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SOME RESULTS ON THE COMPOSITION OF DISTRIBUTIONS

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Abstract. Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The distributions $(x_+^r)^{-1}$ and $(x_-^r)^{-1}$ are evaluated for $r = 1, 2, \ldots$.

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In the following we let N be the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\, \ln^r n : \qquad \lambda > 0,\, r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

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for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [2].

Definition 1. *Let F be a distribution and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a, b) if*

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all test functions φ with compact support contained in (a, b).

The following theorems were proved in [2] and [3] respectively:

Theorem 1. *The distributions $(x_-^\mu)_-^\lambda$ and $(x_+^\mu)_-^\lambda$ exists and*

$$(x_-^\mu)_-^\lambda = (x_+^\mu)_-^\lambda = 0$$

for $\mu > 0$ and $\lambda\mu \neq -1, -2, \ldots$ and

$$(x_-^\mu)_-^\lambda = (-1)^{\lambda\mu}(x_+^\mu)_-^\lambda = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu - 1)!}\delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda \neq -1, -2, \ldots$ and $\lambda\mu = -1, -2, \ldots$.

Theorem 2. *The distribution $(x_+^r)_-^{-s}$ exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x)$$

for $r, s = 1, 2, \ldots,$ where

$$c(\rho) = \int_0^1 \ln t\, \rho(t)\, dt.$$

In the previous theorem, the distribution x_-^{-s} is define by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \ldots$ and not as in Gel'fand and Shilov [4]. We also define the distribution x_+^{-r} by the equation

$$x_+^{-r} = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \ldots$.

We need the following lemma which can be easily proved by induction:

Lemma 1. *If φ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1,1]$, then*

$$
\begin{aligned}
\langle x_+^{-r}, \varphi(x)\rangle &= \int_0^1 x^{-r}\Big[\varphi(x) - \sum_{i=0}^{r-1}\frac{x^i}{i!}\varphi^{(i)}(0)\Big]dx - \sum_{i=0}^{r-2}\frac{\varphi^{(i)}(0)}{i!(r-i-1)} \\
&\quad - \frac{\phi(r-1)}{(r-1)!}\varphi^{(r-1)}(0), \qquad (1)
\end{aligned}
$$

for $r = 1, 2, \ldots,$ where

$$\phi(r) = \begin{cases} \sum_{i=1}^r 1/i, & r \geq 1, \\ 0, & r = 0. \end{cases}$$

We now prove the following theorem.

Theorem 3. *The distribution $(x_+^r)^{-1}$ exists and*

$$(x_+^r)^{-1} = x_+^{-r} + (-1)^r\frac{2c(\rho) - r\phi(r-1)}{r!}\delta^{(r-1)}(x), \tag{2}$$

for $r = 1, 2, \ldots,$ where

$$c(\rho) = \int_0^1 \ln t\rho(t)\,dt.$$

Proof. We put

$$
\begin{aligned}
[(x_+^r)^{-1}]_n &= (x_+^r)^{-1} * \delta_n(x) \\
&= \begin{cases} \int_{-1/n}^{1/n} \ln|x^r - t|\delta_n'(t)\,dt, & x \geq 0, \\ \int_{-1/n}^{1/n} \ln|t|\delta_n'(t)\,dt, & x < 0. \end{cases} \qquad (3)
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{-1}^1 x^k[(x_+^r)^{-1}]_n\,dx &= \int_0^1 x^k\int_{-1/n}^{1/n}\ln|x^r - t|\delta_n'(t)\,dtdx + \\
&\quad + \int_{-1}^0 x^k\int_{-1/n}^{1/n}\ln|t|\delta_n'(t)\,dt\,dx \\
&= \int_{-1/n}^{1/n}\delta_n'(t)\int_0^{n^{-1/r}} x^k\ln|x^r - t|\,dx\,dt + \\
&\quad + \int_{-1/n}^{1/n}\delta_n'(t)\int_{n^{-1/r}}^1 x^k\ln|x^r - t|\,dx\,dt + \\
&\quad + \frac{(-1)^k}{k+1}\int_{-1/n}^{1/n}\ln|t|\delta_n'(t)\,dt
\end{aligned}
$$

$$\begin{aligned} &= \frac{n^{(r-k-1)/r}}{r}\int_{-1}^{1}\rho'(v)\int_{0}^{1}u^{-(r-k-1)/r}\ln|(u-v)/n|\,du\,dv + \\ &\quad + \frac{n^{(r-k-1)/r}}{r}\int_{-1}^{1}\rho'(v)\int_{1}^{n}u^{-(r-k-1)/r}\ln|(u-v)/n|\,du\,dv + \\ &\quad + \frac{(-1)^k n}{k+1}\int_{-1}^{1}\ln|v/n|\rho'(v)\,dv \\ &= I_1 + I_2 + I_3, \qquad (4) \end{aligned}$$

where the substitutions $u = nx^r$ and $v = nt$ have been made.

It is obvious that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = \operatorname*{N-lim}_{n\to\infty} I_3 = 0, \qquad (5)$$

for $k = 0, 1, \ldots, r-2$.

Next we have

$$\begin{aligned} \int_{-1}^{1}\rho'(v)\int_{1}^{n}u^{-(r-k-1)/r}\ln|(u-v)/n|\,du\,dv &= \\ &= \int_{-1}^{1}\rho'(v)\int_{1}^{n}u^{-(r-k-1)/r}[\ln|1-v/u| + \ln u - \ln n]\,du\,dv \\ &= \int_{-1}^{1}\rho'(v)\int_{1}^{n}u^{-(r-k-1)/r}\ln|1-v/u|\,du\,dv, \end{aligned}$$

since $\int_{-1}^{1}\rho'(v)\,dv = 0$. Further

$$\begin{aligned} \int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(r-k-1)/r}\ln|1-v/u|\,du\,dv &= \\ & -\sum_{i=1}^{\infty}\frac{1}{i}\int_{-1}^{1}v^i\rho'(v)\int_{1}^{n}u^{(k+1)/r-i-1}\,du\,dv \\ &= -\sum_{i=1}^{\infty}\frac{r[n^{(k+1)/r-i}-1]}{i(k-ri+1)}\int_{-1}^{1}v^i\rho'(v)\,dv, \end{aligned}$$

and it follows that

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} I_2 &= \frac{1}{r-k-1}\int_{-1}^{1}v\rho'(v)\,dv \\ &= -\frac{1}{r-k-1}, \qquad (6) \end{aligned}$$

for $k = 0, 1, \ldots, r-2$.

Hence

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^{1}x^k[(x_+^r)^{-1}]_n\,dx = -\frac{1}{r-k-1} \qquad (7)$$

for $k = 0, 1, \dots, r-2$, on using equations (4), (5) and (6).

When $k = r-1$, we have with the above substitutions

$$\begin{aligned}
\int_{-1}^{1} x^{r-1}[(x_+^r)^{-1}]_n\,dx &= \int_0^1 x^{r-1}\int_{-1/n}^{1/n} \ln|x^r - t|\delta_n'(t)\,dt\,dx + \\
&\quad + \int_{-1}^{0} x^{r-1}\int_{-1/n}^{1/n} \ln|t|\delta_n'(t)\,dt\,dx \\
&= \int_{-1/n}^{1/n} \delta_n'(t)\int_0^1 x^{r-1}\ln|x^r - t|\,dx\,dt + \\
&\quad + \frac{(-1)^{r-1}}{r}\int_{-1/n}^{1/n} \ln|t|\delta_n'(t)\,dt \\
&= \frac{1}{r}\int_{-1}^{1} \rho'(v)\int_0^n \ln|u-v|\,du\,dv + \\
&\quad - \frac{1}{r}\int_{-1}^{1} \rho'(v)\int_0^n \ln n\,du\,dv + \\
&\quad + \frac{(-1)^{r-1}n}{r}\int_{-1}^{1} \ln|v/n|\rho'(v)\,dv \\
&= J_1 + J_2 + J_3. \qquad (8)
\end{aligned}$$

It is obvious that

$$\mathrm{N}\!-\!\lim_{n\to\infty} J_2 = \mathrm{N}\!-\!\lim_{n\to\infty} J_3 = 0. \qquad (9)$$

Further,

$$\begin{aligned}
\int_0^n \ln|u-v|\,du &= (n-v)\ln|n-v| + v\ln|v| - n \\
&= (n-v)\ln n - (n-v)\sum_{i=1}^{\infty}\frac{v^i}{in^i} + v\ln|v| - n
\end{aligned}$$

and it follows that

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty} J_1 &= -\frac{1}{r}\int_{-1}^{1} v\rho'(v)\,dv + \frac{1}{r}\int_{-1}^{1} v\ln|v|\rho'(v)\,dv \\
&= -\frac{2c(\rho)}{r}. \qquad (10)
\end{aligned}$$

Using equations (8), (9) and (10), we see that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-1}^{1} x^{r-1}[(x_+^r)^{-1}]_n dx = -\frac{2c(\rho)}{r}. \qquad (11)$$

We finally consider the case $k = r$ and let ψ be an arbitrary continuous function. Then

$$\int_{-1}^{0} x^r\psi(x)[(x_+^r)^{-1}]_n\,dx = n\int_{-1}^{0} x^r\psi(x)\,dx \times \int_{-1}^{1} \ln|v/n|\rho'(v)\,dv$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^{0} x^r\psi(x)[(x_+^r)^{-1}]_n\,dx = 0. \tag{12}$$

Next we have

$$\int_{0}^{n^{-1/r}} x^r\psi(x)[(x_+^r)^{-1}]_n\,dx = \frac{1}{rn^{1/r}}\int_{-1}^{1}\rho'(v)\int_0^1 u^{1/r}\psi[(u/n)^{1/r}]\ln|(u-v)/n|\,du\,dv$$

and it follows that

$$\lim_{n\to\infty}\int_{0}^{n^{-1/r}} x^r\psi(x)[(x_+^r)^{-1}]_n\,dx = 0. \tag{13}$$

When $x^r \geq 1/n$, we have

$$\begin{aligned}
[(x_+^r)^{-1}]_n &= \int_{-1/n}^{1/n}\ln|x^r - t|\delta_n'(t)\,dt \\
&= n\int_{-1}^{1}\ln|x^r - v/n|\rho'(v)\,dv \\
&= n\int_{-1}^{1}\Big[\ln x^r - \sum_{i=1}^{\infty}\frac{v^i}{in^ix^{ri}}\Big]\rho'(v)\,dv \\
&= -\sum_{i=1}^{\infty}\int_{-1}^{1}\frac{v^i}{in^{i-1}x^{ri}}\,\rho'(v)\,dv.
\end{aligned}$$

It follows that

$$|[(x_+^r)^{-1}]_n| \leq \sum_{i=1}^{\infty}\int_{-1}^{1}\frac{|v|^i}{in^{i-1}x^{ri}}\,|\rho'(v)|\,dv \leq \sum_{i=1}^{\infty}\frac{K}{in^{i-1}x^{ri}},$$

where

$$K = \int_{-1}^{1}|\rho'(v)|\,dv.$$

If now $n^{-1/r} < \eta < 1$, then

$$\int_{n^{-1/r}}^{\eta}|x^r[(x_+^r)^{-1}]_n|dx \quad\leq\quad K\sum_{i=1}^{\infty}\frac{1}{in^{i-1}}\int_{n^{-1/r}}^{\eta}x^{r(1-i)}dx$$

$$= \begin{cases} K\sum_{i=1}^{\infty} \dfrac{n^{-1/r}}{ri(1-i+1/r)}\left[(n\eta^r)^{1-i+1/r}-1\right], & r \neq 1, \\ K\sum_{\substack{i=1\\ i\neq 2}}^{\infty} \dfrac{n^{-1}}{i(2-i)}\left[(n\eta)^{2-i}-1\right] + K\dfrac{n^{-1}\ln(n\eta)}{2}, & r = 1. \end{cases}$$

It follows that

$$\lim_{n\to\infty} |[(x_+^r)^{-1}]_n| = O(\eta),$$

for $r = 1, 2, \ldots$.

Thus, if ψ is a continuous function

$$\lim_{n\to\infty}\left|\int_{n^{-1/r}}^{\eta} x^r[(x_+^r)^{-1}]_n\psi(x)dx\right| = O(\eta) \tag{14}$$

for $r = 1, 2, \ldots$.

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^r}{r!}\varphi^{(r)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned} \langle[(x_+^r)^{-1}]_n, \varphi(x)\rangle &= \int_{-1}^{1}[(x_+^r)^{-1}]_n\varphi(x)\,dx \\ = \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}&\int_{-1}^{1}x^k[(x_+^r)^{-1}]_n\,dx + \int_0^{n^{-1/r}}\frac{x^r}{r!}[(x_+^r)^{-1}]_n\varphi^{(r)}(\xi x)\,dx + \\ &+ \int_{n^{-1/r}}^{\eta}\frac{x^r}{r!}[(x_+^r)^{-1}]_n\varphi^{(r)}(\xi x)\,dx + \int_{\eta}^{1}\frac{x^r}{r!}[(x_+^r)^{-1}]_n\varphi^{(r)}(\xi x)\,dx + \\ &+ \int_{-1}^{0}\frac{x^r}{r!}[(x_+^r)^{-1}]_n\varphi^{(r)}(\xi x)\,dx. \end{aligned}$$

Using equations (7) and (11) to (14) and noting that the sequence $[(x_+^r)^{-1}]_n$ converges uniformly to x^{-r} on the interval $[\eta, 1]$, it follows that

$$\begin{aligned} \mathrm{N}\!-\!\lim_{n\to\infty}\langle[(x_+^r)^{-1}]_n, \varphi(x)\rangle &= -\sum_{k=0}^{r-2}\frac{1}{(r-k-1)k!}\varphi^{(k)}(0) - \frac{2c(\rho)}{r!}\varphi^{(r-1)}(0) + \\ &\quad + O(\eta) + \int_{\eta}^{1}\frac{\varphi^{(r)}(\xi x)}{r!}dx. \end{aligned}$$

Since η can be made arbitrarily small, it follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle[(x_+^r)^{-1}]_n,\varphi(x)\rangle = -\sum_{k=0}^{r-2}\frac{1}{(r-k-1)k!}\varphi^{(k)}(0)-\frac{2c(\rho)}{r!}\varphi^{(r-1)}(0)+ + \int_0^1\frac{\varphi^{(r)}(\xi x)}{r!}\,dx$$

$$= \int_0^1 x^{-r}\Big[\varphi(x)-\sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\Big]\,dx-\sum_{k=0}^{r-2}\frac{1}{(r-k-1)k!}\varphi^{(k)}(0)-\frac{2c(\rho)}{r!}\varphi^{(r-1)}(0)$$

$$= \langle x_+^{-r},\varphi(x)\rangle+(-1)^r\frac{2c(\rho)-r\phi(r-1)}{r!}\langle\delta^{(r-1)}(x),\varphi(x)\rangle$$

on using equation (1). This proves equation (2) on the interval $[-1,1]$. However, equation (2) clearly holds on any interval not containing the origin, and the proof is complete.

Corollary 3.1 *The distribution $(x_-^r)^{-1}$ exists and*

$$(x_-^r)^{-1} = x_-^{-r}-\frac{2c(\rho)-r\phi(r-1)}{r!}\delta^{(r-1)}(x) \tag{15}$$

for $r = 1, 2, \ldots$.

Proof. Equation (15) follows on replacing x by $-x$ in equation (2).

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