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WEAK CONGRUENCE IDENTITIES AT 0

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Abstract. The aim of the paper is to investigate some local properties of the weak congruence lattice of an algebra, which is supposed to possess the constant **0**, or a nullary term operation. Lattice identities are restricted to the zero blocks of weak congruences. In this way, a local version of the CEP, and local modularity and distributivity of the weak congruence lattices are characterized. In addition, local satisfaction by weak congruences of some general lattice identities is proved.

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1. Introduction

As defined in [2], an algebra $\mathcal{A} = (A, F)$ is said to be **with 0** if it contains an element **0**, which is a nullary term operation on $\mathcal{A}$. If ρ is a binary relation on an algebra $\mathcal{A}$ with **0**, let

$[\mathbf{0}]_\rho := \{a \in A \mid a\rho\mathbf{0}\}$.

Throughout the paper we consider algebras with **0**.

An algebra $\mathcal{A}$ is **congruence modular at 0** ([1]) if for every $\rho, \theta, \sigma \in Con\,\mathcal{A}$,

$$\rho \subseteq \sigma \quad \text{implies} \quad [\mathbf{0}]_{\rho\vee(\theta\cap\sigma)} = [\mathbf{0}]_{(\rho\vee\theta)\cap\sigma}. \tag{1}$$

A **weak congruence relation** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$. The set $Cw\,\mathcal{A}$ of all weak congruences on $\mathcal{A}$ is an algebraic lattice under inclusion. The filter $\Delta\!\uparrow$ generated by the diagonal $\Delta = \{(x,x) \mid x \in A\}$ is the congruence lattice $Con\,\mathcal{A}$. The ideal $\Delta\!\downarrow$ is isomorphic with the subalgebra lattice $Sub\,\mathcal{A}$: every subalgebra $\mathcal{B}$ of $\mathcal{A}$ is represented by the corresponding diagonal relation Δ_B. For more details about weak congruences we refer to [7, 8].

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If ρ and θ are weak congruences on an algebra $\mathcal{A}$ (with $\mathbf{0}$), and $\rho \wedge \theta$ is their meet in $Cw\mathcal{A}$, then obviously

$[\mathbf{0}]_{\rho\wedge\theta} = [\mathbf{0}]_\rho \cap [\mathbf{0}]_\theta$.

As defined in [6], an algebra $\mathcal{A}$ is **weak congruence modular at 0** if (1) holds for every $\rho, \theta, \sigma \in Cw\mathcal{A}$.

$\mathcal{A}$ is said to have the **congruence extension property at 0** (the CEP at $\mathbf{0}$) ([6]) if for every congruence ρ on a subalgebra $\mathcal{B}$ of $\mathcal{A}$ there is a congruence θ on $\mathcal{A}$ such that $\rho \subseteq \theta$ and $[\mathbf{0}]_\rho = B \cap [\mathbf{0}]_\theta$.

Obviously, the CEP at $\mathbf{0}$ is a $\mathbf{0}$-modification of the congruence extension property (the CEP): an algebra $\mathcal{A}$ is said to possess the CEP if for every congruence ρ on a subalgebra $\mathcal{B}$ of $\mathcal{A}$ there is a congruence θ on $\mathcal{A}$ such that $\rho = B^2 \cap \theta$.

$\mathcal{A}$ satisfies the **congruence intersection property** (the CIP) if for every $\rho, \theta \in Cw\mathcal{A}$,

$$\Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge (\Delta \vee \theta).$$

Observe that for a congruence ρ on a subalgebra $\mathcal{B}$ of $\mathcal{A}$, the join $\rho \vee \Delta$ in $Cw\mathcal{A}$ is a congruence on $\mathcal{A}$ obtained by

$$\rho \vee \Delta = \bigcap (\theta \in Con\mathcal{A} \mid \rho \subseteq \theta).$$

In the following we introduce some new notions concerning lattices of weak congruences and prove some of their properties that are used in the sequel.

We say that $\mathcal{A}$ has the **congruence intersection property at 0** (the CIP at $\mathbf{0}$) if for all $\rho, \theta \in Cw\mathcal{A}$,

$$[\mathbf{0}]_{(\rho\wedge\theta)\vee\Delta} = [\mathbf{0}]_{(\rho\vee\Delta)\wedge(\theta\vee\Delta)}.$$

$\mathcal{A}$ has the **strong congruence intersection property at 0** (the SCIP at $\mathbf{0}$) if for all $\rho, \theta \in Cw\mathcal{A}$, $\alpha \in Con\mathcal{A}$,

$$[\mathbf{0}]_{\alpha\vee(\rho\wedge\theta)} = [\mathbf{0}]_{\alpha\vee((\rho\vee\Delta)\wedge(\theta\vee\Delta))}.$$

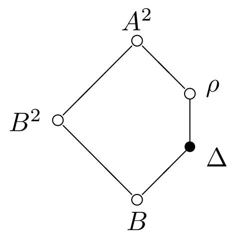
Observe that the above lattice operations are those from the lattice $Cw\mathcal{A}$.

It is easy to see that the CIP implies both the CIP at $\mathbf{0}$ and the SCIP at $\mathbf{0}$, and that SCIP at $\mathbf{0}$ implies CIP at $\mathbf{0}$. The following example demonstrates that SCIP at $\mathbf{0}$ does not imply the CIP.

Example

Let $\mathcal{A} = (A, \{a, b, c\}, *, a)$, where a is a constant, and the binary operation $*$ is given by the table.

$*$	a	b	c
a	b	a	a
b	a	b	c
c	a	b	c



$Cw\mathcal{A}$

The only subalgebra is $B = \{a, b\}$, and the non-trivial congruence on $\mathcal{A}$ is $\rho = \{\{a\}, \{b, c\}\}$. Now,

$(\rho \wedge B^2) \vee \Delta = \Delta$, while $(\rho \vee \Delta) \wedge (\rho \vee B^2) = \rho$,

hence $\mathcal{A}$ fails on the CIP. On the other hand,

$[a]_{(\rho \wedge B^2) \vee \Delta} = [a]_{(\rho \vee \Delta) \wedge (\rho \vee B^2)}$,

and $\mathcal{A}$ possesses the CIP at $\mathbf{0}$. It is easy to see that also the SCIP at $\mathbf{0}$ holds.

Proposition 1. *If algebra $\mathcal{A}$ is $\mathbf{0}$-regular and satisfies the CIP at $\mathbf{0}$, then it satisfies the CIP as well.*

Proof. Straightforward. □

Lemma 1. *The following are equivalent for an algebra $\mathcal{A}$ with $\mathbf{0}$:*

(i) *$\mathcal{A}$ has the CEP at $\mathbf{0}$;*

(ii) *for every $\rho \in Con\,\mathcal{B}$,*

$[\mathbf{0}]_\rho = B \cap [\mathbf{0}]_{\rho \vee \Delta}$;

(iii) *if ρ_1 and ρ_2 are congruences on an arbitrary subalgebra $\mathcal{B}$ of $\mathcal{A}$, then*

$[\mathbf{0}]_{\rho_1 \vee \Delta} = [\mathbf{0}]_{\rho_2 \vee \Delta}$ *implies* $[\mathbf{0}]_{\rho_1} = [\mathbf{0}]_{\rho_2}$;

(iv) *for all $\rho, \theta \in Cw\mathcal{A}$,*

$$[\mathbf{0}]_{(\Delta \wedge \theta) \vee \rho} = [\mathbf{0}]_{(\Delta \vee \rho) \wedge (\theta \vee \rho)}.$$

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ is proved in [6].

$(iii) \Rightarrow (iv)$ Let $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$. We have that $\rho \vee (\Delta \wedge \theta) \in Con(\mathcal{B} \vee \mathcal{C})$.

Since $(\rho \vee \Delta) \wedge (\rho \vee \theta) \subseteq \rho \vee \Delta$, $((\rho \vee \Delta) \wedge (\rho \vee \theta)) \vee \Delta \subseteq \rho \vee \Delta$ and $\rho \subseteq (\rho \vee \Delta) \wedge (\rho \vee \theta)$, $\rho \vee \Delta \subseteq ((\rho \vee \Delta) \wedge (\rho \vee \theta)) \vee \Delta$, we have that $\rho \vee \Delta = ((\rho \vee \Delta) \wedge (\rho \vee \theta)) \vee \Delta$.

Therefore,

$$[\mathbf{0}]_{(\Delta \wedge \theta) \vee \rho \vee \Delta} = [\mathbf{0}]_{\rho \vee \Delta} = [\mathbf{0}]_{((\Delta \vee \rho) \wedge (\theta \vee \rho)) \vee \Delta},$$

and by (iii),

$$[\mathbf{0}]_{(\Delta \wedge \theta) \vee \rho} = [\mathbf{0}]_{(\Delta \vee \rho) \wedge (\theta \vee \rho)}.$$

$(iv) \Rightarrow (ii)$ Let $\rho \in Con\mathcal{B}$. By (iv),

$[\mathbf{0}]_{\rho} = [\mathbf{0}]_{\rho \vee (\Delta \wedge B^2)} = [\mathbf{0}]_{(\rho \vee \Delta) \wedge (\rho \vee B^2)} = [\mathbf{0}]_{(\rho \vee \Delta) \wedge B^2} = B \cap [\mathbf{0}]_{\rho \vee \Delta}$. □

2. Weak congruence distributivity at 0

An algebra $\mathcal{A}$ with $\mathbf{0}$, is said to be congruence distributive at $\mathbf{0}$ ([3]) if for all $\rho, \theta, \sigma \in Con\mathcal{A}$,

(2) $$[\mathbf{0}]_{\rho \wedge (\theta \vee \sigma)} = [\mathbf{0}]_{(\rho \wedge \theta) \vee (\rho \wedge \sigma)}.$$

This condition, however is not equivalent to the dual one:

(3) $$[\mathbf{0}]_{\rho \vee (\theta \wedge \sigma)} = [\mathbf{0}]_{(\rho \vee \theta) \wedge (\rho \vee \sigma)},$$

as pointed out in [5].

We say that an algebra $\mathcal{A}$ with $\mathbf{0}$ is **weak congruence distributive at 0** if for all $\rho, \theta, \sigma \in Cw\mathcal{A}$ the condition (2) is satisfied. If $\mathcal{A}$ fulfills the condition (3), then we say that $\mathcal{A}$ is **weak congruence d-distributive at 0**.

In the following lemma we prove that the weak congruence distributivity at $\mathbf{0}$ follows from the dual condition. Yet these conditions are not equivalent, since neither are the analogue conditions for congruences.

Lemma 2. *If an algebra $\mathcal{A}$ is weak congruence d-distributive at $\mathbf{0}$, then it is also weak congruence distributive at $\mathbf{0}$.*

Proof. Suppose $\rho, \theta, \sigma \in Cw\mathcal{A}$. Then, by (3),

$[\mathbf{0}]_{(\rho \wedge \theta) \vee (\rho \wedge \sigma)} = [\mathbf{0}]_{((\rho \vee (\rho \wedge \sigma)) \wedge \theta) \vee (\rho \wedge \sigma)} = [\mathbf{0}]_{\rho \wedge (\theta \vee (\rho \wedge \sigma))} = [\mathbf{0}]_{\rho} \cap [\mathbf{0}]_{\theta \vee (\rho \wedge \sigma)} = [\mathbf{0}]_{\rho} \cap [\mathbf{0}]_{(\theta \vee \rho) \wedge (\theta \vee \sigma)} = [\mathbf{0}]_{\rho \wedge (\theta \vee \rho) \wedge (\theta \vee \sigma)} = [\mathbf{0}]_{\rho \wedge (\theta \vee \sigma)}$, proving (2). □

In the following proposition we give a characterization of the weak congruence distributivity at $\mathbf{0}$.

Theorem 1. *An algebra $\mathcal{A}$ is weak congruence distributive at $\mathbf{0}$ if and only if the following four conditions are satisfied:*

(i) *$\mathcal{A}$ is subalgebra distributive;*

(ii) *$\mathcal{A}$ is congruence distributive at $\mathbf{0}$;*

(iii) *$\mathcal{A}$ has the CEP at $\mathbf{0}$;*

(iv) *$\mathcal{A}$ has the SCIP at $\mathbf{0}$.*

Proof. $\Rightarrow$ Suppose that $\mathcal{A}$ is weak congruence distributive at $\mathbf{0}$.

Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be subalgebras of $\mathcal{A}$. By the existence of a constant in $\mathcal{A}$, the weak congruences $(B^2 \wedge C^2) \vee D^2$ and $(B^2 \vee D^2) \wedge (C^2 \vee D^2)$ are the squares of subalgebras of $\mathcal{A}$. Hence,

$(B \wedge C) \vee D = [\mathbf{0}]_{(B^2 \wedge C^2) \vee D^2} = [\mathbf{0}]_{(B^2 \vee D^2) \wedge (C^2 \vee D^2)} = (B \vee D) \wedge (C \vee D)$, and **(i)** holds.

(ii) is evident.

(iii) follows by Lemma 1.

We prove **(iv)**. Let $\alpha \in Con\,\mathcal{A}$, $\rho, \theta \in Cw\,\mathcal{A}$. Then,

$[\mathbf{0}]_{\alpha \vee (\rho \wedge \theta)} = [\mathbf{0}]_{\alpha \vee \Delta \vee (\rho \wedge \theta)} = [\mathbf{0}]_{(\alpha \vee \Delta \vee \rho) \wedge (\alpha \vee \Delta \vee \theta)} = [\mathbf{0}]_{\alpha \vee ((\Delta \vee \rho) \wedge (\Delta \vee \theta))}.$

$(\Leftarrow)$

Let $\rho, \theta, \sigma \in Cw\mathcal{A}$. By the distributivity of $Sub\mathcal{A}$, $(\rho \wedge \theta) \vee \sigma$ and $(\rho \vee \sigma) \wedge (\theta \vee \sigma)$ are congruences on the same subalgebra of $\mathcal{A}$. By the SCIP and the congruence distributivity at $\mathbf{0}$,

$[\mathbf{0}]_{(\rho \wedge \theta) \vee \sigma \vee \Delta} = [\mathbf{0}]_{(\rho \wedge \theta) \vee \Delta \vee \sigma \vee \Delta} = [\mathbf{0}]_{((\rho \vee \Delta) \wedge (\theta \vee \Delta)) \vee (\sigma \vee \Delta)} =$
$[\mathbf{0}]_{(\rho \vee \Delta \vee \sigma) \wedge (\theta \vee \Delta \vee \sigma)} = [\mathbf{0}]_{((\rho \vee \sigma) \wedge (\theta \vee \sigma)) \vee \Delta}.$

By the CEP at $\mathbf{0}$ and Lemma 1,

$[\mathbf{0}]_{(\rho \wedge \theta) \vee \sigma} = [\mathbf{0}]_{(\rho \vee \sigma) \wedge (\theta \vee \sigma)}.$ □

Corollary 1. *An algebra $\mathcal{A}$ is a weak congruence both distributive and d-distributive at $\mathbf{0}$ if and only if the following four conditions are satisfied*

(i) *$\mathcal{A}$ is subalgebra distributive;*

(ii) *$\mathcal{A}$ is congruence distributive and dually distributive at $\mathbf{0}$;*

(iii) *$\mathcal{A}$ has the CEP at $\mathbf{0}$;*

(iv) *$\mathcal{A}$ has the SCIP at $\mathbf{0}$.* □

3. Weak congruence modularity at **0**

As defined in [2], an algebra $\mathcal{A}$ is congruence modular at $\mathbf{0}$ if for every $\rho, \theta, \sigma \in Con\mathcal{A}$,

(4) $$\rho \subseteq \sigma \text{ implies } [\mathbf{0}]_{\rho\vee(\theta\wedge\sigma)} = [\mathbf{0}]_{(\rho\vee\theta)\wedge\sigma}.$$

In [6] weak congruence modularity at $\mathbf{0}$ is introduced and investigated. The algebra $\mathcal{A}$ is weak congruence modular at $\mathbf{0}$, if (4) holds for all $\rho, \theta, \sigma \in Cw\mathcal{A}$.

The following proposition was proved in [6]:

Proposition 2. *If the algebra $\mathcal{A}$ is weak congruence modular at $\mathbf{0}$, then*

(i) $\mathcal{A}$ is congruence modular at $\mathbf{0}$;

(ii) $\mathcal{A}$ is subalgebra modular;

(iii) $\mathcal{A}$ has the CEP at $\mathbf{0}$. □

As a converse, we prove the following.

Theorem 2. *Let $\mathcal{A}$ be an algebra with $\mathbf{0}$, satisfying*

(i) *$\mathcal{A}$ is subalgebra modular;*

(ii) *$\mathcal{A}$ is congruence modular at $\mathbf{0}$;*

(iii) *$\mathcal{A}$ has the CEP at $\mathbf{0}$;*

(iv) *$\mathcal{A}$ has the SCIP at $\mathbf{0}$.*

Then, $\mathcal{A}$ is weak congruence modular at $\mathbf{0}$.

Proof. Let $\rho, \theta, \sigma \in Cw\mathcal{A}$ and $\rho \subseteq \sigma$. Let $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$ and $\sigma \in Con\mathcal{D}$, for $\mathcal{B}, \mathcal{C}, \mathcal{D} \in Sub\mathcal{A}$. From $\rho \subseteq \sigma$ it follows that $\mathcal{B} \subseteq Sub\mathcal{D}$, and by the modularity of $Sub\mathcal{A}$, $\rho \vee (\theta \wedge \sigma)$ and $(\rho \vee \theta) \wedge \sigma$ are congruences on the same subalgebra of $\mathcal{A}$. Further on, by the SCIP at $\mathbf{0}$ and the congruence modularity at $\mathbf{0}$,

$$[\mathbf{0}]_{(\rho\vee(\theta\wedge\sigma))\vee\Delta} = [\mathbf{0}]_{(\rho\vee\Delta)\vee(\theta\wedge\sigma)\vee\Delta} = [\mathbf{0}]_{((\rho\vee\Delta)\vee((\theta\vee\Delta)\wedge(\sigma\vee\Delta))} =$$

$$[\mathbf{0}]_{((\rho\vee\Delta)\vee(\theta\vee\Delta))\wedge(\sigma\vee\Delta)} = [\mathbf{0}]_{(\rho\vee\theta\vee\Delta)\wedge(\sigma\vee\Delta)} = [\mathbf{0}]_{((\rho\vee\theta)\wedge\sigma)\vee\Delta}.$$

By the CEP at $\mathbf{0}$,

$$[\mathbf{0}]_{\rho\vee(\theta\wedge\sigma)} = [\mathbf{0}]_{(\rho\vee\theta)\wedge\sigma}.$$ □

Observe that the above theorem is an improvement of Theorem 2 in [6], in which the CIP (and not, as here, the SCIP at $\mathbf{0}$) is one of sufficient conditions for the local modularity of $Cw\mathcal{A}$.

4. Identities at 0

In the following, let $t_1 \approx t_2$ be an arbitrary lattice identity. We say that an algebra $\mathcal{A}$ with $\mathbf{0}$ **satisfies the weak congruence identity** $t_1 \approx t_2$ **at 0** if

$$[\mathbf{0}]_{t_1(\rho_1,\ldots,\rho_n)} = [\mathbf{0}]_{t_2(\rho_1,\ldots,\rho_n)}, \tag{5}$$

for all $\rho_1, \ldots, \rho_n \in Cw\mathcal{A}$. Analogously, $\mathcal{A}$ **satisfies the congruence identity** $t_1 \approx t_2$ **at 0** if the above equality (5) holds for all $\rho_1, \ldots, \rho_n \in Con\,\mathcal{A}$.

Theorem 3. *If an algebra $\mathcal{A}$ with $\mathbf{0}$ is $\mathbf{0}$-regular and satisfies a weak congruence identity $t_1 \approx t_2$ at $\mathbf{0}$, then the same identity $t_1 \approx t_2$ is satisfied in the lattice $Cw\mathcal{A}$.*

Proof. Straightforward. □

Theorem 4. *If an algebra $\mathcal{A}$ with $\mathbf{0}$ satisfies a weak congruence identity $t_1 \approx t_2$ at $\mathbf{0}$, then $\mathcal{A}$ satisfies the same congruence identity $t_1 \approx t_2$ at $\mathbf{0}$; in addition, this identity is also satisfied in the lattice $Sub\,\mathcal{A}$.*

Proof. This follows by the same arguments as the ones in Theorem 2, using mathematical induction on the number of operational symbols in t_1 and t_2. □

The converse of the previous theorem is satisfied for a special class of identities, provided that the algebra $\mathcal{A}$ has the CEP at $\mathbf{0}$ and the SCIP at $\mathbf{0}$.

Let t_1 and t_2 be lattice terms of the following type

$$\bigwedge_{i=1}^{m}\Big(\bigvee_{j=1}^{n_i}\Big(\bigwedge_{k=1}^{p_j}\Big(\bigvee_{l=1}^{q_k} x_{ijkl}\Big)\Big)\Big). \tag{6}$$

Theorem 5. *Let $\mathcal{A}$ be an algebra with $\mathbf{0}$ satisfying the CEP at $\mathbf{0}$ and the SCIP at $\mathbf{0}$, and let t_1 and t_2 be lattice terms of the type* (6)*. Then $\mathcal{A}$ satisfies a weak congruence identity $t_1 \approx t_2$ at $\mathbf{0}$ if and only if $\mathcal{A}$ satisfies the congruence identity $t_1 \approx t_2$ at $\mathbf{0}$ and the identity $t_1 \approx t_2$ is satisfied in $Sub\,\mathcal{A}$.*

Proof. The 'only if' part is proved in Theorem 4. Let $t_1 \approx t_2$ hold in $Sub\,\mathcal{A}$, where $t_1(x_1, \ldots, x_r)$ and $t_2(x_1, \ldots, x_r)$ are terms of the type (6), each with (some) variables from the set $\{x_1, \ldots, x_r\}$. Now, it is easy to see that $t_1(\rho_1, \ldots, \rho_r)$ and $t_2(\rho_1, \ldots, \rho_r)$ are congruences on the same subalgebra of $\mathcal{A}$, for all $\rho_1, \ldots \rho_r \in Cw\mathcal{A}$.

Further,

$[\mathbf{0}]_{t_1(\rho_1,\ldots,\rho_r)\vee\Delta} = [\mathbf{0}]_{t_1(\rho_1\vee\Delta,\ldots,\rho_r\vee\Delta)} = [\mathbf{0}]_{t_2(\rho_1\vee\Delta,\ldots,\rho_r\vee\Delta)} = [\mathbf{0}]_{t_2(\rho_1,\ldots,\rho_r)\vee\Delta}.$

The first and the third equality are satisfied by the SCIP for the terms t_1 and t_2 of type (6). The second equality follows by the fact that $\mathcal{A}$ satisfies the same congruences identity at $\mathbf{0}$.

By the CEP at $\mathbf{0}$,

$[\mathbf{0}]_{t_1(\rho_1,\ldots,\rho_r)} = [\mathbf{0}]_{t_2(\rho_1,\ldots,\rho_r)}.$ □

5. Direct decomposability of weak 0 classes

Let $\mathcal{A}_1, \mathcal{A}_2$ be algebras of the same type and $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. Let $\rho \in Cw\mathcal{A}$. We say that the $\mathbf{0}$-class $[\mathbf{0}]_\rho$ is **directly decomposable** if there exist $\rho_1 \in Cw\mathcal{A}_1$, $\rho_2 \in Cw\mathcal{A}_2$ such that

$$[\mathbf{0}]_\rho = [\mathbf{0}]_{\rho_1} \times [\mathbf{0}]_{\rho_2}.$$

(where $\mathbf{0}$ on the left-hand side is $\mathbf{0} = (\mathbf{0}, \mathbf{0})$).

For $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ we denote by π_1 and π_2 the so called **projection congruences**, i.e., the congruences on $\mathcal{A}$ induced by the projection homomorphisms Pr_1, Pr_2 of $\mathcal{A}$ onto $\mathcal{A}_1$, $\mathcal{A}_2$, respectively.

Moreover, for $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and $\rho \in Cw\mathcal{A}$ we denote

$\sigma_1 = \{\langle (x_1, x_2), (y_1, y_2)\rangle \in \rho;\;\; x_2 = 0 = y_2\};$

$\sigma_2 = \{\langle (x_1, x_2), (y_1, y_2)\rangle \in \rho;\;\; x_1 = 0 = y_1\}.$

Theorem 6. *Let $\mathcal{A}_1, \mathcal{A}_2$ be of the same type such that $f(0, \ldots, 0) = 0$ for each $f \in F$, let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and $\rho \in Cw\mathcal{A}$. Then $[\mathbf{0}]_\rho$ is directly decomposable if and only if*

(7) $$[\mathbf{0}]_\rho = [\mathbf{0}]_{\pi_1\circ\sigma_1} \cap [\mathbf{0}]_{\pi_2\circ\sigma_2}.$$

Proof. First suppose that $[\mathbf{0}]_\rho$ is directly decomposable. Let $(x_1, x_2) \in [\mathbf{0}]_{\pi_1\circ\sigma_1} \cap [\mathbf{0}]_{\pi_2\circ\sigma_2}$. Then,

$(x_1, x_2)\pi_1(x_1, 0)\sigma_1(0, 0);$

$(x_1, x_2)\pi_2(0, x_2)\sigma_2(0, 0),$

thus $(x_1, 0) \in [\mathbf{0}]_\rho$ and $(0, x_2) \in [\mathbf{0}]_\rho$ (by the definition of σ_1, σ_2) and due to direct decomposability of $[\mathbf{0}]_\rho$, also $(x_1, x_2) \in [\mathbf{0}]_\rho$. The converse inclusion is evident, thus (7) is satisfied.

Conversely, suppose (7) and let $(y_1, y_2) \in [\mathbf{0}]_\rho$. Then,

$\langle (y_1, y_2), (0, 0)\rangle \in \pi_1 \circ \sigma_1$ and

$\langle (y_1, y_2), (0, 0)\rangle \in \pi_2 \circ \sigma_2.$

Define

$\rho_1 = \{(x_1, y_1) \in \mathcal{A}_1 \times \mathcal{A}_1; \langle (x_1, 0), (y_1, 0)\rangle \in \rho;$

$\rho_2 = \{(x_2, y_2) \in \mathcal{A}_2 \times \mathcal{A}_2; \langle (0, x_2), (0, y_2)\rangle \in \rho.$

It is clear that ρ_1, ρ_2 are symmetric and transitive relations on $\mathcal{A}_1$, $\mathcal{A}_2$, respectively and, due to $f(0, \ldots, 0) = 0$ for each $f \in F$, they are also compatible, thus $\rho_1 \in Cw\mathcal{A}_1$, $\rho_2 \in Cw\mathcal{A}_2$.

Moreover, $\langle (y_1, y_2), (0, 0)\rangle \in \pi_1 \circ \sigma_1$ gives $(y_1, 0) \in \rho_1$

$\langle (y_1, y_2), (0, 0)\rangle \in \pi_2 \circ \sigma_2$ gives $(y_2, 0) \in \rho_2$, thus

$(y_1, y_2) \in [\mathbf{0}]_{\rho_1} \times [\mathbf{0}]_{\rho_2}$ proving $[\mathbf{0}]_\rho \subseteq [\mathbf{0}]_{\rho_1} \times [\mathbf{0}]_{\rho_2}$.

Suppose $(x_1, x_2) \in [\mathbf{0}]_{\rho_1} \times [\mathbf{0}]_{\rho_2}$. Then, clearly $(x_1, x_2) \in [\mathbf{0}]_{\pi_1 \circ \sigma_1} \cap [\mathbf{0}]_{\pi_2 \circ \sigma_2}$ and, by (7), $(x_1, x_2) \in [\mathbf{0}]_\rho$ proving the converse inclusion. □

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