Novi Sad J. Math.
Vol. 32, No. 2, 2002, 95-108

POSITIVE SOLUTIONS OF NEUTRAL DELAY DIFFERENTIAL EQUATION

Hajnalka Péics[1], János Karsai[2]

Abstract. We consider the scalar nonautonomous neutral delay differential equation with variable delays

$$\frac{d}{dt}\left[x(t)+\sum_{j=1}^{\ell}p_j(t)x(t-\tau_j(t))\right]+\sum_{i=1}^{m}q_i(t)x(t-\sigma_i(t))=0,$$

for $t_0 \le t < T \le \infty$. Using the method of characteristic equations, we give conditions for the existence of positive solutions. Our theorems generalize and extend the results for simpler cases proved by Chuanxi, Ladas [1] and Győri, Ladas [5].

AMS Mathematics Subject Classification (2000): 34K15, 34K25

Key words and phrases: neutral delay differential equation

1 Introduction

Neutral delay differential equations contain the derivative of the unknown function both with and without delays. Some new phenomena can appear, hence the theory of neutral delay differential equations is even more complicated than the theory of non-neutral delay equations. The oscillatory behavior of the solutions of neutral systems is of importance in both the theory and applications, such as the motion of radiating electrons, population growth, the spread of epidemics, in networks containing lossless transmission lines (see [2], [5], [6], [7] and the references therein).

In our paper we consider the scalar nonautonomous neutral delay differential equation with variable delays and coefficients of the form

$$\frac{d}{dt}\left[x(t)+\sum_{j=1}^{\ell}p_j(t)x(t-\tau_j(t))\right]+\sum_{i=1}^{m}q_i(t)x(t-\sigma_i(t))=0, \tag{1}$$

for $t_0 \le t < T \le \infty$, where the next hypotheses are satisfied:

(H_1) $\quad p_j \in C^1[[t_0,T),\mathbf{R}]$, $\tau_j \in C^1[[t_0,T),\mathbf{R}^+]$, $j=1,2,...,\ell$;

[1]University of Novi Sad, Faculty of Civil Engineering, Subotica, Yugoslavia, E-mail: peics@gf.su.ac.yu
[2]University of Szeged, Szeged, Hungary, E-mail: karsai@dmi.u-szeged.hu

(H_2) $\qquad q_i \in C[[t_0,T),\mathbf{R}],\ \sigma_i \in C[[t_0,T),\mathbf{R}^+],\ i=1,2,...,m.$

The oscillatory and asymptotic behavior of the solutions of non-neutral delay differential equations with variable coefficients and variable delays and also for neutral differential equations with constant delays have been studied in many papers (see, for example, [7, 3, 4, 8]). A quite comprehensive treatment of such results is given in the monograph [5] by I. Győri and G. Ladas.

One of the most important methods of such investigations is the method of generalized characteristic equation, which is based on the idea of finding solutions of linear systems in the form

$$x(t) = \exp\left(\int_{t_0}^{t} \alpha(s)ds\right). \tag{2}$$

Our main goal is to apply this method to equation (1) to find conditions for the existence of positive solutions, and to generalize and extend the results proved for special cases of (1).

Before formulating our results, we point to two characteristic results of the recent investigations. Chuanxi and Ladas in [1] investigated the particular case of equation (1) of the form

$$\frac{d}{dt}\left[x(t)+P(t)x(t-\tau)\right]+Q(t)x(t-\sigma)=0 \tag{3}$$

where

$$P\in C^1[[t_0,\infty),\mathbf{R}],\quad Q\in C[[t_0,\infty),\mathbf{R}],\quad \tau\in(0,\infty),\quad \sigma\in[0,\infty), \tag{4}$$

They proved the following result.

Theorem A. *Assume that (4) holds and that there exists a positive number μ such that*

$$|P(t)|\mu e^{\mu\tau}+|\dot{P}(t)|e^{\mu\tau}+|Q(t)|e^{\mu\sigma}\le\mu \quad \text{for} \quad t\ge t_0. \tag{5}$$

Then, for every $t_1 \ge t_0$, equation (3) has a positive solution on $[t_1,\infty)$.

The case of variable delays has been considered for equations of the form

$$\dot{x}(t)+\sum_{i=1}^{m} q_i(t)x(t-\sigma_i(t))=0, \tag{6}$$

where, for $t_0 < T \le \infty$, $q_i \in C[[t_0,T),\mathbf{R}]$, $\quad \sigma_i \in C[[t_0,T),\mathbf{R}^+]$, $\quad i=1,2,...,m$, by many authors. Results that give sufficient conditions for the existence of positive solutions of equation (6) on $[t_0,T)$ can be found in [5].

Theorem B. *Assume that there exists a positive number μ such that*

$$\sum_{i=1}^{m} |q_i(t)|\, e^{\mu\sigma_i(t)} \le \mu$$

for $t_0 \le t < T$*. Then for every*

$$\Phi \in \{\varphi \in C[[t_{-1}, t_0], \mathbf{R}^+] : \varphi(t_0) > 0, \varphi(t) \le \varphi(t_0), t_{-1} \le t \le t_0\},$$

the solution of equation (6) through (t_0, Φ) *remains positive for* $t_0 \le t < T$*.*

2. Notations, definitions

Define

$$T^1_{-1} = \min_{1\le j\le \ell}\left\{\inf_{t_0\le t<T}\{t-\tau_j(t)\}\right\}, \quad T^2_{-1} = \min_{1\le i\le m}\left\{\inf_{t_0\le t<T}\{t-\sigma_i(t)\}\right\}$$

and

$$t_{-1} = \min\{T^1_{-1}, T^2_{-1}\}.$$

A function $x : [t_{-1}, T) \to \mathbf{R}$ is called a solution of equation (1) if x is continuous on $[t_{-1}, T)$ and satisfies equation (1) on (t_0, T). An initial condition for the solutions of equation (1) is given in the form

$$x(t) = \Phi(t), \quad t_{-1} \le t \le t_0, \quad \Phi \in C^1[[t_{-1}, t_0), \mathbf{R}^+]. \tag{7}$$

A solution of the initial value problem (1) and (7) is a continuous function defined on $[t_{-1}, T)$ which coincides with Φ on $[t_{-1}, t_0)$ such that the difference $x(t)+\sum_{j=1}^{\ell} p_j(t)x(t-\tau_j(t))$ is differentiable and satisfies equation (1) on (t_0, T).

The unique solution of the initial value problem (1) and (7) is denoted by $x = x(\Phi)$ and it exists throughout the interval $[t_0, T)$.

The continuous function $x : [t_{-1}, T) \to \mathbf{R}$ is oscillatory if x has arbitrarily large zeros, i.e., for every $a \ge t_{-1}$, there exists a number $c > a$ such that $x(c) = 0$. Otherwise, x is called nonoscillatory.

Rewrite equation (1) as

$$\begin{aligned} \dot{x}(t) \quad &+ \quad \sum_{j=1}^{\ell}[p_j(t)(1-\dot{\tau_j}(t))\dot{x}(t-\tau_j(t)) + \dot{p_j}(t)x(t-\tau_j(t))] + \\ &+ \quad \sum_{i=1}^{m} q_i(t)x(t-\sigma_i(t)) = 0. \end{aligned}$$

The initial value problem for this form is as follows. Let Φ be given by (7). A solution of the initial value problem (1) and (7) is a continuous function defined on $[t_{-1}, T)$ that coincides with Φ on $[t_{-1}, t_0)$, x being continuously differentiable and satisfies equation (1) on (t_0, T) except at the points $k\,r$, where $r = t_0 - t_{-1}$, $k = 0, 1, 2, ...$

On the other hand, if

$$\dot{\Phi}(t_0) = -\sum_{j=1}^{\ell} \dot{p}_j(t)\Phi(t_0-\tau_j(t_0)) - \\ - \sum_{j=1}^{\ell} p_j(t_0)(1-\dot{\tau}_j(t_0))\dot{\Phi}(t_0-\tau_j(t_0)) - \sum_{i=1}^{m} q_i(t)\Phi(t-\sigma_i(t)), \tag{8}$$

then the solution x is continuously differentiable for all $t \geq t_{-1}$. Consequently, relation (8) is necessary and sufficient for the solution x to have a continuous derivative for all $t \geq t_{-1}$.

In the next section we define precisely the generalized characteristic equation associated with the initial value problem (1) and (7). Using the presentation (2) we will obtain the integral equation of the form

$$\alpha(t) + \sum_{j=1}^{\ell} \left[p_j(t)(1-\dot{\tau}_j(t)) \frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)} \alpha(H_j(t)) + \dot{p}_j(t) \frac{\Phi(h_j(t))}{\Phi(t_0)} \right] \times \\ \times \exp\left(- \int_{H_j(t)}^{t} \alpha(s)ds \right) + \sum_{i=1}^{m} q_i(t) \frac{\Phi(g_i(t))}{\Phi(t_0)} \exp\left(- \int_{G_i(t)}^{t} \alpha(s)ds \right) = 0, \tag{9}$$

which is called characteristic equation.

We will use the following notations:

$$h_j(t) = \min\{t_0, t-\tau_j(t)\}, \quad H_j(t) = \max\{t_0, t-\tau_j(t)\}, \quad t \in [t_0, T), \quad j = 1, 2, ..., \ell$$

$$g_i(t) = \min\{t_0, t-\sigma_i(t)\}, \quad G_i(t) = \max\{t_0, t-\sigma_i(t)\}, \quad t \in [t_0, T), \quad i = 1, 2, ..., m.$$

Finally, $[a]_+ := \max(0, a)$ and $[a]_- := \max(0, -a)$ denote the positive and negative part of the real number a, respectively.

3. Main Results

Now we can formulate our main theorem.

Theorem 1. *Suppose that (H_1) and (H_2) hold and let (8) and $\dot{\Phi}(t_0) > 0$ be satisfied. Then the following statements are equivalent:*

(a) *The initial value problem (1) and (7) has a positive solution on $[t_0, T)$.*

(b) *The generalized characteristic equation (9) has a continuous solution on $[t_0, T)$.*

(c) *There exist functions $\beta, \gamma \in C[[t_0, T), \mathbf{R}]$ such that $\beta(t) \leq \gamma(t)$ such that*

$$\beta(t) \leq \delta(t) \leq \gamma(t) \quad \textit{implies} \quad \beta(t) \leq (S\delta)(t) \leq \gamma(t), \tag{10}$$

for every function $\delta \in C[[t_0, T), \mathbf{R}]$ *and* $t_0 \le t < T$*, where*

$$(S\delta)(t) = -\sum_{j=1}^{\ell}\left[p_j(t)(1-\dot{\tau}_j(t))\frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)}\delta(H_j(t)) + \dot{p_j}(t)\frac{\Phi(h_j(t))}{\Phi(t_0)}\right] \times \tag{11}$$

$$\times \quad \exp\left(-\int_{H_j(t)}^{t}\delta(s)ds\right) - \sum_{i=1}^{m} q_i(t)\frac{\Phi(g_i(t))}{\Phi(t_0)}\exp\left(-\int_{G_i(t)}^{t}\delta(s)ds\right).$$

Proof. $(a) \Rightarrow (b)$: Let $x = x(\Phi)$ be the solution of the initial value problem (1) and (7) and suppose that $x(t) > 0$ for $t_0 \le t < T$. It will be shown that the continuous function α defined by

$$\alpha(t) = \frac{\dot{x}(t)}{x(t)}, \quad t_0 \le t < T,$$

is a solution of (9) on $[t_0, T)$. Equation (1) is equivalent to the form

$$\dot{x}(t) \quad + \quad \sum_{j=1}^{\ell}[p_j(t)(1-\dot{\tau}_j(t))\dot{x}(t-\tau_j(t)) + \dot{p_j}(t)x(t-\tau_j(t))] +$$

$$+ \quad \sum_{i=1}^{m} q_i(t)x(t-\sigma_i(t)) = 0.$$

By dividing both sides of the above equation by $x(t)$, we obtain that

$$\frac{\dot{x}(t)}{x(t)} \quad + \quad \sum_{j=1}^{\ell}\left[p_j(t)(1-\dot{\tau}_j(t))\frac{\dot{x}(t-\tau_j(t))}{x(t)} + \dot{p_j}(t)\frac{x(t-\tau_j(t))}{x(t)}\right] +$$

$$+ \quad \sum_{i=1}^{m} q_i(t)\frac{x(t-\sigma_i(t))}{x(t)} = 0.$$

It follows from the definition of α that

$$x(t) = \Phi(t_0)\exp\left(\int_{t_0}^{t}\alpha(s)ds\right),$$

and hence

$$\frac{x(H_j(t))}{x(t)} = \exp\left(-\int_{H_j(t)}^{t}\alpha(s)ds\right), \quad \frac{x(G_i(t))}{x(t)} = \exp\left(-\int_{G_i(t)}^{t}\alpha(s)ds\right),$$

where $j = 1, 2, ..., \ell$, $i = 1, 2, ..., m$, and $t_0 \le t < T$. It is obvious for the same values of j, i and t that

$$\frac{x(t-\tau_j(t))}{x(H_j(t))} = \frac{\Phi(h_j(t))}{\Phi(t_0)}, \quad \frac{x(t-\sigma_i(t))}{x(G_i(t))} = \frac{\Phi(g_i(t))}{\Phi(t_0)}.$$

It remains to prove that

$$\frac{\dot{x}(t-\tau_j(t))}{\dot{x}(H_j(t))} = \frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)}, \quad t_0 \le t < T, \quad j = 1, 2, ..., \ell.$$

Observe that $t - \tau_j(t) \ge t_0$ implies $h_j(t) = t_0$ and $H_j(t) = t - \tau_j(t)$, and hence

$$\frac{\dot{x}(t-\tau_j(t))}{\dot{x}(H_j(t))} = \frac{\dot{x}(H_j(t))}{\dot{x}(H_j(t))} = 1 = \frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)}.$$

On the other hand, $t - \tau_j(t) < t_0$ implies $h_j(t) = t - \tau_j(t)$ and $H_j(t) = t_0$, and hence

$$\frac{\dot{x}(t-\tau_j(t))}{\dot{x}(H_j(t))} = \frac{\dot{x}(h_j(t))}{\dot{x}(t_0)} = \frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)}.$$

Using these equalities and the definition of α, we obtain that equality (9) holds, and hence the proof of the part "$(a) \Rightarrow (b)$" is complete.

$(b) \Rightarrow (c)$: If α is a continuous solution of (9), then take $\beta(t) \equiv \gamma(t) \equiv \alpha(t)$, $t_0 \le t < T$ and the proof is obvious because of the fact that $\alpha = S\alpha$.

$(c) \Rightarrow (a)$: First it must be shown that, under hypothesis (c), equation (9) has a continuous solution $\alpha(t)$ on $[t_0, T)$, and the function x defined by

$$(12) \qquad x(t) = \begin{cases} \Phi(t), & t_{-1} \le t < t_0; \\ \Phi(t_0) \exp\left(\int_{t_0}^t \alpha(s) ds\right), & t_0 \le t < T \end{cases}$$

is a positive solution of the initial value problem (1) and (7).

The continuous solution of equation (9) will be constructed as the limit of a sequence of functions $\{\alpha_k(t)\}$ defined by the following successive approximation. Take any function $\alpha_0 \in C[[t_0, T), \mathbf{R}]$ such that

$$\beta(t) \le \alpha_0(t) \le \gamma(t), \quad t_0 \le t < T$$

and set

$$\alpha_{k+1}(t) = (S\alpha_k)(t), \quad t_0 \le t < T \text{ for } k = 0, 1, 2, ...$$

If follows from the assumption (10) that

$$(13) \qquad \beta(t) \le \alpha_k(t) \le \gamma(t), \quad t_0 \le t < T, \ k = 0, 1, 2, ...,$$

and clearly $\alpha_k \in C[[t_0, T), \mathbf{R}]$. We show that the sequence $\{\alpha_k(t)\}$ converges uniformly on any compact subinterval $[t_0, T_1]$ of $[t_0, T)$. Set

$$M_1 := \max_{t_0 \le t \le T_1} \sum_{j=1}^{\ell} \left| p_j(t)(1 - \dot{\tau}_j(t)) \frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)} \right|, \quad M_2 := \max_{t_0 \le t \le T_1} \sum_{j=1}^{\ell} \left| \dot{p}_j(t) \frac{\Phi(h_j(t))}{\Phi(t_0)} \right|,$$

$$M_3 := \max_{t_0 \le t \le T_1} \sum_{i=1}^{m} \left| q_i(t) \frac{\Phi(g_i(t))}{\Phi(t_0)} \right|, \quad L := \max_{t_0 \le t \le T_1} \left\{ \max \left\{ |\beta(t)|, |\gamma(t)| \right\} \right\},$$

$M := \max\{M_1, M_2, M_3\}, \quad N_1 := Me^{L(T_1 - t_0)}, \quad N := \max\{N_1(L+2), 2LN_1\}.$

Then from (13) we obtain that

$$\max_{t_0 \le t \le T_1} |\alpha_k(t)| \le L, \quad k = 0, 1, 2, ...$$

Using the mean value theorem we have

$$\exp\left(-\int_{H_j(t)}^{t} \alpha_k(s)ds\right) - \exp\left(-\int_{H_j(t)}^{t} \alpha_{k-1}(s)ds\right)$$
$$= e^{-\mu_{k,j}(t)} \int_{H_j(t)}^{t} \left(\alpha_k(s) - \alpha_{k-1}(s)\right) ds,$$

for every $j = 1, 2, ..., \ell$, $k = 0, 1, 2, ...$ and $t_0 \le t \le T_1$, where $\mu_{k,j}(t)$ is between

$$\int_{H_j(t)}^{t} \alpha_k(s)ds \quad \text{and} \quad \int_{H_j(t)}^{t} \alpha_{k-1}(s)ds.$$

Since $H_j(t) \ge t_0$ for $j = 1, 2, ..., \ell$ and $t_0 \le t \le T_1$, $|\mu_{k,j}(t)| \le L(T_1 - t_0)$ and

$$\left| \exp\left(-\int_{H_j(t)}^{t} \alpha_k(s)ds\right) - \exp\left(-\int_{H_j(t)}^{t} \alpha_{k-1}(s)ds\right) \right|$$
$$\le e^{L(T_1 - t_0)} \int_{t_0}^{t} \left|\alpha_k(s) - \alpha_{k-1}(s)\right| ds.$$

Similarly,

$$\left| \exp\left(-\int_{G_i(t)}^{t} \alpha_k(s)ds\right) - \exp\left(-\int_{G_i(t)}^{t} \alpha_{k-1}(s)ds\right) \right|$$
$$\le e^{L(T_1 - t_0)} \int_{t_0}^{t} \left|\alpha_k(s) - \alpha_{k-1}(s)\right| ds$$

for $i = 1, 2, ..., m$, $k = 1, 2, ...$ and $t_0 \le t \le T_1$. Repeating the above arguments, we also have

$$\left| \alpha_k(H_j(t)) \exp\left(-\int_{H_j(t)}^{t} \alpha_k(s)ds\right) - \alpha_{k-1}(H_j(t)) \exp\left(-\int_{H_j(t)}^{t} \alpha_{k-1}(s)ds\right) \right| \le$$

$$\le \left| \alpha_k(H_j(t)) \left[\exp\left(-\int_{H_j(t)}^{t} \alpha_k(s)ds\right) - \exp\left(-\int_{H_j(t)}^{t} \alpha_{k-1}(s)ds\right) \right] \right| +$$

$$+\left|[\alpha_k(H_j(t))-\alpha_{k-1}(H_j(t))]\exp\left(-\int_{H_j(t)}^t \alpha_{k-1}(s)ds\right)\right| \le$$
$$\le Le^{L(T_1-t_0)}\int_{t_0}^t |\alpha_k(s)-\alpha_{k-1}(s)|\,ds + 2Le^{L(T_1-t_0)}.$$

Thus,

$$|\alpha_{k+1}(t)-\alpha_k(t)| \le \sum_{j=1}^{\ell}\left|p_j(t)(1-\dot{\tau}_j(t))\frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)}\right| \times$$
$$\times\left|\alpha_k(H_j(t))\exp\left(-\int_{H_j(t)}^t \alpha_k(s)ds\right)-\alpha_{k-1}(H_j(t))\exp\left(-\int_{H_j(t)}^t \alpha_{k-1}(s)ds\right)\right| +$$
$$+\sum_{j=1}^{\ell}\left|\dot{p}_j(t)\frac{\Phi(h_j(t))}{\Phi(t_0)}\right|\left|\exp\left(-\int_{H_j(t)}^t \alpha_k(s)ds\right)-\exp\left(-\int_{H_j(t)}^t \alpha_{k-1}(s)ds\right)\right| +$$
$$+\sum_{i=1}^{m}\left|q_i(t)\frac{\Phi(g_i(t))}{\Phi(t_0)}\right|\left|\exp\left(-\int_{G_i(t)}^t \alpha_k(s)ds\right)-\exp\left(-\int_{G_i(t)}^t \alpha_{k-1}(s)ds\right)\right| \le$$
$$\le 2LN_1 + N_1(L+2)\int_{t_0}^t |\alpha_k(s)-\alpha_{k-1}(s)|\,ds \le N + N\int_{t_0}^t |\alpha_k(s)-\alpha_{k-1}(s)|\,ds,$$

and now, we can see that

$$|\alpha_{k+1}(t)-\alpha_k(t)| \le N\sum_{i=0}^{k-1}\frac{N(t-t_0)^i}{i!} + 2L\frac{N(t-t_0)^k}{k!}.$$

for $k = 0, 1, 2, ...$ and $t_0 \le t \le T_1$. Since

$$\sum_{i=0}^{\infty}\frac{N(t-t_0)^i}{i!} = e^{N(t-t_0)} \quad \text{and} \quad \lim_{k\to\infty}\frac{N(t-t_0)^k}{k!} = 0 \quad \text{for} \quad t_0 \le t \le T_1,$$

it follows from the Weierstrass criterion that the sequence defined by

$$\alpha_k(t) = \alpha_0(t) + \sum_{j=0}^{k-1}[\alpha_{j+1}(t)-\alpha_j(t)] \;\; (k = 0, 1, 2, ..., \; t_0 \le t \le T_1)$$

converges uniformly, and hence the limit function

$$\alpha(t) = \lim_{k\to\infty}\alpha_k(t) \tag{14}$$

is continuous and solves equation (9) on $[t_0, T_1]$.

Finally, the fact that $x(t)$ defined by (12) is the solution of the initial value problem (1) and (7) can be verified by direct substitution:

$$\begin{aligned}
&\dot{x}(t) = x(t)\alpha(t) = \\
&= -x(t)\sum_{j=1}^{\ell}\left[p_j(t)(1-\dot{\tau}_j(t))\frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)}\alpha(H_j(t)) + \dot{p_j}(t)\frac{\Phi(h_j(t))}{\Phi(t_0)}\right]\times \\
&\times \exp\left(-\int_{H_j(t)}^{t}\alpha(s)ds\right) - x(t)\sum_{i=1}^{m} q_i(t)\frac{\Phi(g_i(t))}{\Phi(t_0)}\exp\left(-\int_{G_i(t)}^{t}\alpha(s)ds\right) = \\
&= -x(t)\sum_{j=1}^{\ell} p_j(t)(1-\dot{\tau}_j(t))\frac{\dot{x}(t-\tau_j(t))}{\dot{x}(H_j(t))}\frac{\dot{x}(H_j(t))}{x(H_j(t))}\frac{x(H_j(t))}{x(t)} - \\
&-x(t)\sum_{j=1}^{\ell}\dot{p_j}(t)\frac{x(t-\tau_j(t))}{x(H_j(t))}\frac{x(H_j(t))}{x(t)} - x(t)\sum_{i=1}^{m} q_i(t)\frac{x(t-\sigma_i(t))}{x(G_i(t))}\frac{x(G_i(t))}{x(t)} = \\
&= -\sum_{j=1}^{\ell}\left[p_j(t)(1-\dot{\tau}_j(t))\dot{x}(t-\tau_j(t)) + \dot{p_j}(t)x(t-\tau_j(t))\right] - \sum_{i=1}^{m} q_i(t)x(t-\sigma_i(t)) = \\
&= -\sum_{j=1}^{\ell}\frac{d}{dt}\left[p_j(t)x(t-\tau_j(t))\right] - \sum_{i=1}^{m} q_i(t)x(t-\sigma_i(t))
\end{aligned}$$

for $t_0 \le t < T$. It completes the proof of Theorem 1. □

The above theorem generalizes Theorem 3.1.1. in [5], proved for the non-neutral equation (6).

Remark. It is clear from the proof that the positive solution of equation (1) has to satisfy the inequality

$$\Phi(t_0)\exp\left(\int_{t_0}^{t}\beta(s)ds\right) \le x(t) \le \Phi(t_0)\exp\left(\int_{t_0}^{t}\gamma(s)ds\right) \tag{15}$$

for $t_0 \le t < T$.

4. Existence of Positive Solutions

Using Theorem 1 we formulate conditions for the existence of positive solutions. Similar results can also be proved for the existence of negative solutions.

Let

$$\mathbf{F} := \{\Phi \in C^1[[t_{-1}, t_0], \mathbf{R}^+] : 0 < \Phi(t) \le \Phi(t_0), 0 < \dot{\Phi}(t) \le \dot{\Phi}(t_0), t_{-1} \le t \le t_0\}.$$

The next theorem is a common generalization of Theorems 1 and 1.

Theorem 2. *Assume that (H_1) and (H_2) hold, and there exists a positive number μ such that*

$$\sum_{j=1}^{\ell} \left[|p_j(t)(1-\dot{\tau}_j(t))|\, \mu + |\dot{p_j}(t)| \right] e^{\mu\tau_j(t)} + \sum_{i=1}^{m} |q_i(t)|\, e^{\mu\sigma_i(t)} \le \mu \tag{16}$$

for $t_0 \le t < T$. Then, for every $\Phi \in \mathbf{F}$ which satisfies the condition (8), the solution $x(\Phi)$ of equation (1) remains positive for $t_0 \le t < T$.

Proof. We show that the conditions of part (c) in Theorem 1 are satisfied with $\beta(t) = -\mu$ and $\gamma(t) = \mu$ for $t_0 \le t < T$. For any continuous function δ, for which $\beta(t) \le \delta(t) \le \gamma$, we have

$$-\mu\tau_j(t) \le -\mu(t - H_j(t)) \le \int_{H_j(t)}^{t} \delta(s)ds \le \mu(t - H_j(t)) \le \mu\tau_j(t) \ \ (j = 1, 2, ..., \ell)$$

and

$$-\mu\sigma_i(t) \le -\mu(t - G_i(t)) \le \int_{G_i(t)}^{t} \delta(s)ds \le \mu(t - G_i(t)) \le \mu\sigma_i(t) \ \ (i = 1, 2, ..., m)$$

for $t_0 \le t < T$. Then, it follows that

$$\begin{aligned} -\mu \ &\le \ \sum_{j=1}^{\ell} \left[|p_j(t)(1-\dot{\tau}_j(t))|\, \mu + |\dot{p_j}(t)| \right] e^{\mu\tau_j(t)} - \sum_{i=1}^{m} |q_i(t)|\, e^{\mu\sigma_i(t)} \le \\ &\le \ (S\delta)(t) \le \\ &\le \ \sum_{j=1}^{\ell} \left[|p_j(t)(1-\dot{\tau}_j(t))|\, \mu + |\dot{p_j}(t)| \right] e^{\mu\tau_j(t)} + \sum_{i=1}^{m} |q_i(t)|\, e^{\mu\sigma_i(t)} \le \mu \end{aligned}$$

for $t_0 \le t < T$. Therefore, by Theorem 1, the solution $x(\Phi)(t)$ of equation (1) through (t_0, Φ) is positive on $[t_0, T)$ and the proof is complete. □

Apply this theorem to some special cases. Introduce the following notations. Let

$$\tau(t) := \max_{j=1,\ell} \tau_j(t), \ \ \sigma(t) := \max_{i=1,m} \sigma_i(t),$$

$$\bar{p}(t) := \sum_{j=1}^{\ell} |p_j(t)(1-\dot{\tau}_j(t))|, \ \ \bar{r}(t) := \sum_{j=1}^{\ell} |\dot{p}_j(t)|, \ \ \bar{q}(t) := \sum_{i=1}^{m} |q_i(t).|$$

Then, inequality (16) follows from the inequality

$$(\bar{p}(t)\mu + \bar{r}(t))e^{\mu\tau(t)} + \bar{q}(t)e^{\mu\sigma(t)} \le \mu, \tag{17}$$

which is identical to (16) for the case of single delays.

Now, consider an even more special case. Let

$$\tau := \sup_{t\in[t_0,T)} \tau(t), \quad \sigma := \sup_{t\in[t_0,T)} \sigma(t),$$

$$\bar{p} := \sup_{t\in[t_0,T)} \bar{p}(t), \quad \bar{r} := \sup_{t\in[t_0,T)} \bar{r}(t), \quad \bar{q} := \sup_{t\in[t_0,T)} \bar{r}(t)$$

be finite. Then, inequality (16) follows from the inequality

$$(\bar{p}\mu + \bar{r})e^{\mu\tau} + \bar{q}e^{\mu\sigma} \le \mu, \tag{18}$$

which is identical to (16) for the case of single constant delays and constant coefficients. In the case $\tau = \sigma = \lambda$, we have

$$e^{\mu\lambda} \le \frac{\mu}{\bar{p}\mu + \bar{r} + \bar{q}}. \tag{19}$$

If $1/\bar{p} > 1$ and $\lambda < \lambda_0$ for some critical λ_0, then (18) has a positive solution. The critical λ_0 can be found by observing that the derivatives of the left and right sides with respect to μ are equal for λ_0. Then, λ_0 is the unique solution of the equation

$$\exp\left(\frac{2A\,\lambda}{A\,\lambda + \sqrt{A\,\lambda}\,\sqrt{4\,\bar{p} + A\,\lambda}}\right) = \frac{2}{2\,\bar{p} + A\,\lambda + \sqrt{A\,\lambda}\,\sqrt{4\,\bar{p} + A\,\lambda}}$$

where $A = \bar{r} + \bar{q}$. Note that in the delay case $\bar{p} = \bar{r} = 0$, we obtain the known result $\lambda_0 = 1/(e\bar{q})$.

In the following theorem we assume an order of the dominance of delays. This condition agrees with several real phenomena.

Theorem 3. *Assume that* (H_1) *and* (H_2) *and the following hold for every* $t \in [t_0, T)$*:*

$$0 \le \tau_1(t) \le \tau_2(t) \le ... \le \tau_\ell(t), \tag{20}$$

$$0 \le \sigma_1(t) \le \sigma_2(t) \le ... \le \sigma_m(t), \tag{21}$$

$$\sum_{j=1}^{\nu} p_j(t)(1 - \dot{\tau}_j(t)) \le 0, \quad \sum_{j=1}^{\nu} \dot{p_j}(t) \le 0, \quad \nu = 1, 2, ..., \ell; \tag{22}$$

$$\sum_{i=1}^{\nu} q_i(t) \le 0 \quad \nu = 1, 2, ..., m; \tag{23}$$

$$\sum_{j=1}^{\ell} [p_j(t)(1 - \dot{\tau}_j(t))]_- < 1. \tag{24}$$

If there exists a positive increasing function $\gamma \in C[[t_0, T), \mathbf{R}]$ *such that*

$$\gamma(t) \geq \frac{\sum_{j=1}^{\ell} [\dot{p_j}(t)]_- + \sum_{i=1}^{m} [q_i(t)]_-}{1 - \sum_{j=1}^{\ell} [p_j(t)(1 - \dot{\tau}_j(t))]_-}, \tag{25}$$

then, for every $\Phi \in \mathbf{F}$ *which satisfies condition (8), equation (1) has a positive increasing solution on* $[t_0, T)$. *This solution satisfies the inequality*

$$x(t) \leq \Phi(t_0) \exp\left(\int_{t_0}^{t} \gamma(s) ds\right). \tag{26}$$

Proof. It will be shown that the statement (c) of Theorem 1 is true with $\beta(t) = 0$ and $\gamma(t)$ for $t_0 \leq t < T$. For any function $\delta \in [[t_0, T], \mathbf{R}]$ between β and γ holds that

$$\begin{aligned}
(S\delta)(t) &\leq \sum_{j=1}^{\ell} [p_j(t)(1 - \dot{\tau}_j(t))]_- \gamma(t) + \sum_{j=1}^{\ell} [\dot{p_j}(t)]_- + \sum_{i=1}^{m} [q_i(t)]_- \\
&\leq \gamma(t) \left[\sum_{j=1}^{\ell} [p_j(t)(1 - \dot{\tau}_j(t))]_-\right] + \gamma(t) \left[1 - \sum_{j=1}^{\ell} [p_j(t)(1 - \dot{\tau}_j(t))]_-\right] \\
&\leq \gamma(t).
\end{aligned}$$

Because of the inequalities

$$H_1(t) \geq H_2(t) \geq ... \geq H_\ell(t), \quad t_0 \leq t < T,$$

$$G_1(t) \geq G_2(t) \geq ... \geq G_m(t), \quad t_0 \leq t < T,$$

the relations (22) and (23) yield

$$\begin{aligned}
(S\delta)(t) \geq & \left[-\sum_{j=1}^{\ell} p_j(t)(1 - \dot{\tau}(t))\right] \min_{1 \leq j \leq \ell} \left\{\frac{\dot{\Phi}(h_j(t))}{\dot{\Phi}(t_0)}\right\} \cdot \\
& \cdot \; 0 \cdot \exp\left(-\int_{H_\ell(t)}^{t} \delta(s) ds\right) + \\
& + \left[-\sum_{j=1}^{\ell} \dot{p_j}(t)\right] \frac{\Phi(h_j(t))}{\Phi(t_0)} \exp\left(-\int_{H_\ell(t)}^{t} \delta(s) ds\right) + \\
& + \left[-\sum_{i=1}^{m} q_i(t)\right] \frac{\Phi(g_i(t))}{\Phi(t_0)} \exp\left(-\int_{G_m(t)}^{t} \delta(s) ds\right) \\
& \geq 0 \quad \text{for} \quad t_0 \leq t < T.
\end{aligned}$$

Therefore, the solution $x(t) = x(\Phi)(t)$ of equation (1) is positive on $[t_0, T)$. As in the proof of Theorem 1, $x(t)$ can be written in the form

$$x(t) = \Phi(t_0) \exp\left(\int_{t_0}^{t} \alpha(s)ds\right) \quad \text{for} \quad t_0 \le t < T,$$

where $\alpha(t)$ is a continuous solution of the characteristic equation (9) such that $0 \le \alpha(t) \le \gamma(t)$ for all $t_0 \le t < T$. Hence, x is an increasing solution of equation (1), and the proof is complete. □

Remark. Note that conditions (20) – (23) show that smaller delays has to be associated with larger coefficients. Conditions (22) and (23) formulate the same property for the functions $p_j(t)(1 - \dot{\tau}_j(t))$, $\dot{p}_j(t)$, and $q_i(t)$. For the numbers $\{a_1, a_2, ..., a_N\}$, the condition

$$\sum_{i=1}^{n} a_i \le 0 \text{ for every } n = 1, 2, ..., N$$

can be expanded to

$$a_1 \le 0,\ a_1 + a_2 \le 0,\ a_1 + a_2 + a_3 \le 0,\ ...,\ \sum_{i=1}^{N} a_i \le 0.$$

For example, this condition holds if $a_1 \le 0$ and $|a_1| \ge \sum_{i=2}^{N} |a_i|$.

The case of single delays is still of importance. In this case, conditions (20), (21) are empty, (22), (23), and (24) turn to $p_1(t)(1 - \dot{\tau}_1(t)) < -1$, $\dot{p}_1(t) \le 0$, and $q_1(t) \le 0$. Finally, (25) becomes

$$\gamma(t) \ge \frac{|\dot{p}_1(t)| + |q_1(t)|}{1 - |p_1(t)(1 - \dot{\tau}_1(t))|}.$$

For $\ell = m = 1$ and constant delays we obtain from our result, as a special case, a theorem of the existence of positive solutions, proved in [1] (Theorem 6.7.2.c). Another special case is $p_j(t) = 0$, for $j = 1, 2, ..., \ell$, $t \in [t_0, T)$. Then, our theorem implies the well known result for the non-neutral equations proved in [1] (Theorem 3.3.3.).

5. Acknowledgment

This paper was completed during the first author's visit to the Department of Medical Informatics at the University of Szeged from October 1, to November 30, 2001 under a fellowship Domus Hungarica. The research of J. Karsai is supported by Hungarian National Foundation for Scientific Research Grant no. T 034275. The authors express their thanks to Professor István Győri for valuable comments and help.

References

[1] Chuanxi, Q., Ladas, G., Existence of Positive Solutions for Neutral Differential Equations, Journal of Applied Mathematics and Simulation 2 (1989), 267-276.

[2] Driver, R. D., A Mixed Neutral System, Nonlinear Analysis, 8 (1984), 155-158.

[3] El-Morshedy, H. A., Gopalsamy, K., Nonoscillation, Oscillation and Convergence of a Class of Neutral Equations, Nonlinear Analysis, 40 (2000), 173-183.

[4] Giang, D. V., Győri, I., Oscillation of a Linear Neutral Delay Differential Equation with Unbounded Time Lag, Differential Equations and Dynamical Systems 1 4(1993), 267-274.

[5] Győri, I., Ladas, G., Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press - Oxford, 1991.

[6] Hale, J., Theory of Functional Differential Equations, Springer-Verlag New York, 1977.

[7] Krisztin, T., Wu, J., Asymptotic Behaviors of Solutions of Scalar Neutral Functional Differential Equations, Differential Equations and Dynamical Systems 4 3/4(1996), 351-366.

[8] Yuecai, F., Yunen, D., Oscillatory and Asymptotic Behaviour of First Order Differential Equation with Piecewise Constant Deviating Arguments, Annales of Differential Equations, 15 4(1999), 345-351.

Received by the editors February 1, 2002