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MINIMAL AND MAXIMAL DESCRIPTION FOR THE REAL INTERPOLATION METHODS IN THE CASE OF QUASI-BANACH TRIPLES

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Abstract. In this note we give a minimal and a maximal description in the sense of Aronszajn-Gagliardo for the real methods in the case of quasi-Banach triples.

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0. Introduction

Our main reference to the theory of interpolation spaces is [7]. Let $\overline{A} = (A_0, A_1, A_2)$ be a (quasi)-Banach triple and $\overline{t} = (t_1, t_2) \in \mathbf{R}_+^2$. The Peetre's K-functional is defined for $a \in A_0 + A_1 + A_2 := \sum(\overline{A})$ by

$$K(t_1, t_2, a; \overline{A}) = \inf_{a=a_0+a_1+a_2} \Big(||a_0||_{A_0} + t_1||a_1||_{A_1} + t_2||a_2||_{A_2} \Big)$$

and similarly the J-functional for $a \in A_0 \cap A_1 \cap A_2 : = \Delta(\overline{A})$ by

$$J(t_1, t_2, a; \overline{A}) = \max \Big(||a||_{A_0}, t_1||a||_{A_1}, t_2||a||_{A_2} \Big).$$

Let $\overline{A} = (A_0, A_1, A_2)$ be a triple of quasi-Banach spaces and $\overline{n} = (n_1, n_2) \in \mathbf{Z}^2$. For $0 < \theta_1, \theta_2 < 1,\ \theta_1 + \theta_2 < 1$ and $0 < q \leq \infty$ we define the real interpolation space $\overline{A}_{(\theta_1,\theta_2),q,K}$ as the set of all $a \in \Sigma(\overline{A})$ which have a finite quasi-norm

$$||a||_{(\theta_1,\theta_2),q,K} = \begin{cases} \left(\sum\limits_{\overline{n}\in\mathbf{Z}^2} \Big(2^{-n_1\theta_1} 2^{-n_2\theta_2} K(2^{n_1}, 2^{n_2}, a; \overline{A}\ ) \Big)^q \right)^{1/q} & \text{if } 0 < q < \infty \\ \sup\limits_{\overline{n}\in\mathbf{Z}^2} \Big\{ 2^{-n_1\theta_1} 2^{-n_2\theta_2} K(2^{n_1}, 2^{n_2}, a; \overline{A}\ ) \Big\} & \text{if } q = \infty \\ , \end{cases}$$

Also we define the real interpolation space $\overline{A}_{(\theta_1,\theta_2),q,J}$ as the set of all $a \in \sum(\overline{A})$ that may be written as $a = \sum\limits_{\overline{n}\in\mathbf{z}^2} u_{\overline{n}},\ u_{\overline{n}} \in \Delta(\overline{A})$ (convergence in $\sum(\overline{A})$) and which have a finite quasi-norm

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$$||a||_{(\theta_1,\theta_2),\ q,\ J} = \inf_{a=\sum u_{\overline{n}}} \left(\sum_{\overline{n}\in\mathbf{Z}^2} \left(2^{-n_1\theta_1} 2^{-n_2\theta_2} J(2^{n_1}, 2^{n_2}, u_{\overline{n}}, \overline{A})\right)^q \right)^{1/q}$$

with the usual interpretation when $q=\infty$.

If $\overline{A} = (A_0, A_1, A_2)$ and $\overline{B} = (B_0, B_1, B_2)$ are Banach triples, we write $T \in \mathcal{L}(\overline{A}, \overline{B})$ to mean that T is a linear operator from $\sum(\overline{A})$ into $\sum(\overline{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j $(j = 0, 1, 2)$. We put

$$||\ T\ ||_{\overline{A},\ \overline{B}} = \max_{j=0,1,2} \{||\ T\ ||_{A_j,\ B_j}\}.$$

Scalar sequence spaces are defined over $\mathbf{Z}^2$ and given any sequence of positive numbers $(w_{\overline{n}})_{\overline{n}\in\mathbf{Z}^2}$ we put

$$l_p(w_{\overline{n}}) = \left\{(a_{\overline{n}}) : ||a_{\overline{n}}||_{l_{p(w_{\overline{n}})}} := ||(w_{\overline{n}} a_{\overline{n}})||_{l_p} < \infty\right\}.$$

Of special interest for us are the triples $\bar{l}_p = \left(l_p, l_p(2^{-n_1}), l_p(2^{-n_2})\right), (0 < p \le 1)$ and $\bar{l}_\infty = \left(l_\infty, l_\infty(2^{-n_1}), l_\infty(2^{-n_2})\right)$.

A maximal description in sense of Aronszajn-Gagliardo [1] for the real method in the case of quasi-Banach couples is given in [2].

In this note we will establish a minimal and a maximal description for the real methods in the case of quasi-Banach triples.

1. Minimal description

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Let $\overline{A} = (A_0, A_1, A_2)$ be a triple of quasi-Banach spaces. Recall that a quasi-norm $||\cdot||$ is said to be a p-norm $(0 < p \le 1)$ if

$$||a+b||^p \le ||a||^p + ||b||^p.$$

Given any quasi-normed space $(A, ||\cdot||)$ the functional

$$|||a||| = \inf \left\{ \left(\sum_{k=1}^n ||a_k||^p \right)^{1/p} : a = \sum_{k=1}^n a_k,\ k \ge 1 \right\}$$

defines a p-norm equivalent to $||\cdot||$. Here p is defined by the equation $(2c)^p = 2$, where c is the constant in the triangle inequality $||\cdot||$ (see, for example [4]). Note also that if $||\cdot||$ is a p-norm then it is also an r-norm for any $0 < r \le p$. Consequently, without loss of generality we may and do work with p-Banach spaces.

Definition 1.1. *Let* $0 < \theta_1, \theta_2 < 1, \theta_1 + \theta_2 < 1$ *and* $0 < q \leq \infty$. *Assume that* $\overline{A} = (A_0, A_1, A_2)$ *is a triple of p-Banach spaces* $(0 < p \leq 1)$. *Put* $r = \min(p, q)$ *and define* $G_{(\theta_1,\theta_2),\ q,\ r}(\overline{A})$ *as the collection of all those* $a \in \sum(\overline{A})$ *which can be represented as a convergent series* $a = \sum_{j=1}^{\infty} T_j\ a_j$ *in* $\sum(\overline{A})$ *with* $a_j \in l_q(2^{-n_1\theta_1 - n_2\theta_2})$, $T_j \in \mathcal{L}\Big((l_p, l_p(2^{-n_1}), l_p(2^{-n_2})), (A_0, A_1, A_2)\Big)$ *and*

$$\left(\sum_{j=1}^{\infty} ||T_j||^r_{\bar{l}_p,\overline{A}}\ ||a_j||^r_{l_q}(2^{-n_1\theta_1 - n_2\theta_2})\right)^{1/r} < \infty.$$

This spaces become an r-Banach spaces endowed with the functional

$$||a||_{G(\theta_1,\theta_2),q,r}(\overline{A}) = \inf\left\{\left(\sum_{j=1}^{\infty}\ ||T_j||^r_{\bar{l}_p,\overline{A}}\ ||a_j||^r_{l_q(2^{-n_1\theta_1 - n_2\theta_2})}\right) : a = \sum_{j=1}^{\infty} T_j\ a_j\right\}$$

Theorem 1.2. *Let* $\overline{A} = (A_0, A_1, A_2)$ *be a triple of p-Banach spaces, let* $0 < \theta_1, \theta_2 < 1,\ \theta_1 + \theta_2 < 1$ *and* $0 < q \leq \infty$. *Put* $r = \min(p, q)$. *Then*

$$(A_0, A_1, A_2)_{(\theta_1,\theta_2),q,J} = G_{(\theta_1,\theta_2),q,r}(A_0, A_1, A_2)$$

Proof. Let $a \in (A_0, A_1, A_2)_{(\theta_1,\theta_2),q,J}$. Then $a = \sum_{\overline{n}\in\mathbf{Z}^2} u_{\overline{n}}$ (converge in $\sum(\overline{A})$) and $\sum_{\overline{n}\in\mathbf{Z}^2}\left[2^{-n_1\theta_1 - n_2\theta_2}\ J(2^{n_1}, 2^{n_2},\ u_{\overline{n}}; \overline{A})\right]^q < \infty$. Let T be the operator defined by

$$T\left((b_{\overline{n}})^2_{\overline{n}}\right) = \sum_{\overline{n}\in\mathbf{Z}^2} u_{\overline{n}}\ \frac{b_{\overline{n}}}{J(2^{n_1}, 2^{n_2}, u_{\overline{n}}, \overline{A})}.$$

Since A_0 is a p-Banach space, we have

$$||T\left((b_{\overline{n}})_{\overline{n}}\right)||_{A_0} \leq \left(\sum_{\overline{n}\in\mathbf{Z}^2}\Bigg|\Bigg|\ u_{\overline{n}}\ \frac{b_{\overline{n}}}{J(2^{n_1}, 2^{n_2}, u_{\overline{n}}, \overline{A})}\ \Bigg|\Bigg|^p_{A_0}\right)^{1/p} \leq \left(\sum_{\overline{n}\in\mathbf{Z}^2}\Big|b_{\overline{n}}\Big|^p\right)^{1/p} = ||b\,||_{l_p}.$$

Similarly

$$||T\left((b_{\overline{n}})_{\overline{n}}\right)||_{A_1} \leq ||(b_{\overline{n}})||_{l_{p(2^{-n_1})}}\ ,\ ||T\left((b_{\overline{n}})_{\overline{n}}\right)||_{A_2} \leq ||(b_{\overline{n}})_{\overline{n}}||_{l_p(2^{-n_2})}.$$

Thus

$$T \in \mathcal{L}(\bar{l}_p, \overline{A}) \quad \text{and} \quad ||T||_{\bar{l}_p,\overline{A}} \leq 1.$$

It follows from

$$T\Big(\left(J\left(2^{n_1}, 2^{n_2}, u_{\overline{n}}; \overline{A}\right)\right)_{\overline{n}}\Big) = a \quad \text{and}$$

$$||T||_{\bar{l}_p,\overline{A}}\Big|\Big|\left(J\left(2^{n_1}, 2^{n_2}, u_{\overline{n}}; \overline{A}\right)\right)\Big|\Big|_{l_q\left(2^{-n_1\theta_1 - n_2\theta_2}\right)} \leq$$

$$\leq \left(\sum_{\overline{n}\in\mathbf{Z}^2} \left(2^{-n_1\theta_1-n_2\theta_2} J\left(2^{n_1},2^{n_2},u_{\overline{n}};\overline{A}\right)\right)^q \right)^{1/q}$$

that

$$a \in G_{(\theta_1,\theta_2),q,r}(\overline{A}) \quad \text{with} \quad ||a||_{G_{(\theta_1,\theta_2),q,r}}(\overline{A}) \leq ||a||_{(\theta_1,\theta_2),q,J}.$$

Conversely, let a be in $G_{(\theta_1,\theta_2),q,r}(\overline{A})$ and have the form $a = T(\xi)$, where $T \in \mathcal{L}(\bar{l}_p, \overline{A})$ and $\xi = (\xi_{\overline{n}})_{\overline{n}\in\mathbf{Z}^2} \in l_q\left(2^{-\theta_1 n_1-\theta_2 n_2}\right)$.

Denote $T(e_{\overline{n}})$ by $v_{\overline{n}}$, where $e_{\overline{n}}$ are standard basis vectors of l_p. Then $||v_{\overline{n}}||_{A_0} \leq$ $\leq ||T||_{\bar{l}_p,\overline{A}}$, $2^{n_1}||v_{\overline{n}}||_{A_1} \leq ||T||_{\bar{l}_p,\overline{A}}$, $2^{n_2}||v_{\overline{n}}||_{A_2} \leq ||T||_{\bar{l}_p,\overline{A}}$ and $\sum\limits_{\overline{n}\in\mathbf{Z}^2} \xi_{\overline{n}} v_{\overline{n}} = a$ (convergence in $\sum(\overline{A})$).

If we put now $\xi_{\overline{n}} v_{\overline{n}} = u_{\overline{n}} \in \Delta(\overline{A})$, then $\sum\limits_{\overline{n}\in\mathbf{Z}^2} u_{\overline{n}} = a$ and

$$J(2^{n_1},2^{n_2},u_{\overline{n}};\overline{A}) \leq |\xi_{\overline{n}}| \; ||T||_{\bar{l}_p,\overline{A}}.$$

Hence $a = \sum\limits_{\overline{n}\in\mathbf{Z}^2} u_{\overline{n}}$ is a J-representation of a with

$$||a||_{\theta_1,\theta_2,q,J} \leq \left(\sum_{\overline{n}\in\mathbf{Z}^2} \left(2^{-n_1\theta_1-n_2\theta_2} J\left(2^{n_1},2^{n_2},u_{\overline{n}};\overline{A}\right)\right)^q \right)^{1/q} \leq$$

$$\leq ||T||_{\bar{l}_p,\ \overline{A}} \left(\sum_{\overline{n}\in\mathbf{Z}^2} \left(2^{-n_1\theta_1-n_2\theta_2}\,|\xi_{\overline{n}}|\right)^q \right)^{1/q} = ||T||_{\bar{l}_p,\ \overline{A}}\; ||\xi||_{l_q(2^{-n_1\theta_1-n_2\theta_2})}.$$

If a is now any element of $G_{(\theta_1,\theta_2),q,r}(\overline{A})$ and $a = \sum\limits_{j=1}^{\infty} T_j\xi_j$ is an arbitrary representation of a with

$$\left(\sum_{j=1}^{\infty} ||T_j||^r_{\bar{l}_p,\ \overline{A}}\; ||\xi_j||^r_{l_q(2^{-n_1\theta_1-n_2\theta_2})} \right)^{1/r} < \infty$$

then, using that $(A_0,A_1,A_2)_{\theta,q,J}$ is r-normed we obtain that

$$||a||_{(\theta_1,\theta_2),q,J} \leq \left(\sum_{j=1}^{\infty} \left|\left|T_j\ \xi_j\right|\right|^r_{(\theta_1,\theta_2),\ q,\ J} \right)^{1/r} \leq$$

$$\leq \left(\sum_{j=1}^{\infty} \left|\left|T_j\right|\right|^r_{\bar{l}_p,\ \overline{A}} \left|\left|\xi_j\right|\right|^r_{l_q(2^{-n_1\theta_1-n_2\theta_2})} \right)^{1/r}.$$

Consequently, $a \in (A_0,A_1,A_2)_{(\theta_1,\theta_2),\ q,\ J}$ and $||a||_{(\theta_1,\theta_2),\ q,\ J} \leq ||a||_{(\theta_1,\theta_2),\ q,\ r}$.
This completes the proof. □

2. Maximal description

In this case quasi-linear operators are needed.
Let T be a mapping from a quasi-Banach space A into a scalar sequence space $\mathcal{M}$. We say that T is quasi-linear with constant $C \geq 1$ if

$$\Big| T(a+b) \Big| \leq C\Big(\Big| Ta \Big| + \Big| Tb \Big| \Big), \ a,b \in A$$

$$\Big| T(\lambda a) \Big| = \Big| \lambda \Big| \Big| Ta \Big| \quad a \in A, \ \lambda \in \underline{K} \quad (\underline{K}\text{-scalar field}).$$

Given any quasi-Banach triple $\overline{A} = (A_0, A_1, A_2)$ and $C \geq 1$ we denote by $\mathcal{L}_C(\overline{A}, \ \bar{l}_\infty)$ the collection of all those quasi-linear operators $T : \sum(\overline{A}) \longrightarrow \sum(\bar{l}_\infty)$ with the constant C whose restriction to A_i $(i = 0,1,2)$ defines a bounded operator from A_0, A_1, A_2 into $l_\infty, l_\infty(2^{-n_1}), l_\infty(2^{-n_2})$ respectively.

Definition 2.1 *Let* $0 < \theta_1, \theta_2 < 1, \ \theta_1 + \theta_2 < 1$ *and* $0 < q \leq \infty$. *Given any quasi-Banach triple* $\overline{A} = (A_0, A_1, A_2)$ *we define* $H_{(\theta_1,\theta_2), \ q, \ C}(\overline{A})$ *as the collection of all those* $a \in \sum(\overline{A})$ *such that* $Ta \in l_q(2^{-n_1\theta_1 - n_2\theta_2})$ *for any* $T \in \mathcal{L}_C(\overline{A}, \ \bar{l}_\infty)$ *and quasi-norm*

$$||a||_{H_{(\theta_1, \ \theta_2), \ q, \ C}(\overline{A})} = \sup\Big\{ \ || \ Ta \ ||_{l_q(2^{-n_1\theta_1 - n_2\theta_2})} : \ || \ T \ ||_{\overline{A}, \bar{l}_\infty} \leq 1 \Big\}$$

is finite.

Theorem 2.2. *Let* $\overline{A} = (A_0, A_1, A_2)$ *be a quasi-Banach triple, let* $0 < \theta_1, \theta_2 < 1, \ \theta_1 + \theta_2 < 1$ *and* $0 < q \leq \infty$. *Assume that the constant in the triangle inequality of* A_i *is* $C_i (i = 0,1,2)$ *and put* $C = \max(C_0, C_1, C_2)$. *Then*

$$(A_0, A_1, A_2)_{(\theta_1, \ \theta_2), \ q, \ K} = H_{(\theta_1, \ \theta_2), \ q, \ C}(A_0, A_1, A_2).$$

Proof. Let $\overline{n} = (n_1, n_2) \in \mathbf{Z}^2$ and $a, b \in \sum(\overline{A})$. Given any decompositions $a = a_0 + a_1 + a_2, \ b = b_0 + b_1 + b_2$, with $a_i, b_i \in A_i$ $(i = 0,1,2)$, it follows from

$$\begin{aligned} K(2^{n_1}, 2^{n_2}, a+b, \overline{A}) \leq \ & || \ a_0 + b_0 \ ||_{A_0} + 2^{n_1} || \ a_1 + b_1 ||_{A_1} + 2^{n_2} || \ a_2 + b_2 \ ||_{A_2} \\ \leq \ & C \left[(|| \ a_0 \ ||_{A_0} + 2^{n_1} || \ a_1 ||_{A_1} + 2^{n_2} || \ a_2 \ ||_{A_2}) + (|| \ b_0 \ ||_{A_0} \right. \\ & \left. + (2^{n_1} || \ b_1 ||_{A_1} + 2^{n_2} || \ b_2 \ ||_{A_2}) \right] \end{aligned}$$

that

$$K(2^{n_1}, 2^{n_2}, a+b, \overline{A}) \leq C[K(2^{n_1}, 2^{n_2}, a, \overline{A}) + K(2^{n_1}, 2^{n_2}, b, \overline{A})].$$

Let T be the operator defined by

$$Ta = \big(K(2^{n_1}, 2^{n_2}, a, \overline{A})\big)_{\overline{n} \in \mathbf{Z}^2}.$$

Then $T = \mathcal{L}_C(\overline{A},\ \bar{l}_\infty)$. Moreover, $||\ Ta\ ||_{l_\infty} \leq ||\ a\ ||_{A_0},\ a \in A_0,\ ||\ Ta\ ||_{l_\infty(2^{-n_1})} \leq$

$\leq ||\ a\ ||_{A_1},\ a \in A_1$, and $||\ Ta\ ||_{l_\infty(2^{-n_2})} \leq ||\ a\ ||_{A_2},\ a \in A_2$. So $||\ T\ ||_{\overline{A},\ l_\infty} \leq 1$. Now,

for any $a \in H_{((\theta_1,\ \theta_2),\ q,\ C}(\overline{A})$ we have

$$||\ a\ ||_{(\theta_1,\ \theta_2),\ q,\ K} = \left|\left| \left(K\left(2^{n_1}, 2^{n_2}, a, \overline{A}\ \right)\right)_{\overline{n}}\ \right|\right|_{l_q(2^{-n_1\theta_1 - n_2\theta_2})} =$$

$$= ||\ Ta\ ||_{l_q(2^{-n_1\theta_1 - n_2\theta_2})} \leq ||\ a\ ||_{H_{(\theta_1,\ \theta_2),\ q,\ C}(\overline{A})}.$$

This shows the embedding $H_{(\theta_1,\ \theta_2),\ q,\ C}(\overline{A}) \hookrightarrow \overline{A}_{(\theta_1,\ \theta_2),\ q,\ K}$.

Conversely, given any $T \in \mathcal{L}_C(\overline{A},\ \bar{l}_\infty)$ with $||T||_{\overline{A},\ \bar{l}_\infty} \leq 1$, we can represent Ta as

$(T_{\overline{n}}a)_{\overline{n}\in\mathbf{Z}^2}$, where $T_{\overline{n}} \in \mathcal{L}_C\left((A_0, A_1, A_2), (\underline{K}, \underline{K}, \underline{K})\right)$ with $||T_{\overline{n}}||_{A_0, \underline{K}} \leq ||T||_{A_0,\ l_\infty} \leq 1$,

$||T_{\overline{n}}\ \ ||_{A_1,\ 2^{-n_1}\ \underline{K}} \leq ||T||_{A_1,\ l_\infty(2^{-n_1})} \leq 1$ and $||T_{\overline{n}}\ \ ||_{A_2,\ 2^{-n_2}\ \underline{K}} \leq ||T||_{A_2,\ l_\infty(2^{-n_2})} \leq 1$.

If $a \in \sum(\overline{A})$ and $a = a_0 + a_1 + a_2$ with $a_i \in A_i$, then we get

$$\begin{aligned} |T_{\overline{n}}a\ | \leq &\ C^2\left(|T_{\overline{n}}a_0\ | + |T_{\overline{n}}a_1\ | + |T_{\overline{n}}a_2\ |\right) \\ \leq &\ C^2\left(\ ||\ a_0\ ||_{A_0} + 2^{n_1}||\ a_1\ ||_{A_1} + 2^{n_2}||\ a_2\ ||_{A_2}\ \right) \end{aligned}$$

Whence

$$|T_{\overline{n}}a\ | \leq C^2 K\left(2^{n_1}, 2^{n_2}, a;\ \overline{A}\right).$$

Now, for any $a \in \overline{A}_{(\theta_1,\ \theta_2),\ q,\ K}$ we obtain

$$\left|\left|\ Ta\ \right|\right|_{l_q(2^{-n_1\theta_1 - n_2\theta_2})} = \left(\sum_{\overline{n}\in\mathbf{Z^2}} \left(2^{-n_1\theta_1 - n_2\theta_2}\ \left|T_{\overline{n}}a\ \right|\right)^q\right)^{1/q} \leq$$

$$\leq C^2\left(\sum_{\overline{n}\in\mathbf{Z^2}} \left(2^{-n_1\theta_1 - n_2\theta_2} K\left(2^{n_1}, 2^{n_2}, a, \overline{A}\ \right)\right)^q\right)^{1/q} = C^2||\ a\ ||_{(\theta_1,\ \theta_2),\ q,\ K}.$$

This shows the embedding $\overline{A}_{(\theta_1,\ \theta_2),\ q,\ K} \hookrightarrow H_{(\theta_1,\ \theta_2),\ q,\ C}(\overline{A})$ and completes the proof. □

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