such as Reynold's number, Péclet's number, or Hartmann's number, which are used in various branches of hydrodynamics and magnetohydrodynamics.

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Even in the case where only the approximate solution of the singularly perturbed boundary value problem is required, classical numerical methods, such as finite difference schemes and finite element methods, exhibit unsatisfactory behaviour. The typical behaviour observed is that the maximum pointwise error increases as the mesh is refined, until the mesh size is comparable in size with the perturbation parameter.

Numerical methods that behave uniformly well for all values of the singular perturbation parameter are said to be ε-uniformly convergent. There are two classes of ε-unfirormly convergent finite difference methods:

- "fitted operator" methods, where on given meshes with arbitrary distribution of nodes appropriate difference approximations are constructed,
- "fitted mesh" methods, where on an appropriate constructed mesh given the difference method is used.

In the case when $f(x, y)$ is linear in y, fitted operator methods have been constructed and analysed in, for example, [5],[10],[12]. Fitted operator methods in spline approximations are examined in [18]-[24]. Several types of fitted mesh methods for the linear case have been introduced and analysed in [3],[7],[8],[25].

Bakhvalov was the first to use special grids to solve singularly perturbed problems. Meshes of Bakhvalov type require detailed information about the solution, they are uniformly spaced outside the layers and are characterized by a gradual transition from the coarse to a very fine mesh at the layers.

Uniformly convergent methods for the semilinear problem (1) have also been examined. Problem (1) has a unique solution $y \in C^4(I)$ (see Lorenz [9]),whose *a priori* estimates give the following lemma:

Lemma 1. *There exists a unique solution y(x) of problem (1). This solution*

$$y(x) = v(x) + g(x), \tag{2}$$

satisfies

$$|g^{(j)}(x)| \leq M, \quad |v^{(j)}(x)| \leq M\varepsilon^{-j}(e^{-x\frac{\beta}{\varepsilon}} + e^{(x-1)\frac{\beta}{\varepsilon}}), \quad j = 1, 2, 3, 4. \tag{3}$$

Proof. See Vulanović [25]. □

Vulanović [25] has considered problem (1). He obtained the second order ε-uniform convergence of a central difference scheme on a special mesh of Bakhvalov's type. Herceg [8] introduced a difference scheme for problem (1) and achieved the fourth order uniform convergence on the mesh of Bakhvalov's type under extra conditions of the problem. Vulanović [26] has considered a modification of [8]. He used a special graded mesh which varies more smoothly than Herceg's. He then proves the uniform fourth order convergence. A different modification of [8] has been recently considered in [17]. The authors use a new approach, the piecewise equidistant meshes, introduced by Shishkin [13], the construction of which is remarkably simple. They are much simpler than the graded mesh of Bakhvalov type and, surprisingly, one can also achieve uniform convergence of standard difference operators. Sun and Stynes proved almost fourth order uniform convergence in the discrete maximum norm for Herceg's scheme on a mesh of Shishkin type, without extra conditions.

Sun and Stynes [16] considered a simple central difference scheme for the problem which may have non-unique solutions. They consider this scheme on a mesh of Shishkin type and prove that it is almost second order accurate in the discrete maximum norm.

Our paper is devoted to the construction of ε-uniform approximations using collocation with classical quadratic splines $u(x) \in C^1(I)$ on Shishkin meshes.

Throughout the paper, M denotes any positive constant that may take different values in different formulas, but always independent of ε and number of nodes.

2. Construction of the method

For a given positive integer n, we denote an arbitrary mesh by $\Delta : 0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$. We use the notation: $h_i = x_{i+1} - x_i$, $x_{i+1/2} = x_i + h_i/2$, $I_i = [x_i, x_{i+1}]$ for $i = 0, ..., n-1$. For the given function $y(x)$, let $y_i = y(x_i)$ for $i = 0, ..., n$, and $\bar{y}_i = (y_i + y_{i+1})/2$, and $\hbar_i = (h_i + h_{i-1})/2$ for $i = 0, ..., n-1$.

An approximate solution of problem (1) we seek in the form of the quadratic spline

$$(4) \quad u(x) = u_i(x) = u_i + u_i'(x - x_i) + u_i''(x - x_i)^2/2, \quad x \in I_i, \quad i = 0, ..., n.$$

The truncation error of a polynomial-based discretization becomes infinity when $\varepsilon \to 0$ unless it is constructed on a special mesh. In this paper we introduced the slightly modified piecewise equidistant mesh of Shishkin type (see [13], and [14]) appropriate to problem (1).

For a given positive integer $n = 2^k$, $k \geq 2$, the interval $[0, 1]$ is divided into three subintervals $[0, \delta]$, $[\delta, 1-\delta]$, $[1-\delta, 1]$. On each of these subintervals the equidistant meshes are used. The transition point δ from the fine to the coarse mesh is defined by $\delta = min\{1/4, 4b^{-1}\varepsilon \ln n\}$, where $b = min\{\beta, 1\}$. We shall assume that $\delta = 4b^{-1}\varepsilon \ln n$, since in the opposite case the method can be analyzed using standard techniques. Set $i_0 = n/4$, then $x_{i_0} = \delta$, $x_{n-i_0} = 1 - \delta$ are the transition points and the mash spacing is given by

$$(5) \qquad \tilde{h}_1 = h_i = x_{i+1} - x_i = 16b^{-1}\varepsilon n^{-1} \ln n,$$

for $i = 0, 1, ..., i_0 - 1, n - i_0, ..., n - 1$, and

$$\tilde{h}_2 = h_i = x_{i+1} - x_i = 2\,(1 - 2\,\delta)\, n^{-1}, \quad n^{-1} \leq h_i \leq 2n^{-1},$$

for $i = i_0, ..., n - i_0 - 1$. Set $\tilde{h}_3 = b^{-1}\varepsilon n \ln n$. If $\tilde{h}_3 \le \tilde{h}_2/2$, we shall modify the mesh adding two points to the Shishkin mesh: $\tilde{x}_{i_0} = x_{i_0} + \tilde{h}_3$ and $\tilde{x}_{n-i_0} = x_{n-i_0} - \tilde{h}_3$. Then, we put $n = n + 2$ and renumerate the points. If $\tilde{h}_3 > \tilde{h}_2/2$, the Shiskin mesh remains unchanged.

To derive the spline difference scheme we use the quadratic spline $u(x)$ as the approximation function, the points $x_{i+1/2}$, $i = 0, ..., n-1$, as the collocation points and the equations

$$-\varepsilon^2 u_i'' + f(x_{i+1/2}, \bar{u}_i) = 0, \qquad i = 0, ..., n-1, \tag{6}$$

as the collocation equations.

By (6) and the requirements $u(x) \in C^1(I)$, $u_0 = 0$ and $u_n = 0$ we obtain the scheme:

$$\begin{cases} r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1} = q_i^- f_i^- + q_i^+ f_i^+, \\ \\ u_0 = 0, \quad u_n = 0, \quad i = 1, ..., n-1, \end{cases} \tag{7}$$

where

$$\begin{cases} r_i^- = \frac{h_i}{\hbar_i}, \; r_i^+ = \frac{h_{i-1}}{\hbar_i}, \; r_i^c = -2, \; q_i^- = \frac{h_{i-1}^2 h_i}{2\hbar_i \varepsilon^2}, \; q_i^+ = \frac{h_i^2 h_{i-1}}{2\hbar_i \varepsilon^2}, \\ \\ f_i^- = f(x_{i-1/2}, \bar{u}_{i-1}), \; f_i^+ = f(x_{i+1/2}, \bar{u}_i). \end{cases} \tag{8}$$

3. Truncation error

The scheme (7) can be written in the form

$$F u^n = 0, \tag{9}$$

where $F = -A + B$, A is an $(n+1)$ x $(n+1)$ tridiagonal matrix defined by:

$$A = \begin{bmatrix} 1 & 0 & 0 & & & & \\ r_1^- & r_1^c & r_1^+ & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & & r_{n-1}^- & r_{n-1}^c & r_{n-1}^+ \\ & & & & 0 & 0 & 1 \end{bmatrix},$$

$B : \boldsymbol{R}^{n+1} \to \boldsymbol{R}^{n+1}$ is the mapping:

$$(Bu)_i = \begin{cases} 0, & \text{for } i = 0, \\ \\ q^- f_i^- + q^+ f_i^+, & \text{for } i = 1, \cdots, n-1, \\ \\ 0, & \text{for } i = n. \end{cases}$$

Since $(Fy^n)_0 = (Fy^n)_n = 0$ we shall bound $|(Fy^n)_i|$ for $i = 1, ..., n-1$ in this section.

Lemma 2. *Let y be the solution of problem* (1)*. The truncation error Fy^n of F approximating L at y on the Shishkin mesh satisfies*

$$||Fy^n||_\infty \le G_1(n,\varepsilon),$$

where

$$G_1(n,\varepsilon) = M(\varepsilon n^{-3}\ln n + n^{-4}\ln^4 n).$$

Proof. Suppose first that x_i is inside the $[0,\delta]\cup[1-\delta,1]$ where the mesh is equidistant, i.e., $i\in\{1,...,i_0-1\}\cup\{n-i_0+1,...,n-1\}$. Using (1), the Taylor expansion gives

$$(10)\qquad (Fy^n)_i = \frac{h_{i-1}h_i}{\hbar_i}\Big[\frac{h_i^2 y_i''' - h_{i-1}^2 y_{i-1}'''}{12} + \frac{h_i^3 y^{(iv)}(\beta_i^+)}{16} - \frac{h_i^3 y^{(iv)}(\eta_i^0)}{24} +$$

$$+\frac{h_{i-1}^3 y^{(iv)}(\beta_i^-)}{16} + \frac{h_{i-1}^3 y^{(iv)}(\eta_{i-1}^0)}{24} - \frac{h_{i-1}^3 y^{(iv)}(\eta_{i-1}^1)}{6}\Big] - T_i(y),$$

where

$$T_i(y) = \frac{h_i^4}{16\varepsilon^2}\Big(y''(\gamma_i^+)\frac{\partial f}{\partial y}(x_{i+1/2},\theta_i) + y''(\gamma_i^-)\frac{\partial f}{\partial y}(x_{i-1/2},\theta_{i-1})\Big),$$

θ_i is a point between $y_{i+1/2}$ and $\bar{y}_i$; $\beta_i^+\in(x_i,x_{i+1/2})$; $\beta_i^-\in(x_{i-1/2},x_i)$; $\gamma_i^+\in(x_i,x_{i+1/2})$; $\gamma_i^-\in(x_{i-1/2},x_i)$; $\eta_i^0,\ \eta_i^1\in(x_i,x_{i+1})$

From (5), (10) and Lemma 1 we have that

$$|(Fy^n)_i| \le Mn^{-4}\ln^4 n, \quad \text{for} \quad i\in\{1,...,i_0-1\}\cup\{n-i_0+1,..,n-1\}.$$

and

$$|(Fy^n)_i| \le Mn^{-4}, \quad \text{for} \quad i\in\{i_0+3,...,n-i_0-3\}.$$

Let us now estimate $|\ (Fy^n)_i\ |$ for $i\in I_0$, where $I_0=\{\ i_0,\ i_0+1,\ i_0+2,\ n-i_0-2,\ n-i_0-1,\ n-i_0\}$. Then,

$$(11)\qquad |(Fy^n)_i| \le M\psi_i(y)$$

where

$$\psi_i(y) = \frac{h_{i-1}^2 h_i}{2\bar{h}_i\varepsilon^2}B_{i-1}(y) + \frac{h_{i-1}h_i^2}{2\bar{h}_i\varepsilon^2}B_i(y) + \frac{h_i}{\bar{h}_i}|R_2(x_i,x_{i-1},y)| +$$

$$(12)\qquad +\frac{h_{i-1}}{\bar{h}_i}|R_2(x_i,x_{i+1},y)| + \frac{h_{i-1}h_i}{\bar{h}_i}|R_1(x_i,x_{i-1},y')|,$$

$$R_k(a,b,g) = \frac{1}{k!}\int_a^b (b-s)^k g^{(k+1)}(s)ds = \frac{1}{(k+1)!}(b-a)^{k+1}g^{(k+1)}(\rho),\ \rho\in(a,b),$$

and

$$B_i(y) = f(x_i, y_i) - f(x_{i+1}, y_{i+1}) = \varepsilon^2 h_i y'''(x_i + \alpha_i h_i),\ 0 < \alpha_i < 1.$$

It is easy to prove that

$$|\psi_i(g)| \le M\varepsilon n^{-3} \ln n.$$

We will estimate $\psi_i(v)$ by considering each term separately. The modification of the Shishkin mesh is introduced because difficulties appear in the estimation of the second and the fourth term in (12).

From the Taylor expansion

$$e^{-\frac{h_i b}{\varepsilon}} = \sum_{k=1}^{p-1} (-1)^{k-1} \frac{(h_i b)^{k-1}}{\varepsilon^{k-1}(k-1)!} + (-1)^p \frac{(h_i b)^p}{\varepsilon^p} e^{-\frac{\theta_{p,i} h_i}{\varepsilon} b}, \qquad 0 < \theta_{p,i} < 1,$$

we can explicitly express (see [22])

$$\theta_{p,i} = -\frac{\varepsilon}{h_i b} ln\{(-1)^{p-1}[(1 - e^{-\frac{h_i b}{\varepsilon}}) \frac{\varepsilon^p p!}{h_i^p b^p} + \sum_{k=1}^{p-1} (-1)^k \frac{p! \varepsilon^{p-k}}{k! h_i^{p-k} b^{p-k}}]\} \tag{13}$$

While using this form of $\theta_{p,i}$ we estimated the value of $\tilde{h}_3$ in order to achieve the same error bound as in the boundary layers.

For example, consider for the function $v(x)$ the second term in (12) :

$$\kappa 2_i(v) := \frac{h_i^3 h_{i-1} b^3}{2\hbar_i \varepsilon^3} e^{-\frac{x_i b}{\varepsilon}} e^{-\frac{\theta_{1,i} h_i b}{\varepsilon}},$$

where from (13) we find

$$\theta_{1,i} = -\frac{\varepsilon}{h_i b} ln((1 - e^{-\frac{h_i b}{\varepsilon}}) \frac{\varepsilon}{h_i b}).$$

Let $i = i_0$, then $h_i = \tilde{h}_3$. The requirement

$$|\kappa 2_i(v)| \le M n^{-4} \ln^4 n$$

is fulfilled if

$$h_i \le \tilde{h}_3 = b^{-1} n \ln n. \tag{14}$$

For $i \in \{i_0 + 1, i_0 + 2\}$ the proof follows from the inequality

$$\max(\kappa 2_{i_0+1}(v),\ \kappa 2_{i_0+2}(v)) \le \kappa 2_{i_0}(v).$$

Let us consider now the fourth term in (12) for the function $v(x)$:

$$\kappa 4_i(v) = \frac{h_{i-1}}{\hbar_i} R_2(x_i, x_{i+1}, v) = \frac{h_{i-1} h_i^3 b^3}{3! \varepsilon^3} e^{-\frac{x_i b}{\varepsilon}} e^{-\frac{\theta_{3,i} h_i b}{\varepsilon}},$$

where $\theta_{3,i}$ is defined by (13). For $i = i_0$, the estimate

$$|\kappa 4_i(v)| \le M n^{-4} \ln^4 n$$

follows from (14). For $i \in \{i_0+1, i_0+2\}$ the proof follows from the inequality

$$\max(\kappa 4_{i_0+1}(v), \kappa 4_{i_0+2}(v)) \le \kappa 4_{i_0}(v).$$

It is easy to prove that the other terms in (12) satisfy the same inequality, and we have

$$(15) \qquad |\psi_i(v)_i| \le M n^{-4} \ln^4 n, \quad \text{for} \quad i \in \{i_0, i_0+1, i_0+2\}.$$

For the remained points $i \in \{n-i_0, n-i_0-1, n-i_0-2\}$ the proof is similar.

In the case when the mesh is not modified we use estimates of the form

$$\frac{h_i^p b^p}{\varepsilon^p} \, e^{-\frac{\theta_{p,i} h_i b}{\varepsilon}} \;\le\; M \, e^{-\frac{\theta_{p,i} h_i b}{2\varepsilon}}$$

to obtain (15). □

4. Convergence results at the mesh points

Let $S(z_0, r)$ denote the open ball in R^{n+1} with the center z_0 and diameter $2r$.

Theorem 1. *Let Lemma* 2 *hold. Then, there exists a constant* M *independent of* ε *and* n *and there exists the constant* n_1 *which is dependent of* M *and independent of* ε*, so that the scheme* (7), (8) *for* $n \ge n_1$ *has a solution* $u^n \in R^{n+1}$ *satisfying*

$$(16) \qquad |y(x_i) - u_i| \;\le\; M\, G(\varepsilon, n), \quad i = 0, 1, \ldots, n;$$

where

$$G(\varepsilon, n) :=\; \varepsilon n^{-1} \ln^{-1} n + n^{-2} \ln^2 n.$$

The solution u^n *is the unique solution in* $S(y^n, MG(\varepsilon, n))$.

Proof. Let $\zeta = F'(z)$ be the $(n+1) \times (n+1)$ tridiagonal matrix and let $M(\zeta)$ be the comparison matrix for the matrix ζ, i.e.,

$$M(\zeta) = \begin{cases} |\zeta_{ij}|, & i = j \\ -|\zeta_{ij}|, & i \ne j \end{cases}$$

Since $M(\zeta)$ is the nonsingular $H-$matrix we have that ([2], [4], [11], [15])

$$|(\zeta^{-1})_{ij}| \;\le\; M\, ((M(\zeta))^{-1})_{ij}.$$

The following relations between the elements of the matrices $(M(\zeta))^{-1} = [m_{i,j}]$ and $\zeta^{-1} = [g_{i,j}]$ hold:

$$\max_i \sum_{j=0}^{n} |g_{ij}| \le \max_i \sum_{j=0}^{n} |m_{ij}| \le M n^{-2} \ln^2 n, \quad \text{for} \;\; 0 \le i \le i_0 \;\; \text{and} \;\; n - i_0 \le i \le n,$$

$$\max_i \sum_{j=0}^{n} |g_{ij}| \leq \max_i \sum_{j=0}^{n} |m_{ij}| \leq M \quad \text{for} \quad i_0+1 \leq i \leq n-i_0-1.$$

From [11] (pp. 138) and Lemma 2 we obtain for $i = 0, 1, ..., n$:

$$|y(x_i) - u_i| \leq \max_{0 \leq i \leq n} \sum_{j=0}^{n} |g_{ij}||\tau_j(y)| \leq MG(\varepsilon, n).$$

□

5. Numerical results

In this section we present some numerical experiments for scheme (7), applied to the problem (1) on a special mesh of Shishkin type. Our example is taken from [17].

Example 1

$$-\varepsilon^2 y'' + (1+y)(1+(1+y)^2) = 0, \qquad x \in [0,1], \qquad u(0) = 0, \qquad u(1) = 0.$$

The reduced solution is $u_0 = -1$.

We use a double-mesh method [5] to compute the experimental rates of convergence. In order to do this, we shall compute not only u^n (the solution to (7) on Shishkin's mesh Δ^n), but also another approximation solution $\tilde{u}^n$ (the solution to (7) on Shishkin's mesh $\tilde{\Delta}^n$, defined in [17]). The corresponding spline (4), which provides approximate values for the exact solution between the grid points of the mesh Δ^n, we shall denote as $u^n(x)$.

Let $\tilde{\Delta}^n$ be the Shishkin mesh with the mesh parameter δ altered slightly to

$$\tilde{\delta} = \min\{1/4, 4b^{-1}\varepsilon ln(n/2)\}.$$

Then, for $i = 0, 1, \ldots, n$ the i-th point of the mesh Δ^n coincides with the $(2i)$-th point of the mesh $\tilde{\Delta}^n$.

Assuming the convergence order $(n^{-1}lnn)^r$ for some r on Shishkin's mesh, in Table 1. we estimate the rate r for each fixed $\varepsilon = 2^{-l}$, $l = 1, ..., 15$ from (see [17]:)

$$Ord_{u^n} = \frac{lnE^n_{u^n} - lnE^{2n}_{u^n}}{ln\frac{2lnn}{ln2n}} = \frac{lnE^n_{u_n} - lnE^{2n}_{u_n}}{ln\frac{2k}{k+1}} \quad \text{for } n = 2^k \text{ and } k = 6, 7 \ldots, 11,$$

where,

$$E^n_{u^n} = \max_{0 \leq i \leq n} |u^n_i - \tilde{u}^{2n}_{2i}|.$$

We solve the nonlinear system of equations using Newton's method with the initial guess $u^{n,0} = (0, u_0(x_1), \ldots, u_0(x_{n-1}), 0)^T$, where $u_0(x)$ is the reduced solution. The stopping criterion used is

$$\max\{\|Fu^{n,m}\|_\infty, \|u^{n,m} - u^{n,m-1}\|_\infty\} < 0.1n^{-2}$$

where, $u^{n,m}$, for $m = 1, 2, ...$, are successive approximations to u^n computed iteratively. For each n and ε in Table 1 it takes only about 5 iterations to satisfy this condition.

l	n					
	64	128	256	512	1024	
1	6.60(–5)	1.65(–5)	4.12(–6)	1.03(–6)	2.58(–7)	$E^n_{u^n}$
	2.57	2.48	2.41	2.36		Ord_{u^n}
4	2.85(–3)	7.13(–4)	1.78(–4)	4.45(-5)	1.11(–5)	$E^n_{u^n}$
	2.57	2.48	2.41	2.36		Ord_{u^n}
6	4.72(–2)	1.19(–2)	2.85(–3)	7.13(–4)	1.78(–4)	$E^n_{u^n}$
	2.57	2.55	2.41	2.36		Ord_{u^n}
8	5.06(–2)	1.79(–2)	5.53(–3)	1.74(–3)	5.34(–4)	$E^n_{u^n}$
	1.92	2.11	2.01	2.01		Ord_{u^n}
10	5.06(–2)	1.79(–2)	5.53(–3)	1.74(–3)	5.34(–4)	$E^n_{u^n}$
	1.92	2.11	2.01	2.01		Ord_{u^n}
12	5.06(–2)	1.79(–2)	5.53(–3)	1.74(–3)	5.34(–4)	$E^n_{u^n}$
	1.92	2.11	2.01	2.01		Ord_{u^n}
15	5.06(–2)	1.79(–2)	5.53(–3)	1.74(–3)	5.34(–4)	$E^n_{u^n}$
	1.92	2.11	2.01	2.01		Ord_{u^n}

Table 1: Errors $E^n_{u^n}$ at the mesh points and convergence rates Ord_{u^n}

Remark 1 *The additional points to Shishkin's mesh have more theoretical than practical importance. In Table 2 we present the exact values of* $n(\varepsilon)$*, such that for* $n < n(\varepsilon)$ *the Shishkin mesh has to be modified.*

ε	2^{-8}	2^{-20}	2^{-25}	2^{-30}
$n(\varepsilon)$	8	128	256	1024

Table 2: The values of $n(\varepsilon)$ for $b = 1$, and $M = 1$.

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