Novi Sad J. Math.
Vol. 33, No. 1, 2003, 109–115

HANSEN–PATRICK'S FAMILY IS OF LAGUERRE'S TYPE

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Abstract. We studied a class of cubically convergent iterative methods for simple and multiple zeros of analytic functions and to investigate their mutual dependence and showed that one-parameter Hansen-Patrick's family (1977) is not a new one but it simply follows directly from the classical Laguerre method. In addition, new derivations of Laguerre's method for simple and multiple zeros are given and some special cases are presented.

AMS Mathematical Subject Classification (2000): 65H05

Key words and phrases: solving equations, iterative methods, one–parameter families, Laguerre's method

1. Introduction

Let f be an analytic function in some complex domain S with a simple or multiple zero ζ. The problem of extracting zeros is extensively investigated in the literature and many efficient iterative methods were developed. Among them the third-order methods as Euler's, Laguerre's, Halley's and Ostrowski's method have an important role. Such a wide range of methods poses the question of their mutual dependence and the equivalency of some methods. The justification of such a study is the recent paper of Petković and Herceg [11] where it was shown that seven families of iterative methods, presented by different formulas and derived in various manners (from 1946 to 1996) are actually equivalent.

One attempt of unifying the class of methods with cubic convergence was presented in the paper by Hansen and Patrick [3] by the family

$$\hat{z} = z - \frac{(\alpha+1)f(z)}{\alpha f'(z) \pm \sqrt{f'(z)^2 - (\alpha+1)f(z)f''(z)}}, \tag{1}$$

where α ($\neq -1$) is a parameter and $\hat{z}$ denotes a new approximation. This family has a cubic convergence and produces some well-known methods.

The purpose of this paper is to revisit some known zero-finding iterative methods and investigate their interdependence. We show that the Laguerre

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iteration

$$(2)\qquad \hat{z} = L(f,\nu;z) := z - \frac{\nu f(z)}{f'(z) \pm \sqrt{(\nu-1)^2 f'(z)^2 - \nu(\nu-1)f(z)f''(z)}},$$

where ν ($\neq 0,1$) is a parameter, can be derived from Halley's method and present how some special cases of Laguerre's method reduce to the well-known methods by suitable choice of the parameter ν (Section 2). In Section 3 we show that Hansen-Patrick's family (1) is not a new one but it directly follows from the classical Laguerre method (2). According to Henrici [5, p. 532], the argument of the square root appearing in (1) and (2) *is to be chosen to differ by less than* $\pi/2$ *from the argument of* $f'(z)$. Finally, in Section 4 we give a new derivation of a class of third-order methods for multiple zeros and present some special cases.

2. A new derivation of Laguerre's method

Extensive studies of Laguerre's method (2) can be found in [5] and [7] (see, also, [1], [3], [6], [8]). Two modifications of Laguerre's method, which provide simultaneous determination of all simple zeros of a polynomial and have the order of convergence at least four, were presented in [4]. Further improvements of these methods were proposed in [12]. These simultaneous methods are realized in ordinary ("point") complex arithmetic and possess the order of convergence at least four. Initial conditions that provide the guaranteed convergence of the basic variant of Laguerre's simultaneous method are considered in [13]. Interval versions of Laguerre's method for the simultaneous inclusion of simple complex zeros of a polynomial are presented in [9] and [10].

In this section we will give a new and simple derivation of Laguerre's iteration formula (2). Let be given an analytic function f whose simple zero ζ is sought and let z be an approximation to this zero. Then the improved approximation $\hat{z}$ can be obtained by the classical Halley iteration, given by the iteration formula

$$(3)\qquad \hat{z} = z - \frac{f(z)}{f'(z) - \dfrac{f(z)f''(z)}{2f'(z)}}\,.$$

The order of convergence of the Halley method is three, the same as the Laguerre method (2). It is known that Laguerre's and Halley's methods converge globally and monotonically in the case when f is a polynomial with all real roots (e.g., see Davies and Dawson [2]). Moreover, Laguerre's metod shows extremely good behavior when $|z|$ is large, see Parlett [8].

Let us now apply Halley's method (3) to the function νf, where ν is a parameter ($\nu \neq 0,1$). We obtain

$$\hat{z} \;=\; z - \frac{\nu f(z)}{\nu f'(z) - \nu\dfrac{f(z)f''(z)}{2f'(z)}}$$

$$= z - \frac{\nu f(z)}{f'(z) + (\nu-1)f'(z) - \nu \dfrac{f(z)f''(z)}{2f'(z)}}\,. \tag{4}$$

By using the approximation

$$\sqrt{1-x} \approx 1 - \frac{x}{2},$$

for reasonably small $|x|$, and applying it to the appropriate part of the denominator of (4), we obtain

$$\begin{aligned}
\hat{z} &= z - \frac{\nu f(z)}{f'(z) + (\nu-1)f'(z)\left[1 - \dfrac{1}{2}\dfrac{\nu}{\nu-1}\dfrac{f(z)f''(z)}{f'(z)^2}\right]} \\
&= z - \frac{\nu f(z)}{f'(z) + (\nu-1)f'(z)\sqrt{1 - \dfrac{\nu}{\nu-1}\dfrac{f(z)f''(z)}{f'(z)^2}}} \\
&= z - \frac{\nu f(z)}{f'(z) \pm \sqrt{(\nu-1)^2 f'(z)^2 - \nu(\nu-1)f(z)f''(z)}} \\
&= L(f,\nu;z)\,.
\end{aligned}$$

The last formula represents the Laguerre iteration (2). Obviously, in this derivation we have assumed that z is a reasonably good approximation to the zero ζ so that the quantity

$$\left|\frac{\nu}{\nu-1}\frac{f(z)f''(z)}{f'(z)^2}\right|$$

is sufficiently small. The choice of the sign in front of the square root is performed according to Henrici's criterion given in Introduction.

3. Hansen-Patrick's family is of Laguerre's type

In [3] Hansen and Patrick derived a family of zero-finding methods (1) through an extensive procedure. Now we will show that this family is not new; actually, it can be obtained quite easily from Laguerre's method (2) by a special choice of the parameter ν. Substituting $\nu = 1/\alpha + 1$ in (2) we obtain

$$\begin{aligned}
\hat{z} &= z - \frac{\dfrac{\alpha+1}{\alpha}f(z)}{f'(z) \pm \sqrt{\dfrac{1}{\alpha^2}f'(z)^2 - \dfrac{\alpha+1}{\alpha}f(z)f''(z)}} \\
&= z - \frac{(\alpha+1)f(z)}{\alpha f'(z) \pm \sqrt{f'(z)^2 - (\alpha+1)f(z)f''(z)}}\,,
\end{aligned}$$

which is equivalent to Hansen-Patrick's formula (1).

Special cases of Laguerre's method

Starting from the Laguerre iteration formula (2) we can obtain some well-known methods by a suitable choice of the parameter ν. We will demonstrate this by several examples.

I) Taking $\nu = 1$, formula (2) reduces to **Newton's method**

$$\hat{z} = L(f, 1; z) = z - \frac{f(z)}{f'(z)},$$

which has the quadratic convergence.

II) Setting $\nu = 2$ in (2), we directly obtain **Euler's method**

$$\hat{z} = L(f, 2; z) = z - \frac{2f(z)}{f'(z) \pm \sqrt{f'(z)^2 - 2f(z)f''(z)}},$$

which is of the third order.

III) For $\nu = 0$ we will show that (2) reduces to **Halley's method**. We start from Laguerre's formula (2) written in the equivalent form

$$\hat{z} = z - \frac{\nu f\Big(f' \pm \sqrt{(\nu-1)^2 f'^2 - \nu(\nu-1)ff''}\Big)}{f'^2 - (\nu-1)^2 f'^2 + \nu(\nu-1)ff''},$$

wherefrom we have

$$\hat{z} = z - \frac{f\Big(f' \pm \sqrt{(\nu-1)^2 f'^2 - \nu(\nu-1)ff''}\Big)}{(2-\nu)f'^2 + (\nu-1)ff''}.$$

Putting $\nu = 0$ in the last formula, we get in a limit process

$$\hat{z} = L(f; 0; z) = z - \frac{f(z)}{f'(z) - \dfrac{f(z)f''(z)}{2f'(z)}}.$$

IV) If we let $\nu \to \infty$ in Laguerre's formula, we obtain the third-order **Ostrowski method**

$$\hat{z} = L(f, \infty; z) = z - \frac{f(z)}{\sqrt{(f'(z))^2 - f''(z)f(z)}}.$$

4. Laguerre's class of methods for multiple zeros

Let ζ be the zero of f of the (known) order m. We introduce the function

$$F(z) = f(z)^{1/m}$$

for which ζ is a simple zero. In this section we will omit sometimes the argument z for simplicity. The first two derivatives of F are

$$\begin{aligned} F' &= \frac{1}{m} f^{1/m-1} f' = \frac{Ff'}{mf}, \\ F'' &= F \cdot \frac{f'^2(1-m) + mff''}{m^2 f^2}. \end{aligned}$$

Applying Laguerre's iteration (2) to the function F, we obtain the iterative procedure:

$$\begin{aligned} \hat{z} &= z - \frac{\nu F}{F' \pm \sqrt{(\nu-1)^2 F'^2 - \nu(\nu-1) F F''}} \\ &= z - \frac{\nu F}{\dfrac{Ff'}{mf} \pm \sqrt{(\nu-1)^2 \dfrac{F^2 f'^2}{m^2 f^2} - \nu(\nu-1) F^2 \dfrac{f'^2(1-m) + mff''}{m^2 f^2}}}. \end{aligned}$$

After short rearrangement we get a class of iterative methods of Laguerre's type for finding a multiple zero of the function f, given by the iteration formula

$$\hat{z} = L_m(f, \nu; z) = z - \frac{m\nu f}{f' \pm \sqrt{(m\nu-1)(\nu-1)f'^2 - m\nu(\nu-1)ff''}}. \tag{5}$$

Substituting formally $m\nu$ by a new parameter λ, from (5) one arrives at the counterpart of Laguerre's method

$$\hat{z} = z - \frac{\lambda f}{f' \pm \sqrt{\left(\dfrac{\lambda - m}{m}\right)\left[(\lambda-1)f'^2 - \lambda ff''\right]}} \tag{6}$$

which was known to Bodewig [1].

Some special cases

From the family (5) (or (6)) it is possible to obtain some special cases of iterative methods for finding multiple zeros of functions, which are well known in the literature. Let us note that all multiple zero counterparts of the methods for simple zeros (presented in Section 3) are obtained for the same value of the parameter ν.

I) Taking $\nu = 1$ in (5) we obtain the second-order **Newton method** for multiple zeros, known also as Schröder's method,

$$\hat{z} = L_m(f, 1; z) = z - m\frac{f(z)}{f'(z)} .$$

II) Putting $\nu = 2$ in (5) we obtain the third-order **Euler method** for multiple zeros

$$\hat{z} = L_m(f, 2; z) = z - \frac{2mf(z)}{f' \pm \sqrt{(2m-1)f'(z)^2 - 2mf(z)f''(z)}} .$$

III) Starting from the equivalent form of the method (5), written as

$$\hat{z} = z - \frac{f' \pm \sqrt{(m\nu - 1)(\nu - 1)f'^2 - m\nu(\nu - 1)ff''}}{(m + \nu - m\nu)f'^2 + m(\nu - 1)ff''} \cdot mf ,$$

and letting $\nu = 0$, one obtains in a limit process

$$\hat{z} = L_m(f, 0; z) = z - \frac{f(z)}{\frac{m+1}{2m}f'(z) - \frac{f(z)f''(z)}{2f'(z)}} .$$

This is **Halley's method** for multiple zeros of the third order.

IV) If we let $\nu \to \infty$ in (5), then we obtain the well-known **Ostrowski method** of the third order

$$\hat{z} = L_m(f, \infty; z) = z - \frac{\sqrt{m}f}{\sqrt{f'^2 - f''f}} .$$

V) Taking the parameter ν to be $\nu = \dfrac{1}{\alpha}+1$, the iteration formula (5) becomes

$$\hat{z} = L_m(f, 1/\alpha + 1; z) = z - \frac{m(\alpha + 1)f}{\alpha f' \pm \sqrt{(m(\alpha + 1) - \alpha)f'^2 - m(\alpha + 1)ff''}} .$$

This is actually **Hansen-Patrick's family** for multiple zeros derived in [3] in a more complex form ($\alpha \to m\alpha$)

$$\hat{z} = z - \frac{m(m\alpha + 1)f}{m\alpha f' \pm \sqrt{m(\alpha(m-1)+1)f'^2 - m(m\alpha + 1)ff''}}$$

starting from the function $f(z) = (z - \zeta)^m g(z)$, $g(\zeta) \neq 0$.

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Received by the editors October 4, 2002