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W–CURVES IN MINKOWSKI SPACE–TIME

Miroslava Petrović–Torgašev[1], Emilija Šućurović[1]

Abstract. In this paper we complete a classification of W–curves in Minkowski space–time. Namely, we classify all spacelike curves with constant curvatures in E^4_1.

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1. Introduction

It is well–known that to each unit speed curve $\alpha : I \to E^n$ in the Euclidean space E^n whose successive derivatives $\alpha'(s), \alpha''(s), \dots, \alpha^{(n)}(s)$ are linearly independent vectors, one can associate the orthonormal frame $\{V_1, V_2, V_3, \dots, V_n\}$ and $n-1$ functions $k_1, \dots, k_{n-1} : I \to R$ called the Frenet curvatures, such that the following Frenet formulas hold ([6]):

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-1}' \\ V_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & \dots & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \dots & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \dots & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-1} \\ V_n \end{bmatrix}.$$

In particular, the first curvature k_1 is also called the curvature k, and the second curvature k_2 is also called the torsion τ. Recall that a curve α is called a W–curve (or a helix), if it has constant Frenet curvatures. W–curves in the Euclidean space E^n have been studied intensively. The simplest examples are circles as planar W–curves and helices as non–planar W–curves in E^3. A parameterization of a unit speed W–curve in E^{2k+1} is given by

$$\gamma(s) = \gamma_0 + ase_0 + \sum_{i=1}^{k} r_i\big(\cos(a_i s)e_{2i-1} + \sin(a_i s)e_{2i}\big), \tag{1.1}$$

where $\{e_0, e_1, \dots, e_{2k}\}$ is an orthonormal basis of E^{2k+1}, $a \in R$, $a_1 < a_2 < \dots < a_k$ are positive real numbers satisfying the equation $a^2 + \sum_{i=1}^{k}(r_i a_i)^2 = 1$. If

[1]Institute of Mathematics, Faculty of Science, Radoja Domanovića 12, 34000 Kragujevac, Yugoslavia, *e–mail*: miraptuis0.uis.kg.ac.yu, emilija$uis0.uis.kg.ac.yu

$a \neq 0$ the curve γ lies fully in E^{2k+1}. Otherwise, γ lies fully in E^{2k} and on a hypersphere in that space. We remark that a W–curve is closed if and only if $a = 0$ and $a_i = \frac{p_i}{r}, p_i \in N, r \in R_0^+$. Further, we mention that W–curves in E^n are the examples of the finite type curves ([3]). In particular, closed W–curves in E^4 are spherical 2–type curves ([4]).

All W–curves in the Minkowski 3–space E_1^3 are completely classified in [10]. For example, the only planar spacelike W–curves are circles and hyperbolas. In this paper, we classify all spacelike W–curves in the Minkowski space–time E_1^4. Since all three curvatures k_1, k_2 and k_3 are constant, the classification is reduced mainly to differential equations with constant coefficients and a method well developed by B. Y. Chen.

The examples of null W–curves in the Minkowski space–time E_1^4 are given in [1]. Timelike W–curves in the same space have been studied in [8].

2. Preliminaries

Let E_1^4 denote the 4–dimensional Minkowski space–time, i.e. the Euclidean space E^4 with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \tag{2.1}$$

where $(x_1, \dots x_4)$ is a rectangular coordinate system of E_1^4. Since g is indefinite metric, recall that a vector v in E_1^4 can have one of three causal characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$, and null if $g(v,v) = 0$ and $v \neq 0$. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v and w are said to be orthogonal if $g(v,w) = 0$.

An arbitrary curve $\alpha : I \to E_1^4$ in the space E_1^4 can locally be spacelike, timelike or null, if respectively all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null. Next, α is a unit speed curve if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Recall that a curve α in E_1^n is said to be of k–type for some natural number k, if its position vector $\alpha(s)$ can be written as a finite sum of eigenfunctions s, $\cos(ps)$, $\sin(ps)$, $\cosh(qs)$, $\sinh(qs)$ of its Laplace operator $\Delta = \pm\frac{d^2}{ds^2}$ which has exactly k mutually different eigenvalues $\{\lambda_1, \dots, \lambda_k\}$. In particular, if one of the eigenvalues is equal to zero, α is said to be of null k–type. Therefore, α is a 2–type curve in E_1^4 if and only if it has one of the following forms:

$$(i) \quad \alpha(s) = a_0 + \sum_{i=1}^{2}(a_i \cos(p_i s) + b_i \sin(p_i s));$$

$$(ii) \quad \alpha(s) = a_0 + \sum_{i=1}^{2}(a_i \cosh(p_i s) + b_i \sinh(p_i s));$$

$$(iii) \quad \alpha(s) = a_0 + a_1 \cos(p_1 s) + b_1 \sin(p_1 s) + a_2 \cosh(p_2 s) + b_2 \sinh(p_2 s);$$

where a_0, a_1, a_2, b_1, $b_2 \in E_1^4$ are constant vectors and $0 < p_1 < p_2$, p_1, $p_2 \in N$.

In particular, α is a null 2–type curve in E_1^4 if and only if it has one of the following forms:

$$(iv) \quad \alpha(s) = a_0 + b_0 s + a_1 \cos(ps) + b_1 \sin(ps);$$

$$(v) \quad \alpha(s) = a_0 + b_0 s + a_1 \cosh(ps) + b_1 \sinh(ps);$$

where a_0, a_1, b_0, $b_1 \in E_1^4$ are constant vectors and $p \in N$.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the spacelike curve α, where s is a pseudo arclength parameter. Then $T(s)$ is a spacelike tangent vector, so depending on the causal character of the principal normal vector $N(s)$ and the binormal vector $B_1(s)$, we have the following Frenet formulas ([10]):

Case (1). N and B_1 are spacelike;

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T,T) = g(N,N) = g(B_1,B_1) = 1, \quad g(B_2,B_2) = -1.$$

Case (2). N is spacelike, B_1 is timelike;

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T,T) = g(N,N) = g(B_2,B_2) = 1, \quad g(B_1,B_1) = -1.$$

Case (3). N is spacelike, B_1 is null;

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 satisfy the equations

$$g(T,T) = g(N,N) = 1, \quad g(B_1,B_1) = g(B_2,B_2) = 0,$$

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0, \quad g(B_1,B_2) = 1.$$

Case (4). N is timelike, B_1 is spacelike;

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T,T) = g(B_1,B_1) = g(B_2,B_2) = 1, \quad g(N,N) = -1.$$

Case (5). N is null, B_1 is spacelike;

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B}_1 \\ \dot{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the curvature k_1 can only take two values: 0 if α is a straight line, or 1 in all other cases. In this case, the vectors T, N, B_1, B_2 satisfy the equations

$$g(T,T) = g(B_1,B_1) = 1, \quad g(N,N) = g(B_2,B_2) = 0,$$
$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(B_1,B_2) = 0, \quad g(N,B_2) = 1.$$

The notion of causal character of vectors has a natural generalization to vector subspaces. A subspace V of E_1^4 can be spacelike, timelike or lightlike if respectively $g|_V$ is positive definite, $g|_V$ is nondegenerate of index 1 or $g|_V$ is degenerate. For a subspace V of the Minkowski space–time E_1^4, recall that $V^\perp$ is a subspace defined by $V^\perp = \{v \in E_1^4 : v \perp V\}$. Then the following simple property holds: a subspace V is timelike (spacelike) if and only if $V^\perp$ is spacelike (timelike) ([7]). Moreover, if V is a timelike (spacelike) subspace, then $E_1^4 = V \oplus V^\perp$, where $\oplus$ denotes the direct sum of subspaces. Next, a subspace V is lightlike if and only if $V^\perp$ is lightlike, but then $V \oplus V^\perp$ is not all of E_1^4.

Recall some of the most important hyperquadrics in E_1^4. The pseudo–Riemannian sphere and the pseudo–hyperbolic space in E_1^4 are defined respectively by

$$S_1^3(c,r) = \{x \in E_1^4 : g(x-c, x-c) = r^2\}, \tag{2.2}$$

$$H^3(c,-r) = \{x \in E_1^4 : g(x-c, x-c) = -r^2\}, \tag{2.3}$$

where $r > 0$ is a radius and $c \in E_1^4$ is a center of the mentioned hyperquadrics. Finally, the light cone $C(c)$ with the vertex at a point c in E_1^4 is defined by

$$C(c) = \{x \in E_1^4 : g(x-c, x-c) = 0\}. \tag{2.4}$$

3. A classification of spacelike W–curves

First we give some introductory results which characterize spacelike curves in the Minkowski space–time E_1^4. In [10] it is proved that a spacelike curve α in E_1^3 with $g(\ddot{\alpha},\ddot{\alpha}) \neq 0$ has the second curvature $k_2 \equiv 0$ if and only if α is a planar curve. Therefore, it is easy to prove that in E_1^4 the following analogous theorem holds.

Theorem 3.1. *Let α be a spacelike unit speed curve in E_1^4 with curvature $k_1 > 0$. Then α has $k_2 \equiv 0$ if and only if α lies fully in a 2–dimensional subspace of E_1^4.*

The following theorems characterize spacelike curves with respect to their third curvature k_3.

Theorem 3.2. *Let α be a spacelike unit speed curve in E_1^4 with a spacelike principal normal N, a spacelike binormal B_1 and with curvatures $k_1 > 0$, $k_2 \neq 0$. Then α has $k_3 \equiv 0$ if and only if α lies fully in a spacelike hyperplane of E_1^4.*

Proof. If α has $k_3 \equiv 0$, then by using the Frenet equations we obtain $\dot{\alpha} = T$, $\ddot{\alpha} = k_1 N$, $\dddot{\alpha} = -k_1^2 T + \dot{k}_1 N + k_1 k_2 B_1$, $\ddddot{\alpha} = -3k_1\dot{k}_1 T + (\ddot{k}_1 - k_1^3 - k_1 k_2^2)N + (2\dot{k}_1 k_2 + k_1 \dot{k}_2)B_1$. Next, all higher-order derivatives of α are linear combinations of vectors $\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}$, so by using the MacLaurin expansion for α given by

$$\alpha(s) = \alpha(0) + \dot{\alpha}(0)\, s + \ddot{\alpha}(0)\,\frac{s^2}{2!} + \dddot{\alpha}\,(0)\,\frac{s^3}{3!} + \dots, \tag{3.1}$$

we conclude that α lies fully in a spacelike hyperplane of the space E_1^4, spanned by
$\{\dot{\alpha}(0), \ddot{\alpha}(0), \dddot{\alpha}\,(0)\}$.

Conversely, assume that α satisfies the asumptions of the theorem and lies fully in a spacelike hyperplane π of E_1^4. Then there exist points $p, q \in E_1^4$, such that α satisfies the equation of π given by $g(x(s) - p, q) = 0$, where $q \in \pi^{\perp}$ is a timelike vector. Differentiation of the last equation yields $g(\dot{\alpha}, q) = g(\ddot{\alpha}, q) = g(\dddot{\alpha}, q) = 0$. Therefore, $\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha} \in \pi$. Since $T = \dot{\alpha}$, $N = \frac{\ddot{\alpha}}{||\ddot{\alpha}||}$, it follows that $g(T, q) = g(N, q) = 0$. Next, differentiation of the equation $g(N, q) = 0$ gives $g(\dot{N}, q) = 0$. From the Frenet equations we obtain $B_1 = \frac{1}{k_2}(\dot{N} + k_1 T)$, so $g(B_1, q) = 0$. Since $B_2(s)$ is the unique timelike unit vector perpendicular to $\{T, N, B_1\}$, it follows that $B_2(s) = \frac{q}{||q||}$. Thus $\dot{B}_2(s) = k_3 B_1 = 0$ for each s and therefore $k_3 \equiv 0$. □

Theorem 3.3. *Let α be a spacelike unit speed curve in E_1^4 with a spacelike (timelike) principal normal N, a timelike (spacelike) binormal B_1 and with curvatures $k_1 > 0, k_2 \neq 0$. Then α has $k_3 \equiv 0$ if and only if α lies fully in a timelike hyperplane of E_1^4.*

We omit the proof, as it is analogous to the proof of Theorem 3.2.

Next, recall that a spacelike curve with a spacelike principal normal N and a null binormal B_1 is called *a partially null spacelike curve*.

Theorem 3.4. *A partially null spacelike unit speed curve α in E_1^4 with curvatures $k_1 > 0, k_2 \neq 0$ lies fully in a lightlike hyperplane of E_1^4 and has $k_3 \equiv 0$.*

Proof. By using the Frenet formulas for this case, we obtain $\dot{\alpha} = T, \ddot{\alpha} = k_1 N, \dddot{\alpha}= -k_1 T + \dot{k}_1 N + k_1 k_2 B_1$, $\ddddot{\alpha}= -3k_1 \dot{k}_1 T + (\ddot{k}_1 - k_1^3)N + (2\dot{k}_1 k_1 + k_1 \dot{k}_2 + k_1 k_2 k_3)B_1$. Thus $\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}$ are linearly independent vectors, while $\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}, \ddddot{\alpha}$ are not linearly independent. Moreover, all higher-order derivatives of α are linear combinations of $\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}$, so by using the MacLaurin expansion (3.1) it follows that α lies fully in a lightlike hyperplane π of E_1^4, spanned by $\{\dot{\alpha}(0), \ddot{\alpha}(0), \dddot{\alpha}(0)\}$. Therefore, we may assume that there exist points $p, q \in E_1^4$, such that α satisfies the equation of π given by $g(x(s) - p, q) = 0$, where $q \in \pi^{\perp}$ is a null vector. Since q is a null vector perpendicular to B_1, it follows that $q = \lambda B_1, \lambda \in R_0$. Then $\dot{q} = \lambda k_3 B_1 = 0$ and thus $k_3 \equiv 0$. □

Remark 3.1. Also by making a null rotation from one null tetrad to another null tetrad, we can make $k_3 \equiv 0$. For more details, see [2].

Next, recall that a spacelike curve with a null principal normal is called *a pseudo null spacelike curve*. Such curves are characterized by the following theorem.

Theorem 3.5. *A pseudo null spacelike unit speed curve α in E_1^4 with curvatures $k_1 > 0, k_2 \neq 0$ lies fully in the space E_1^4.*

Proof. The Frenet formulas imply the equations $\dot{\alpha} = T, \ddot{\alpha} = N$, $\dddot{\alpha}= k_2 B_1$, $\ddddot{\alpha}= k_2 k_3 N + \dot{k}_2 B_1 - k_2^2 B_2$. Therefore, the vectors $\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}, \ddddot{\alpha}$ are linearly independent. On the other hand, all higher order derivatives of α may be expressed as linear combinations of vectors $\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}, \ddddot{\alpha}$. Thus by using the MacLaurin expansion (3.1) for α, we conclude that α lies fully in the space E_1^4, spanned by $\{\dot{\alpha}(0), \ddot{\alpha}(0), \dddot{\alpha}(0), \ddddot{\alpha}(0)\}$. □

Theorem 3.6. *Let α be a spacelike unit speed curve in E_1^4, with a spacelike principal normal N and a spacelike binormal B_1. Then α has:*
(i) $k_1 = c_1$, $k_2 = c_2$, $k_3 = 0$, c_1, $c_2 \in R_0$ if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\lambda^2}(0, c_2 \lambda s, c_1 \sin(\lambda s), c_1 \cos(\lambda s)), \quad \lambda^2 = c_1^2 + c_2^2; \tag{3.2}$$

(ii) $k_1 = c_1$, $k_2 = c_2$, $k_3 = c_3$, c_1, c_2, $c_3 \in R_0$ if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\lambda_1}(V_1 \sinh(\lambda_1 s) + V_2 \cosh(\lambda_1 s)) + \frac{1}{\lambda_2}(V_3 \sin(\lambda_2 s) - V_4 \cos(\lambda_2 s)) \tag{3.3}$$

with $\lambda_1^2 = \frac{-K+\sqrt{K^2+4c_1^2c_3^2}}{2}$, $\lambda_2^2 = \frac{K+\sqrt{K^2+4c_1^2c_3^2}}{2}$, $K = c_1^2 + c_2^2 - c_3^2$, *where* V_1, V_2, V_3, V_4 *are mutually orthogonal vectors satisfying the equations* $g(V_1,V_1) = -g(V_2,V_2) = \frac{\lambda_2^2-c_1^2}{\lambda_1^2+\lambda_2^2}$, $g(V_3,V_3) = g(V_4,V_4) = \frac{\lambda_1^2+c_1^2}{\lambda_1^2+\lambda_2^2}$.

Proof. (i) If α has constant curvatures, then by using the Frenet formulas we find $\dddot{T} + (c_1^2 + c_2^2)\dot{T} = 0$. Solving this equation, we easily obtain $T = A + B\cos(\sqrt{c_1^2+c_2^2}s) + C\sin(\sqrt{c_1^2+c_2^2}s)$, where A, B, $C \in E_1^4$ are constant vectors. Next, the equation $g(T,T) = 1$ implies that $g(B,B) = g(C,C)$, $g(A,B) = g(A,C) = g(B,C) = 0$, $g(A,A) = 1 - g(B,B)$. On the other hand, the equation $g(\dot{T},\dot{T}) = c_1^2$ gives $g(B,B) = \frac{c_1^2}{c_1^2+c_2^2}$. Therefore, we may take $A = (0, \frac{c_2}{\sqrt{c_1^2+c_2^2}}, 0, 0)$, $B = (0, 0, \frac{c_1}{\sqrt{c_1^2+c_2^2}}, 0)$, $C = (0, 0, 0, \frac{c_1}{\sqrt{c_1^2+c_2^2}})$ and up to isometries of E_1^4 the curve α has the form (3.2).

Conversely, if α has the form (3.2), then it lies fully in a spacelike hyperplane of E_1^4, with the equation $x_1 = 0$. Then the Theorem 3.2 implies that $k_3 \equiv 0$. Next, from the Frenet equations we get $g(\dot{T},\dot{T}) = k_1^2$, $g(\dot{N},\dot{N}) = k_1^2 + k_2^2$. Since $\dot{T} = \ddot{\alpha}$ and $N = \frac{\ddot{\alpha}}{||\ddot{\alpha}||}$, we find $g(\dot{T},\dot{T}) = c_1^2$, $g(\dot{N},\dot{N}) = c_1^2 + c_2^2$. Accordingly, $k_1 = c_1$ and $k_2 = c_2$.

(ii) First assume that α has constant curvatures different from zero. Then from the Frenet formulas we obtain the equation $\ddddot{T} + (c_1^2+c_2^2-c_3^2)\ddot{T} - c_1^2c_3^2T = 0$. Solving the previous equation, we find

$$T = V_1\cosh(\lambda_1 s) + V_2\sinh(\lambda_1 s) + V_3\cos(\lambda_2 s) + V_4\sin(\lambda_2 s)$$

where V_1, V_2, V_3, $V_4 \in E_1^4$ are constant vectors, $\lambda_1^2 = \frac{-K+\sqrt{K^2+4c_1^2c_3^2}}{2}$, $\lambda_2^2 = \frac{K+\sqrt{K^2+4c_1^2c_3^2}}{2}$, $K = c_1^2 + c_2^2 - c_3^2$. Next, the equation $g(T,T) = 1$ implies $g(V_1,V_1) = -g(V_2,V_2)$, $g(V_3,V_3) = g(V_4,V_4)$, $g(V_1,V_1)+g(V_3,V_3) = 1$, $g(V_i,V_j) = 0$ for $i \neq j$ $(i,j \in \{1,2,3,4\})$. Finally, by using the equation $g(\dot{T},\dot{T}) = c_1^2$, we get $g(V_3,V_3) = \frac{\lambda_1^2+c_1^2}{\lambda_1^2+\lambda_2^2}$. Accordingly, α has the form (3.3).

Conversely, if α can be parameterized by (3.3), then it has spacelike principal normal N and spacelike binormal B_1, so that the Frenet formulas imply $g(\dot{T},\dot{T}) = k_1^2$, $g(\dot{N},\dot{N}) = k_1^2 + k_2^2$. Since $\dot{T} = \ddot{\alpha}$ and $N = \frac{\ddot{\alpha}}{||\ddot{\alpha}||}$, we get $g(\dot{T},\dot{T}) = c_1^2$, $g(\dot{N},\dot{N}) = c_1^2 + c_2^2$. Thus $k_1 = c_1$ and $k_2 = c_2$. Finally, by the Frenet equations we get $g(\dot{B}_1,\dot{B}_1) = k_2^2 - k_3^2$, and on the other hand since $\dot{B}_1 = \frac{1}{c_2}(\ddot{N} + c_1^2N)$ we obtain $g(\dot{B}_1,\dot{B}_1) = c_2^2 - c_3^2$. Consequently, $k_3 = c_3$. □

Remark 3.2. The curve (3.2) lies on a circular cylinder in E_1^4 with the equation $x_3^2 + x_4^2 = \frac{c_1^2}{(c_1^2+c_2^2)^2}$. The curve (3.3) lies on some hyperquadric in E_1^4. More precisely, if $c_3^2 > c_2^2$, $c_3^2 < c_2^2$, or $c_3^2 = c_2^2$, then respectively α lies on pseudo–Riemannian sphere with the equation $-x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{c_3^2-c_2^2}{c_1^2c_3^2}$, pseudo–hyperbolic space with the same equation or light cone with the equation $-x_1^2 +$

$x_2^2 + x_3^2 + x_4^2 = 0$.

By the Theorem 3.3, all spacelike W–curves with a spacelike (timelike) principal normal N and a timelike (spacelike) binormal B_1 which have $k_3 \equiv 0$ lie fully in E_1^3, so their classification is given in [10]. In the next two theorems we consider the remaining cases and omite the proofs, since they are very similar with the proof of the Theorem 3.6.

Theorem 3.7. *A spacelike unit speed curve α in E_1^4 with a spacelike principal normal N and a timelike binormal B_1 has $k_1 = c_1$, $k_2 = c_2$, $k_3 = c_3$, c_1, c_2, $c_3 \in R_0$ if and only if α can be parameterized by*

$$(3.4)\quad \alpha(s) = \frac{1}{\lambda_1}(V_1 \sinh(\lambda_1 s) + V_2 \cosh(\lambda_1 s)) + \frac{1}{\lambda_2}(V_3 \sin(\lambda_2 s) - V_4 \cos(\lambda_2 s)),$$

with $\lambda_1^2 = \frac{-K+\sqrt{K^2+4c_1^2c_3^2}}{2}$, $\lambda_2^2 = \frac{K+\sqrt{K^2+4c_1^2c_3^2}}{2}$, $K = c_1^2 - c_2^2 - c_3^2$, *where* V_1, V_2, V_3, V_4 *are mutually orthogonal vectors satisfying the equations* $g(V_1, V_1) = -g(V_2, V_2) = \frac{\lambda_2^2 - c_1^2}{\lambda_1^2+\lambda_2^2}$, $g(V_3, V_3) = g(V_4, V_4) = \frac{c_1^2+\lambda_1^2}{\lambda_1^2+\lambda_2^2}$.

Remark 3.3. The curve (3.4) lies on pseudo–Riemannian sphere with the equation $-x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{c_2^2+c_3^2}{c_1^2c_3^2}$.

Theorem 3.8. *A spacelike unit speed curve α in E_1^4 with a timelike principal normal N has $k_1 = c_1$, $k_2 = c_2$, $k_3 = c_3$, c_1, c_2, $c_3 \in R_0$ if and only if α can be parameterized by*

$$(3.5)\quad \alpha(s) = \frac{1}{\lambda_1}(V_1 \sinh(\lambda_1 s) + V_2 \cosh(\lambda_1 s)) + \frac{1}{\lambda_2}(V_3 \sin(\lambda_2 s) - V_4 \cos(\lambda_2 s)),$$

with $\lambda_1^2 = \frac{-K+\sqrt{K^2+4c_1^2c_3^2}}{2}$, $\lambda_2^2 = \frac{K+\sqrt{K^2+4c_1^2c_3^2}}{2}$, $K = c_3^2 - c_1^2 - c_2^2$, *where* V_1, V_2, V_3, V_4 *are mutually orthogonal vectors satisfying the equations* $g(V_1, V_1) = -g(V_2, V_2) = \frac{\lambda_2^2 + c_1^2}{\lambda_1^2+\lambda_2^2}$, $g(V_3, V_3) = g(V_4, V_4) = \frac{\lambda_1^2-c_1^2}{\lambda_1^2+\lambda_2^2}$.

Remark 3.4. The curve (3.5) lies on some hyperquadric in E_1^4. If $c_2^2 > c_3^2$, $c_2^2 < c_3^2$, $c_2^2 = c_3^2$, then respectively α lies on pseudo–Riemannian sphere with the equation $-x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{c_2^2-c_3^2}{c_1^2c_3^2}$, pseudo–hyperbolic space with the same equation or light cone with the equation $-x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$.

By the Theorem 3.4, a partially null spacelike curve α has $k_3(s) = 0$ for each s. In the following theorem, we classify all partially null spacelike W–curves in E_1^4.

Theorem 3.9. *A partially null spacelike unit speed curve α in E_1^4 has $k_1 = c_1 \in R_0$, $k_2 = \text{constant} \neq 0$ if and only if α is a part of a partially null spacelike* *helix*

$$\alpha(s) = (as, as, \frac{1}{c_1}\sin(c_1 s), \frac{1}{c_1}\cos(c_1 s)), \quad a \in R_0. \tag{3.6}$$

Proof. First assume that α has non–zero constant curvatures. Then by the Frenet equations we find $\dddot{T} + c_1^2 \dot{T} = 0$. Solving this equation, we get $T = V_1 + V_2\cos(c_1 s) + V_3 \sin(c_1 s)$, where V_1, V_2, $V_3 \in E_1^4$ are constant vectors. Next, the equation $g(T,T) = 1$ implies that $g(V_1,V_1) + g(V_2,V_2) = 1$, $g(V_1,V_2) = g(V_1,V_3) = g(V_2,V_3) = 0$, $g(V_2,V_2) = g(V_3,V_3)$. Finally, by using the equation $g(\dot{T},\dot{T}) = c_1^2$, we obtain $g(V_2,V_2) = 1$. Therefore, we may take $V_1 = (a,a,0,0,)$, $a \in R_0$, $V_2 = (0,0,1,0)$, $V_3 = (0,0,0,1)$, so up to isometries of E_1^4 the curve α has the form (3.6).

On the other hand, if α can be parameterized by (3.6), then we obtain that $\ddot{\alpha} = (0,0,-c_1\sin(c_1 s), c_1\cos(c_1 s))$ and thus $g(\ddot{\alpha},\ddot{\alpha}) = c_1^2$. However, from the Frenet formulas we get $g(\dot{T},\dot{T}) = g(\ddot{\alpha},\ddot{\alpha}) = k_1^2$. It follows that $k_1 = c_1$. Next, since $N = \frac{\ddot{\alpha}}{||\ddot{\alpha}||}$, we obtain that $\overset{\cdots\cdot}{\alpha} = (0,0,c_1^3\sin(c_1 s), -c_1^3\cos(c_1 s)) = -c_1^3 N$. However, by the Frenet equations we get $\overset{\cdots\cdot}{\alpha} = -k_1^3 N + k_1\dot{k}_2 B_1$. It follows that $k_1\dot{k}_2 = 0$ and therefore $k_2 = \text{constant} \neq 0$. □

Remark 3.5. The curve (3.6) lies on a circular cylinder in E_1^4 with the equation $x_3^2 + x_4^2 = \frac{1}{c_1^2}$.

Theorem 3.10. *Let α be a pseudo null spacelike unit speed curve in E_1^4. Then α has:*

(i) $k_1 = 1$, $k_2 = c_2$, $k_3 = 0$, $c_2 \in R_0$, if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\sqrt{2c_2}}(\cosh(\sqrt{c_2}s), \sinh(\sqrt{c_2}s), \sin(\sqrt{c_2}s), \cos(\sqrt{c_2}s)); \tag{3.7}$$

(ii) $k_1 = 1$, $k_2 = c_2$, $k_3 = c_3$, c_2, $c_3 \in R_0$ if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\lambda_1}(V_1\sinh(\lambda_1 s) + V_2\cosh(\lambda_1 s)) + \frac{1}{\lambda_2}(V_3\sin(\lambda_2 s) - V_4\cos(\lambda_2 s)), \tag{3.8}$$

with $\lambda_1^2 = K + \sqrt{K^2 + c_2^2}$, $\lambda_2^2 = -K + \sqrt{K^2 + c_2^2}$, $K = c_2 c_3$, where V_1, V_2, V_3, V_4 are mutually orthogonal vectors satisfying the equations $g(V_1,V_1) = -g(V_2,V_2) = \frac{\lambda_2^2}{\lambda_1^2+\lambda_2^2}$, $g(V_3,V_3) = g(V_4,V_4) = \frac{\lambda_1^2}{\lambda_1^2+\lambda_2^2}$.

Proof. (i) First assume that α has $k_1 = 1$, $k_2 = c_2$, $k_3 = 0$. Then by using the Frenet equations we find $\overset{\cdots\cdot}{T} - c_2^2 T = 0$. Solving the previous equation, we obtain that

$$T = V_1\cosh(\sqrt{c_2}s) + V_2\sinh(\sqrt{c_2}s) + V_3\cos(\sqrt{c_2}s) + V_4\sin(\sqrt{c_2}s),$$

where V_1, V_2, V_3, $V_4 \in E_1^4$ are constant vectors. Further, the equation $g(T,T) = 1$ implies that $g(V_1,V_1) = -g(V_2,V_2)$, $g(V_3,V_3) = g(V_4,V_4)$, $g(V_1,V_1)+g(V_3,V_3) = 0$, $g(V_i,V_j) = 0$ for $i \neq j$, $(i,j \in \{1,2,3,4\})$. Finally, by using the equation $g(\dot{T},\dot{T}) = 0$, we find $g(V_3,V_3) = \frac{1}{2}$. Consequently, we may take $V_1 = (0, \frac{1}{\sqrt{2}}, 0, 0)$, $V_2 = (\frac{1}{\sqrt{2}}, 0, 0, 0)$, $V_3 = (0, 0, \frac{1}{\sqrt{2}}, 0)$, $V_4 = (0, 0, 0, \frac{1}{\sqrt{2}})$. Accordingly, up to isometries of E_1^4 the curve α has the form (3.7).

Conversely, if α can be parameterized by (3.7), then we find $g(\ddot{\alpha},\ddot{\alpha}) = 0$ and therefore $k_1 = 1$. Next, we find $g(\dddot{\alpha},\dddot{\alpha}) = g(\dot{N},\dot{N}) = c_2^2$. However, the Frenet formulas give $g(\dot{N},\dot{N}) = k_2^2$. It follows that $k_2 = c_2$. Finally, the Frenet equations imply $g(\dot{B}_1,\dot{B}_1) = -2k_2k_3$ and on the other hand since $B_1 = \frac{\dddot{\alpha}}{||\dddot{\alpha}||}$, we obtain that $g(\dot{B}_1,\dot{B}_1) = 0$. Therefore, $k_3 = 0$.

(ii) Suppose that α has constant curvatures $k_1 = 1$, $k_2 = c_2$, $k_3 = c_3$. Then by the Frenet formulas we find $\dddot{T} - 2c_2c_3\ddot{T} - c_2^2 T = 0$. Solving this equation, we obtain

$$T = V_1 \cosh(\lambda_1 s) + V_2 \sinh(\lambda_1 s) + V_3 \cos(\lambda_2 s) + V_4 \sin(\lambda_2 s),$$

where V_1, V_2, V_3, $V_4 \in E_1^4$ are constant vectors, $\lambda_1^2 = K + \sqrt{K^2 + c_2^2}$, $\lambda_2^2 = -K + \sqrt{K^2 + c_2^2}$, $K = c_2c_3$. Next, the equation $g(T,T) = 1$ implies that $g(V_1,V_1) = -g(V_2,V_2)$, $g(V_3,V_3) = g(V_4,V_4)$, $g(V_1,V_1) + g(V_3,V_3) = 1$, $g(V_i,V_j) \neq 0$ for $i \neq j$ $(i,j \in \{1,2,3,4\})$. Finally, from the equation $g(\dot{T},\dot{T}) = 0$, we get $g(V_1,V_1) = \frac{\lambda_2^2}{\lambda_1^2+\lambda_2^2}$. Consequently, α has the form (3.8).

On the other hand, if α can be parameterized by (3.8), then we find that $g(\ddot{\alpha},\ddot{\alpha}) = 0$ and thus $k_1 = 1$. Further, we find that $g(\dddot{\alpha},\dddot{\alpha}) = c_2^2$ and from the Frenet formulas we get $g(\dot{N},\dot{N}) = k_2^2$. It follows that $k_2 = c_2$. Finally, since $B_1 = \frac{\dddot{\alpha}}{||\dddot{\alpha}||}$, we obtain $g(\dot{B}_1,\dot{B}_1) = -c_2c_3$. However, the Frenet equations imply $g(\dot{B}_1,\dot{B}_1) = -k_2k_3$ and consequently $k_3 = c_3$. □

Remark 3.6. The curve (3.7) lies on a light cone in E_1^4 with the equation $-x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$. The curve (3.8) lies on some hyperquadric in E_1^4. If $c_2c_3 > 0$ or $c_2c_3 < 0$, then respectively α lies on pseudo–Riemannian sphere with the equation $-x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{2c_3}{c_2}$ or on pseudo–hyperbolic space with the same equation.

Finally, note that some of a W–curves are the curves of 2–type. The proof of the following theorem follows immediately from definition of 2–type curves.

Theorem 3.11. *The curves (3.3), (3.4), (3.5) and (3.8) for which λ_1, $\lambda_2 \in N$ are a 2–type curves. The curve (3.7) for which $\sqrt{c_2} \in N$ is a 2–type curve. The curves (3.2) and (3.6) for which respectively $\lambda \in N$, $c_1 \in N$ are a null 2–type curves.*

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