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IDEALS AND DIVISIBILITY IN A RING WITH RESPECT TO A FUZZY SUBSET

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Abstract. Ideals of a ring generated by a fuzzy subset and an element of a ring are defined and their properties are discussed. The notions of units, associates, prime element, irreducible element, etc. in classical ring theory are generalized with respect to a fuzzy subset and analogous results are obtained . Images and pre-images of translational invariant fuzzy subset under ring homomorphisms are studied.

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1. Introduction

The notion of fuzzy subset of a set was introduced by Zadeh [4]. Rosenfeld [3] introduced the concept of a fuzzy subgroup of a group and established many important properties. The notion of a fuzzy ideal of a ring was introduced by Liu [1]. Ray [2] introduced the concept of translational invariant fuzzy subset. The purpose of this paper is to generalize some of the classical results of ring theory using the notion of a translational invariant fuzzy subset.

2. Preliminaries

Throughout this paper R is an arbitrary ring with binary operations ' + ' and ' . ' . The operation ' . ' is suppressed and indicated by juxtaposition. A fuzzy subset P of any set S is a mapping from S into $[0,1]$. Let ' * ' be a binary operation in S.

Definition 2.1. *P is said to be left translational invariant with respect to ' * ' if $P(x) = P(y) \Rightarrow P(a * x) = P(a * y)\ \forall x, y,\ \ a \in S$.*

Definition 2.2. *P is said to be right translational invariant with respect to ' * ' if $P(x) = P(y) \Rightarrow P(x * a) = P(y * a)\ \ \forall x, y, a \in S$.*

Definition 2.3. *P is said to be translational invariant with respect to ' * ' if P is both left and right translational invariant with respect to * .*

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Remark 2.1. If P is commutative , i.e., $P(x * y) = P(y * x) \ \ \forall x, y \in S$, then P is left translational invariant if and only if P is right translational invariant.

Example 2.1. Consider the ring $Z_6 = \{0, 1, 2, 3, 4, 5\}$, the ring of integers modulo 6 .

Let P be a fuzzy subset of Z_6 defined as follows:

$$\begin{aligned} P(0) &= P(3) = 1 \\ P(1) &= P(4) = .5 \\ P(2) &= P(5) = .3 \end{aligned}$$

It can be easily verified that P is a translational invariant fuzzy subset of Z_6 with respect to addition and multiplication modulo 6.

3. Ideals of a ring generated by an element and a fuzzy subset

Throughout this section P is a fuzzy subset of R satisfying $P(x) = P(-x) \ \ \forall x \in R$.

Proposition 3.1. *Suppose P is left translational invariant with respect to both ' + ' and '. '. Then for any $a \in R$, the set*

$$L(a, P) = \{r \in R : \ P(r) = P(xa), \quad \textit{for some } x \in R\}$$

is a left ideal of R.

Proof. Let $s, r \in L(a, P)$.
Then $P(s) = P(xa)$ and $P(r) = P(ya)$ for some $x, y \in R$. Now

$(i) \qquad P(s) = P(xa) \Rightarrow P(s - r) = P(xa - r) = P(r - xa)$

Also

$(ii) \qquad P(r) = P(ya) \Rightarrow P(r - s) = P(ya - s) = P(s - ya)$

(i) and (ii) implies $P(r - xa) = P(s - ya) \Rightarrow P(r - s) = P((x - y)a)$. Thus $r - s \in L(a, P)$, since $x - y \in R$. Also for any $u \in R$, $P(us) = P(u(xa)) = P((ux)a) \Rightarrow us \in L(a, P)$, since $ux \in R$. Hence $L(a, P)$ is a left ideal of R. □

Analogously we can prove:

Proposition 3.2. *Suppose P is right translational invariant with respect to both '+ ' and ' . ' . Then for any $a \in R$, the set $R(a, P) = \{r \in R : \ P(r) = P(ax)$, for some $x \in R\}$ is a right ideal of R.*

Remark 3.1. If P is commutative , then $L(a, P) = R(a, P) \ \forall a \in R$.

Remark 3.2. We observe that for any $a \in R$, the ideal $Ra = \{ra : r \in R\}$ of R is contained in the left ideal $L(a, P)$. Also for any $a \in R$, the ideal $aR = \{ar : r \in R\}$ of R is contained in the right ideal $R(a, P)$.

If R is a commutative ring with identity then the principal ideal $< a >= aR = Ra$ is a subset of $L(a, P) = R(a, P)$.

Example 3.1. Let Z be the ring of integers. We define $P : Z \to [0, 1]$ as follows:

$$\begin{aligned} P(x) &= 1, \quad \text{if } x \text{ is even} \\ &= .5, \quad \text{otherwise.} \end{aligned}$$

Then $< 6 >= \{\ldots, -12, -6, 0, 6, 12, \ldots\}$ and $L(6, P) =$All even integers. We observe that $< 6 > \subsetneq L(6, P) \subsetneq Z$.

Definition 3.1. *$L(a, P)$ is called left P-principal ideal of R generated by a and P, and $R(a, P)$ is called right P-principal ideal of R generated by a and P.*

Definition 3.2. *If $L(a, P) = R(a, P)$, then the ideal is denoted by $I(a, P)$ and is called P-principal ideal of R generated by a and P.*

Definition 3.3. *R is called P-principal ideal ring if P is commutative and every ideal of R is a P-principal ideal generated by some $a \in R$ and P.*

Example 3.2. We consider Z_2, the ring of integers modulo 2. Let $P : Z_2 \to [0, 1]$, such that $P(0) = 1$ and $P(1) = .5$. Then Z_2 is a P-principle ideal ring .

Definition 3.4. *Let R be a ring with identity e and $P(0) \neq P(e)$. An element $a \in R$ with $P(a) \neq P(0)$ is called a P-unit of R if there exists an element $u \in R$ such that $P(u) \neq P(0)$ and $P(au) = P(ua) = P(e)$.*

Proposition 3.3. *If R contains the identity e and a is a P-unit of R, then $L(a, P) = R(a, P) = R$.*

Proof. As a is a P-unit of R, there exists $u \in R$ such that $P(u) \neq P(0)$ and $P(au) = P(ua) = P(e)$. Let $x \in R$. Then

$$P(e) = P(au) \Rightarrow P(ex) = P(aux) \Rightarrow P(x) = P(aux) \Rightarrow x \in R(a, P),$$

since $ux \in R$. Therefore $R \subseteq R(a, P)$. Similarly, $R \subseteq L(a, P)$. Hence $L(a, P) = R(a, P) = R$.

Proposition 3.4. *Let $a, b \in R$. Then*

$$a \in L(b, P) \Rightarrow L(a, P) \subseteq L(b, P) \quad \textit{and} \quad a \in R(b, P) \Rightarrow R(a, P) \subseteq R(b, P).$$

Proof. Let $a \in L(b, P)$, then $P(a) = P(xb)$, for some $x \in R$. Let $r \in L(a, P)$. Then $P(r) = P(ya)$ for some $y \in R$.

Now $P(a) = P(xb) \Rightarrow P(ya) = P(yxb) \Rightarrow P(r) = P(yxb) \Rightarrow r(L(b, P)$. Hence $L(a, P) \subseteq L(b, P)$.

Similarly , we can prove $a \in R(b, P) \Rightarrow R(a, P) \subseteq R(b, P)$. □

Remark 3.3. We observe that $L(0,P) = \{r \in R: \ P(r) = P(0)\}$.

Proposition 3.5. *Let $a, b \in R$. Then $P(a) = P(b)$ implies $L(a,P) = L(b,P)$ and $R(a,P) = R(b,P)$.*

Proof. Let $P(a) = P(b)$. Suppose $x \in L(a,P)$. Then $P(x) = P(ra)$ for some $r \in R$, so $P(x) = P(rb)$. Hence $x \in L(b,P)$. Thus $L(a,P) \subseteq L(b,P)$.

Next, let $y \in L(b,P)$. Then $P(y) = P(sb)$ for some $s \in R$, and so $P(y) = P(sa)$. Hence $y \in L(a,P)$. Thus $L(b,P) \subseteq L(a,P)$. Consequently, $L(a,P) = L(b,P)$. Similarly we can prove $R(a,P) = R(b,P)$. □.

In the next two sections R is assumed to be a commutative ring with the identity e and P is assumed to be a translational invariant fuzzy subset of R satisfying $P(x) = P(-x), \forall x \in R$. Henceforth, the ideal generated by an element a with respect to P will be denoted by $I(a,P)$.

4. P- divisors of zero, P-associates

Definition 4.1. *An element $a \in R$ with $P(a) \neq P(0)$ is said to be a P-divisor of zero if there exists some $b \in R$ with $P(b) \neq P(0)$ such that $P(ab) = P(0)$.*

Henceforth we shall assume that R contains no P-divisor of zero and $P(e) \neq P(0)$. Let $S = \{a \in R: \ P(a) \neq P(0)\}$.

Definition 4.2. *Let $a, b \in R$ and $P(a) \neq P(0)$. We say that a divides b with respect to P or a is a P- divisor of b, written as $(a/b)_P$, if there exists $c \in R$ such that $P(b) = P(ac) = P(ca)$.*

Theorem 4.1. *Let $a, b \in R$ and $P(a) \neq P(0)$. Then $(a/b)_P$ if and only if $I(b,P) \subseteq I(a,P)$.*

Proof. Suppose that $(a/b)_P$. Then $P(b) = P(ca)$ for some $c \in R$, which implies that $b \in I(a,P)$ and therefore $I(b,P) \subseteq I(a,P)$. Conversely, let $I(b,P) \subseteq I(a,P)$. As R contains identity e, $P(b) = P(eb) \Rightarrow b \in I(b,P) \subseteq I(a,P)$. Therefore, $P(b) = P(ca)$, for some $c \in R$. Also $P(a) \neq P(0)$. Hence $(a/b)_P$. □

Definition 4.3. *Let $a, b \in S$. We say that a and b are P-associates if $(a/b)_P$ and $(b/a)_P$.*

Proposition 4.2. *Let $a, b \in S$. Then a, b are P-associates if and only if $P(a) = P(bu)$ for some P-unit $u \in R$.*

Proof. Let a, b be P-associates. Then $(a/b)_P$ and $(b/a)_P$. So $P(b) = P(ad)$ and $P(a) = P(bc)$ for some $c, d \in R$. Hence

$$\begin{aligned}
&P(a) = P(bc) = P(adc) \\
&\Rightarrow P(a - adc) = P(0) \\
&\Rightarrow P(a(e - dc)) = P(0) \\
&\Rightarrow P(e - dc) = P(0), \text{ since } P(a) \neq P(0) \text{ and } R \text{ is without P-divisor of zero.} \\
&\Rightarrow P(dc) = P(e) \\
&\Rightarrow c \text{ and } d \text{ are P -units.}
\end{aligned}$$

Hence $P(a) = P(bc)$, where c is a P-unit in R. Conversely, suppose that $P(a) = P(bu)$, for some P-unit u in R. Now, $P(a) = P(bu) \Rightarrow (b/a)_P$. Since u is a P-unit, there exists $v \in S$ such that $P(uv) = P(vu) = P(e)$. Hence $P(a) = P(bu) \Rightarrow P(av) = P(buv) = P(be) = P(b)$. This shows that $(a/b)_P$. Thus we find $(a/b)_P$ and $(b/a)_P$. Hence a, b are P-associates. □

Corollary 4.3. *Let $a, b \in S$. If a, b are P-associates then $I(a, P) = I(b, P)$.*

Proof. Suppose a and b are P-associates. Then by Proposition 4.2, $P(a) = P(ub)$, for some P-unit $u \in R$. Then, $a \in I(b, P)$, and so $I(a, P) \subseteq I(b, P)$. Since u is a P-unit of R, and $P(a) \neq P(0)$ there exists $v \in S$ such that $P(uv) = P(e) = P(vu)$. Hence $P(buv) = P(be) = P(b) \Rightarrow P(av) = P(b)$ and so $b \in I(a, P)$. Therefore $I(b, P) \subseteq I(a, P)$. □

Remark 4.1. The relation of being P-associates is an equivalence relation on S.

Definition 4.4. *Suppose $a \in S$ and a is not a P-unit. Then a is said to be P-irreducible if $P(a) = P(bc)$ implies either b or c is a P-unit.*

Definition 4.5. *Suppose $a \in S$ and a not a P-unit. Then a is said to be P-prime if $(a/bc)_P$ implies $(a/b)_P$ or $(a/c)_P$.*

Proposition 4.4. *In the ring R any P-prime is P-irreducible.*

Proof. Let a be P-prime. Suppose $P(a) = P(bc)$. We can write $P(bc) = P(ae)$. Hence $(a/bc)_P$. Since a is P-prime , either $(a/b)_P$ or $(a/c)_P$.

Suppose $(a/b)_P$. Then $P(b) = P(ad)$ for some $d \in R$. Now

$$\begin{aligned}
& P(a) = P(bc) = P(adc) \\
\Rightarrow\ & P(a(e - dc)) = P(0) \\
\Rightarrow\ & P(e - dc) = P(0), \quad \text{since } P(a) \neq P(0) \text{ and } R \text{ is without P-divisor of zero.} \\
\Rightarrow\ & P(dc) = P(e) \\
\Rightarrow\ & c \text{ and } d \text{ are P -units.}
\end{aligned}$$

Similar is the case if $(a/bc)_P$.

Hence a is P-irreducible. □

Theorem 4.5. *Suppose $a \in S$ and a is not a P-unit. Then*

(i) The element a is P-irreducible if and only if the ideal $I(a, P)$ is maximal among all ideals $I(b, P)$, where $b \in R$ and $P(a) \neq P(b)$.

(ii) Let $a \in S$ and $I(a, P) \neq R$. Then a is P-prime if and only if the ideal $I(a, P)$ is a non-zero prime ideal.

Proof. (i) Suppose a is P-irreducible. Let $I(a, P) \subseteq I(b, P) \neq R$ for some $b \in R$ with $P(b) \neq P(0)$. As R contains the identity , $a \in I(a, P) \subseteq I(b, P)$ and so $P(a) = P(cb)$ for some $c \in R$. Since a is P-irreducible, either b is a P-unit or c is a P-unit. Since $I(b, P) \neq R$, by Proposition 3.3, we find that b is not a P-unit. Hence c is a P-unit . So there exists $u \in S$ such that $P(cu) = P(uc) = P(e) \Rightarrow P(bcu) = P(be) = P(b)$. Again, $P(a) = P(cb)$ implies $P(au) = P(cbu) = P(bcu) = P(b)$. Hence $P(b) = P(au)$. This implies $b \in I(a, P)$ and so $I(b, P) \subseteq I(a, P)$. Consequently, $I(b, P) = I(a, P)$. Thus $I(a, P)$ is maximal.

Conversely, assume $I(a, P)$ is maximal. Assume that $P(a) = P(cd)$ where $c, d \in R$. Then $a \in I(d, P)$ and so $I(a, P) \subseteq I(d, P)$. Hence by our hypothesis either $I(a, P) = I(d, P)$ or $I(d, P) = R$. If $I(a, P) = I(d, P)$, then $d \in I(d, P) = I(a, P)$. Therefore $P(d) = P(ra)$ for some $r \in R$. This gives $P(cd) = P(cra)$. Thus we have $P(a) = P(cra)$ and so $P(a(e - cr)) = P(0)$. Since R is without P-divisors of zero and $P(a) \neq P(0)$, we have $P(e - cr) = P(0)$, i.e., $P(cr) = P(e)$. This shows that c is a P-unit. If $I(d, P) = R$, then as $e \in R$, $e \in I(d, P) = R$. Hence $P(e) = P(ds)$ for some $s \in R$. Thus $P(a) = P(dc)$ implies either c or d is a P-unit. Hence a is P-irreducible . This proves (i).

(ii) Suppose a is P-prime in R. Let $x, y \in R$ and $xy \in I(a, P)$. Then $P(xy) = P(ar)$ for some $r \in R$. Which shows that $(a/xy)_P$. As a is P-prime, either $(a/x)_P$ or $(a/y)_P$. If $(a/x)_P$, then $P(x) = P(ac)$ for some $c \in R$, and so $x \in I(a, P)$. If $(a/y)_P$, then $P(y) = P(ad)$ for some $d \in R$, and so $y \in I(a, P)$. Thus $xy \in I(a, P)$ implies either $x \in I(a, P)$ or $y \in I(a, P)$. Since $P(a) \neq P(0)$, we must have $a \neq 0$. As $e \in R$, it follows that $a \in I(a, P)$. Hence $I(a, P) \neq \{0\}$. Consequently $I(a, P)$ is a non-zero prime ideal of R. Conversely, let $I(a, P)$ be a non-zero prime ideal of R. Let $x, y \in R$ and $(a/xy)_P$. Then $P(xy) = P(ac) = P(ca)$, for some $c \in R$. Hence $xy \in I(a, P)$. Since $I(a, P)$ is a prime ideal of R, either $x \in I(a, P)$ or $y \in I(a, P)$.

If $x \in I(a, P)$, then $P(x) = P(da)$ for some $d \in R$. Hence $(a/x)_P$.

If $y \in I(a, P)$, then $P(y) = P(ra)$ for some $r \in R$. Hence $(a/y)_P$. Thus $(a/xy)_P$ implies either $(a/x)_P$ or $(a/y)_P$. Hence a is P-prime. □

5. Images and inverse images under ring homomorphisms

In this section we discuss the invariance of translational invariace property of a fuzzy subset under ring homomorphism. Also we study the algebraic nature of P-ideals under ring homomorphism.

Definition 5.1. *Let f be a function from a ring R into a ring R' and let P be a fuzzy subset of R. Then P is called f-invariant if $f(x) = f(y) \Rightarrow P(x) = P(y)$, where $x, y \in R$.*

Proposition 5.1. *Let f be a homomorphism of a ring R into a ring R'. Let Q be a translational invariant fuzzy subset of R'. Then $f^{-1}(Q)$ is a translational invariant fuzzy subset of R.*

Proof. Let $a, b \in R$ and $f^{-1}(Q)(a) = f^{-1}(Q)(b)$. Then $Q(f(a)) = Q(f(b))$. Let $x \in R$ and $f(x) = y \in R'$. Since Q is a translational invariant fuzzy subset of R' and $Q(f(a)) = Q(f(b))$, we have $Q(f(a)+y) = Q(f(b)+y)$ and $Q(f(a)y) = Q(f(b)y)$, $Q(yf(a)) = Q(yf(b))$. Now $Q(f(a)+y) = Q(f(b)+y)$ implies $Q(f(a)+f(x)) = Q(f(b)+f(x))$, and so $Q(f(a+x)) = Q(f(b+x))$. Hence $f^{-1}(Q)(a+x) = f^{-1}(Q)(b+x)$. On the other hand, from $Q(f(a)y) = Q(f(b)y)$ and $Q(yf(a)) = Q(yf(b))$, we get $Q(f(a)f(x)) = Q(f(b)f(x))$ and $Q(f(x)f(a)) = Q(f(x)f(b))$, and so $Q(f(ax)) = Q(f(bx))$ and $Q(f(xa)) = Q(f(xb))$. Thus we have $f^{-1}(Q)(ax) = f^{-1}(Q)(bx)$ and $f^{-1}(Q)(xa) = f^{-1}(Q)(xb)$ $\forall a, b, x \in R$. Consequently $f^{-1}(Q)$ is translational invariant fuzzy subset of R. □

Proposition 5.2 *Let f be a homomorphism of a ring R onto a ring R'. Let P be a translational invariant fuzzy subset of R. If P is f-invariant, then $f(P)$ is a translational invariant fuzzy subset of R'.*

Proof. Suppose P is f-invariant. Then $\forall x, y \in R$, $f(x) = f(y)$ implies $P(x) = P(y)$. Now for any $a \in R'$, $f(P)(a) = \sup\{P(x) : x \in R, f(x) = a\}$, since f is onto. Let $x, y \in R$ and $f(x) = a$, $f(y) = a$. Then $f(x) = f(y)$, and so $P(x) = P(y)$. Hence $f(P)(a) = P(x)$, where $x \in R$ and $f(x) = a$. Thus $\forall a \in R'$, $f(P)(a) = P(x)$, where $x \in R$ and $f(x) = a$. Now, let $a, b \in R'$, and $f(P)(a) = f(P)(b)$. Then $P(x) = P(y)$, where $x, y \in R$, and $f(x) = a$, $f(y) = b$. Let $c \in R'$ be such that $f(z) = c$, where $z \in R$. Then, $a + c = f(x) + f(z) = f(x + z)$ and $b + c = f(y) + f(z) = f(y + z)$. Hence $f(P)(a + c) = P(x + z)$ and $f(P)(b+c) = P(y+z)$. Again, $ac = f(x)f(z) = f(xz)$, $ca = f(z)f(x) = f(zx)$, $bc = f(y)f(z) = f(yz)$, and $cb = f(z)f(y) = f(zy)$.

Hence $f(P)(ac) = P(xz)$, $f(P)(ca) = P(zx)$, $f(P)(bc) = P(yz)$, and $f(P)(cb) = P(zy)$. Since P is translational invariant and $P(x) = P(y)$, we have $P(x + z) = P(y + z)$, $P(xz) = P(yz)$, and $P(zx) = P(zy)$. Hence $f(P)(a + c) = f(P)(b + c)$, $f(P)(ac) = f(P)(bc)$, and $f(P)(ca) = f(P)(cb)$. Thus if $a, b \in R'$ and $f(P)(a) = f(P)(b)$, then $f(P)(a + c) = f(P)(b + c)$, $f(P)(ac) = f(P)(bc)$, and $f(P)(ca) = f(P)(cb)$ $\forall c \in R'$. Hence $f(P)$ is a translational invariant fuzzy subset of R'. □

Theorem 5.3. *Let f be a homomorphism of a ring R onto a ring R' and P be a translational invariant fuzzy subset of R. If P is f-invariant then,*

$$f(I(a, P)) = I(f(a), f(P)), \quad \forall a \in R.$$

Proof. Suppose P is f-invariant. Let $y \in I(f(a), f(P))$. Then $f(P)(y) = f(P)(sf(a))$ for some $s \in R'$. Since $y, s \in R'$ and f is onto, there exist $x, r \in R$ such that $f(x) = y$ and $f(r) = s$. Thus $f(P)f(x) = f(P)(f(r)f(a)) = f(P)(f(ra))$. Since P is translational invariant, by what we have proved in Proposition 5.2 , we get $f(P)(f(x)) = P(x)$ and $f(P)(f(ra)) = P(ra)$. Thus $P(x) = P(ra)$, which implies $x \in I(a, P)$, and so $f(x) \in f(I(a, P))$, i.e., $y \in f(I(a, P))$. Consequently, $I(f(a), f(P)) \subseteq f(I(a, P))$. Again, let $y \in f(I(a, P))$. Then there exists $x \in I(a, P)$ such that $f(x) = y$. Also, $x \in I(a, P)$ implies $P(x) = P(ar)$ for some $r \in R$. Now,

$$\begin{aligned} f(P)(y) = & \sup\left\{P(x) : x \in f^{-1}(y)\right\} \\ = & P(x), \text{since } P \text{ is f-invariant} \\ = & P(ar). \end{aligned}$$

Also, if $f(r) = s$ we have $f(P)(f(a)s) = f(P)(f(a)f(r)) = f(P)(f(ar)) = \sup\left\{P(x'), \text{ such that } x' \in f^{-1}(f(ar))\right\} = P(ar)$, since P is f-invariant. Thus $f(P)(y) = f(P)(f(a)s)$ which implies $y \in I(f(a), f(P))$. Hence $f(I(a, P)) \subseteq I(f(a), f(P))$, $a \in R$. Consequently, $f(I(a, P)) = I(f(a), f(P))$, $a \in R$.

Proposition 5.4. *Let f be a homomorphism of a ring R onto a ring R'. Let Q be a translational invariant fuzzy subset of S. Let $a' \in R'$. Then $\forall a, b \in f^{-1}(a')$, $I(a, f^{-1}(Q)) = I(b, f^{-1}(Q))$, provided $f^{-1}(a')$ contains more than one element.*

Proof. Let $x \in I(a, f^{-1}(Q))$. Then $f^{-1}(Q)(x) = f^{-1}(Q)(ra)$ for some $r \in R$ and so $f^{-1}(Q)(x) = Q(f(ra))$. Thus $f^{-1}(Q)(x) = Q(f(a)f(r))$. Since $a, b \in f^{-1}(a')$, $f(a) = f(b) = a'$ and hence we have $f^{-1}(Q)(x) = Q(f(b)f(r)) = Q(f(br)) = f^{-1}(Q)(br)$. This shows that $x \in I(b, f^{-1}(Q))$. Hence $I(a, f^{-1}(Q)) \subseteq I(b, f^{-1}(Q))$. Now let $y \in I(b, f^{-1}(Q))$. Then $f^{-1}(Q)(y) = f^{-1}(Q)(br')$ for some $r' \in R$, and so $f^{-1}(Q)(y) = Q(f(br')) = Q(f(b)f(r'))$. Since $a, b \in f^{-1}(a')$, $f(a) = a' = f(b)$ and hence we have $f^{-1}(Q)(y) = Q(f(a)f(r')) = Q(f(ar')) = f^{-1}(Q)(ar')$. This shows that $y \in I(a, f^{-1}(Q))$. Hence $I(b, f^{-1}(Q)) \subseteq I(a, f^{-1}(Q))$. Consequently, $I(a, f^{-1}(Q)) = I(b, f^{-1}(Q))$ $\forall a, b \in f^{-1}(a')$.

Theorem 5.5. *Let f be an isomomorphism of a ring R onto a ring R'. Let Q be a translational invariant fuzzy subset of R'. Then*

$$I(f^{-1}(y), f^{-1}(Q)) = f^{-1}(I(y, Q)) \quad \forall y \in R'.$$

Proof. Let $x \in I(f^{-1}(y), f^{-1}(Q))$. Then

$$\begin{aligned} f^{-1}(Q)(x) & = f^{-1}(Q)(f^{-1}(y)r) \quad \text{for some } r \in R. \\ & = f^{-1}(Q)(f^{-1}(y)f^{-1}(s)), \quad \text{where } s \in R' \text{ such that } f(r) = s. \\ \Rightarrow Q(f(x)) & = f^{-1}(Q)(f^{-1}(ys)), \quad \text{since } f \text{ is bijective.} \\ & = Q(f(f^{-1}(ys)) \\ & = Q(ys) \\ & \Rightarrow f(x) \in I(y, Q) \\ & \Rightarrow x \in f^{-1}(I(y, Q)). \end{aligned}$$

Hence $I(f^{-1}(y), f^{-1}(Q)) \subseteq f^{-1}(I(y, Q)) \;\; \forall y \in R'$. Again, let $a \in f^{-1}(I(y, Q))$ then $f(a) \in I(y, Q) \Rightarrow Q(f(a)) = Q(ys)$, for some $s \in R'$. Also, $y, s \in R'$ and f is onto implies there exist $x, r \in R$ such that $f(x) = y$ and $f(r) = s$. Now , $Q(f(a)) = Q(ys) \Rightarrow Q(f(a)) = Q(f(x)f(r)) = Q(f(xr)) \Rightarrow f^{-1}(Q)(a) = f^{-1}(Q)(xr) = f^{-1}(Q)(f^{-1}(y)r)$ which implies $a \in I(f^{-1}(y), f^{-1}(Q))$. Thus, $f^{-1}(I(y, Q)) \subseteq I(f^{-1}(y), f^{-1}(Q)), \forall y \in R'$. Consequently, $I(f^{-1}(y), f^{-1}(Q)) = f^{-1}(I(y, Q)), \forall y \in R'$. □

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References

[1] Liu , W.J., Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Set and Systems 8 (1982), 133-139.

[2] Ray, A.K., Quotient group of a group generated by a subgroup and a fuzzy subset, The Journal of Fuzzy Mathematics Vol. 7 , No. 2 (1999), 459-463.

[3] Rosenfeld, A., Fuzzy Groups, J. Math. Anal. Appl. 35(1971), 512-517.

[4] Zadeh, L.A., Fuzzy Sets, Inform. And Control 8 (1965), 338-353.

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