Novi Sad J. Math.
Vol. 32, No. 2, 2002, 9-12

A NOTE ON INFINITE FORCING

Milan Z. Grulović[1]

Abstract. We consider one possible generalization of the notion of reduced product of infinite forcing systems.

AMS Mathematics Subject Classification (2000): Primary 03C25, Secondary 03C52, 03C62

Key words and phrases: Infinite forcing, Reduces products

1. Preliminaries

Throughout the article L is a first order language. As in our previuos papers on this subject the basic logical symbols will be $\neg$ (negation), $\wedge$ (conjunction) and $\exists$ (existential quantifier); the others are defined by the basic ones in the standard way. A theory T of the language L is a consistent deductively closed set of sentences. The class of models of a theory T will be denoted by $\mu(T)$. Models (of the language L) will be denoted by $\mathbf{A}, \mathbf{B}, \ldots$, while their domains will be $A, B, \ldots$. We recall ([6]) that if $\mathbf{A}_i$, $i \in I$, is a family of models and if D is a filter over I, the reduced product of the given family of models *modulo* D will be standardly denoted by $\prod_D \mathbf{A}_i$. On the other hand the elements of the reduced product $\mathbf{A} = \prod_D \mathbf{A}_i$ will be $f_A^1, f_A^2, \ldots, g_A^1, g_A^2, \ldots$, where $f^1, f^2, \ldots, g^1, g^2, \ldots$ (the elements of $\prod_I A_i$) are their representatives. Such notation (though not standard) simplifies the formulation of the definitions and propositions.

By an n-infinite forcing system we understand a class of models of the same language with the inclusion relation together with the n-infinite forcing relation between the models of the class and the sentences defined in them ([4]).

2. Ultraproducts of n_i-infinite forcing systems

The aim of this presentation is to contribute a bit to the examination of reduced products of infinite forcing systems introduced in [6]. A step futher that we are going to make is the generalization of these products in the sense that instead of infinite forcing systems we will be dealing generally with n_i-infinite forcing system (in paricular, 0-infinite forcing system is "the classical" infinite forcing system).

Let $\{\Sigma_i \mid i \in I\}$ be a family of classes of models of the (same) language L, each class Σ_i being in connection with n_i-infinite forcing relation defined "in it"

[1]Institute of Mathematics, Faculty of Science, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

([4]) (it is assumed that the number n_i is in a function of i). Futhermore, let D be a proper filter over the index set I and let Σ_D be a class of models whose elements are reduced products of models from the classes Σ_i, $i \in I$, modulo D – $\prod_D \mathbf{A}_i$. The relation $=_D$ will be defined (as before):

$\prod_D \mathbf{A}_i =_D \prod_D \mathbf{B}_i$ iff $\{i \in I \mid \mathbf{A}_i = \mathbf{B}_i\} \in D$,

but as for the relation $\leq_D$ we will now have (in accordance with the "new situation"):

$\prod_D \mathbf{A}_i \leq_D \prod_D \mathbf{B}_i$ iff $\{i \in I \mid \mathbf{A}_i \prec_{n_i} \mathbf{B}_i\} \in D$,

where, of course, $\mathbf{A}_i \prec_{n_i} \mathbf{B}_i$ means that $\mathbf{A}_i$ is n_i-elementary submodel of $\mathbf{B}_i$. We preserve the notation $X_{\mathbf{A},\mathbf{B}}$ for the set $\{i \in I \mid \mathbf{A}_i \prec_{n_i} \mathbf{B}_i\}$.

The point in the definition of $\leq_D$ is that in the case of n-infinite forcing the relation between models which is of interest is not just the ordinary inclusion extension but the "n-elementary extension".

With the (unchanged) notation (and text) from [6] (for it is just the matter of infinite forcing relations we have at disposal) we repeat

Definition 2.1 *The relation* $\mathbf{A}$ *D-infinitely forces* $\phi(f_A^1, \ldots, f_A^k)$*, denoted by* $\mathbf{A} \,\|{=}_D\, \phi(f_A^1, \ldots, f_A^k)$*, between a model* $\mathbf{A} = \prod_D \mathbf{A}_i$ $(\in \Sigma_D)$ *and a sentence* $\phi(f_A^1, \ldots, f_A^k)$ *of the language* $L(A)$ *is defined inductively as follows:*

(1) if $\phi(f_A^1, \ldots, f_A^k)$ *is atomic, then* $\mathbf{A} \,\|{=}_D\, \phi(f_A^1, \ldots, f_A^k)$ *iff* $\{i \in I \mid \mathbf{A}_i \,\|{=}_{n_i}\, \phi(f^1(i), \ldots, f^k(i))\} \in D$*, where* $\|{=}_{n_i}$ *is the appropriate Robinson's infinite forcing relation "of the class* Σ_i*";*

(2) if $\phi \equiv \psi \wedge \theta$*, then* $\mathbf{A} \,\|{=}_D\, \phi$ *iff* $\mathbf{A} \,\|{=}_D\, \psi$ *and* $\mathbf{A} \,\|{=}_D\, \theta$*;*

(3) if $\phi \equiv \exists v\, \psi(v, f_A^1, \ldots, f_A^k)$*, then* $\mathbf{A} \,\|{=}_D\, \phi$ *iff there exists* $f_A \in A$ *such that* $\mathbf{A} \,\|{=}_D\, \psi(f_A, f_A^1, \ldots, f_A^k)$

and

(4) if $\phi(f_A^1, \ldots, f_A^k) \equiv \neg\psi(f_A^1, \ldots, f_A^k)$*, then* $\mathbf{A} \,\|{=}_D\, \phi$ *iff no* $\mathbf{B}$ *"greater" than* $\mathbf{A}$ $(\mathbf{A} \leq_D \mathbf{B})$ *D-infinitely forces* $\psi(g_B^1, \ldots, g_B^k)$*, where* $g_B^1, \ldots, g_B^k$ *are the elements of* $\mathbf{B}$ *such that* $X_j \overset{\text{def}}{=} \{i \in I \mid f^j(i) = g^j(i)\} \in D$*,* $j = 1, \ldots, k$*.*

Surely, the definition is correct, that is independent of the choice of the "representatives" both of models and of elements of these models, for in comparison with the "standard" reduced product of infinite forcing systems nothing essentially is changed. Equally well, the basic properties of such "expanded" D-infinite forcing relation correspond to the properties of the "standard" D-infinite forcing relation. So we have

Lemma 2.2 *Let* $\mathbf{A}, \mathbf{B} \in \Sigma_D$ *and let* $\phi(f_A^1, \ldots, f_A^k), \psi$ *be sentences defined in* $\mathbf{A}$*. It holds:*

(1) $\mathbf{A}$ *cannot D-infinitely forces both* ϕ *and* $\neg\phi$*;*

(2) if $\mathbf{A} \leq_D \mathbf{B}$ *and* $\mathbf{A} \,\|{=}_D\, \phi(f_A^1, \ldots, f_A^k)$*, then* $\mathbf{B} \,\|{=}_D\, \phi(g_B^1, \ldots, g_B^k)$ *for each* g_B^j*,* $j = 1, \ldots, k$*, such that* $\{i \in I \mid f^j(i) = g^j(i)\} \in D$*.*

(3) if $\mathbf{A} \,\|{=}_D\, \phi$*, then* $\mathbf{A} \,\|{=}_D\, \neg\neg\phi$*;*

$\mathbf{A} \Vdash_D \neg\phi$ *iff* $\mathbf{A} \Vdash_D \neg\neg\neg\phi$;
(4) *if* $\mathbf{A} \Vdash_D \phi$ *or* $\mathbf{A} \Vdash_D \psi$, *then* $\mathbf{A} \Vdash_D \neg(\neg\phi \wedge \neg\psi)$ *(that is* $\mathbf{A} \Vdash_D \phi \vee \psi$*)*;

(5) *if* $\mathbf{A} \Vdash_D \neg\exists v \neg\psi(v)$, *then* $\mathbf{A} \Vdash_D \neg\neg\psi(f_A)$ *for each* $f_A \in A$.

The Łos theorem is preserved as well.

Theorem 2.3 *Let* U *be an ultrafilter over the index set* I, *let* $\mathbf{A} \in \Sigma_U$ *and let* $\phi(f_A^1, \ldots, f_A^k)$ *be a sentence defined in* $\mathbf{A}$. *It holds:*

$$\mathbf{A} \Vdash_U \phi(f_A^1, \ldots, f_A^k) \quad \textit{iff} \quad \{i \in I \mid \mathbf{A}_i \Vdash_{n_i} \phi(f^1(i), \ldots, f^k(i))\} \in U.$$

Surely, when U is a principal ultrafilter nothing new is obtained, more precisely we obtain the "isomorphic image" of the corresponding forcing system. On the other hand if, for some natural number n, $X_n \stackrel{\text{def}}{=} \{i \in I \mid n_i = n\} \in U$, we have some form of $U-n$-infinite forcing system. However one could find it more appropriate to define $U-n$-infinite forcing system using the "$U-n$-elementary submodel relation" (let us denote it by $\mathbf{A} = \prod_U \mathbf{A}_i \preceq_{U-n} \prod_U \mathbf{B}_i = \mathbf{B}$) defined by: $\mathbf{A} \preceq_{U-n} \mathbf{B}$ iff for any Σ_n- or Π_n-sentence $\phi(f_A^1, \ldots, f_A^k)$ defined in the language $L(A)$ $\mathbf{A} \models \phi(f_A^1, \ldots, f_A^k) \iff \mathbf{B} \models \phi(g_B^1, \ldots, g_B^k)$, where $X_j \stackrel{\text{def}}{=} \{i \in I \mid f^j(i) = g_j^i\} \in U$, $j = 1, \ldots, k$ (compare with the relation $\preceq_U$ introduced in [6]). The question is whether these definitions coincide or, if not, under what conditions they coincide. For it is clear that the relation $\leq_U$ is a subset of the relation $\preceq_{U-n}$ (when $X_n \in U$), but at the moment we cannot offer any (counter)example which would show that we have in fact a proper subset (a word of warning: in [6] the notation $\leq_U$ was defined by: $\mathbf{A} \leq_U \mathbf{B}$ iff $X \stackrel{\text{def}}{=} \{i \in I \mid \mathbf{A}_i \leq \mathbf{B}_i\} \in U$).

The definition of generic models remains the same. Hence (we recall)

Definition 2.4 *Let* D *be a proper filter over the index set* I. *A model* $\mathbf{A}$ $(= \prod_D \mathbf{A}_i)$ *of the class* Σ_D *is* D*-infinitely generic iff for any sentence* $\phi(f_A^1, \ldots, f_A^n)$ *defined in* $\mathbf{A}$ *either* $\mathbf{A} \Vdash_D \phi(f_A^1, \ldots, f_A^n)$ *or* $\mathbf{A} \Vdash_D \neg\phi(f_A^1, \ldots, f_A^n)$.

All the properties of generic models given in [6] remain valid (of course, after the necessary slightly reformulations) and we will not bother ourselves with repeating the proofs. Instead we are going to give the proof of the result corresponding to 2.2 in [3].

Theorem 2.5 *Let* U *be an ultrafilter over* I *and let* $\Sigma_i = \mu(T_i \cap \Pi_{n_i+1})$. *If we put* $\Sigma_U^F \stackrel{\text{def}}{=} Th(\{\mathbf{A} \in \Sigma_U \mid \mathbf{A}$ *is* U*-infinitely generic*$\})$, *then*

$$\Sigma_U^F = \prod_U T_i^{F_{n_i}},$$

where $T_i^{F_{n_i}}$ *is* n_i*-infinite forcing companion of the theory* T_i *(*[4]*)*.

Proof. See proof of 3.10 from [6]. □

Corollary 2.6 *Let T be a theory, U a nonprincipal ultrafilter over ω and let, for each $n \in \omega$, $\Sigma_n = \mu(T \cap \Pi_{n+1})$. Then $\Sigma_U^F = T$.*

Proof. A direct consequence of the previous theorem and the fact that $T^{F_n} \cap \Pi_{n+1} = T \cap \Pi_{n+1}$ ([4]). □

References

[1] Barwise J., Robinson A., Completing Theories by Forcing, Annals of Mathematical Logic – Vol. 2, No. 2 (1970), 119-142.

[2] Chang C. C., Keisler H. J., Model Theory, North-Holland, Amsterdam - London, 1973.

[3] Grulović M. Z., On Reduced Products of Forcing Systems, Publications de l'Institute Mathématique, Nouvelle serie, tome 41 (55), 1987, 17-20.

[4] Grulović M. Z., A Word on n-Infinite Forcing, Facta Universitatis, series: Mathematics and Informatics, University of Niš (to appear)

[5] Grulović M. Z., Comments on Ultraproducts of Forcing Systems, Publ. Inst. Math. Beograd (N.S) 69(83) (2001),

[6] Grulović M. Z., Reduced Products of Infinite Forcing Systems, NSJOM vol. 32, no. 1 (2002), 93-100

[7] Hirschfeld J., Wheeler, W.H., Forcing, Arithmetic, Division Rings, Lecture Notes in Mathematics 454, Springer-Verlag, Berlin-Heidelberg- New York, 1975.

[8] Keller J. P., Abstract Forcing and Applications, Ph. D. thesis, New York University, 1977.

[9] Robinson A., Infinite Forcing in Model Theory, Proceedings of the Second Scandinavian Logic Symposium, Oslo 1970, Amsterdam, North-Holland Publishing Company, 1971., 317-340.

Received by the editors October 5, 2000