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SPECIAL MORPHISMS OF GROUPOIDS

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Abstract. The aim of this paper is to give the most important algebraic properties of special morphisms in the category of groupoids. Also some various topics of their are established.

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Introduction

Groupoids were introduced and named by H. Brandt ([2]) in 1926, in a paper on the composition of quadratic forms in four variables. A groupoid is, roughly speaking, a set with a not everywhere defined binary operation which would a group if the operation were defined everywhere.

There are various equivalent definitions for groupoids (see [6], [10],[11], [14]) and various ways of thinking of them.

In 1950, in his paper ([4]) on connections, C. Ehresmann added further structures (topological and differentiable) to groupoids, thereby introducing them as a tool in differentiable topology and geometry.

The differentiable groupoids endowed with supplementary structures (for example: Lie groupoids, symplectic groupoids, contact groupoids, Riemannian groupoids, measure groupoids) has used by C. Albert,P. Dazord, M. V. Karasev, P. Libermann, K. Mackenzie, G. W. Mackey, J. Pradines, J. Renault, A. Weinstein, in a series of papers for applications to differential topology and geometry, symplectic geometry, Poisson geometry, quantum mechanics, quantization theory, ergodic theory.

1. The category of groupoids

The purpose of this section is to construct the category of groupoids and give several properties characterizing them.

Definition 1.1. *([11]) A* **groupoid** Γ **over** Γ_0 *or* **groupoid with the base** Γ_0 *is a 7-tuple* $(\Gamma, \alpha, \beta, \epsilon, i, \mu; \Gamma_0)$ *formed by: the sets* Γ *and* Γ_0*, the surjections* $\alpha, \beta : \Gamma \longrightarrow \Gamma_0$*, called respectively the* **source** *and the* **target** *map, an*

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injection $\epsilon : \Gamma_0 \longrightarrow \Gamma, u \longrightarrow \epsilon(u) = \widetilde{u}$, *called the* **inclusion map**, *a map* $i : \Gamma \longrightarrow \Gamma, \quad x \longrightarrow i(x) = x^{-1}$, *called the* **inversion map** *and a* **(partial) composition law** $\mu : \Gamma_{(2)} \longrightarrow \Gamma, (x, y) \longrightarrow \mu(x, y) = x \cdot y = xy$, *with the domain* $\Gamma_{(2)} = \{(x, y) \in \Gamma \times \Gamma \quad | \quad \beta(x) = \alpha(y)\}$, *such that the following axioms are satisfied:*

*(G1) (***associative law***) For arbitrary* $x, y, z \in \Gamma$ *the triple product* $(xy)z$ *is defined iff* $x(yz)$ *is defined. In case either is defined, we have* $\quad (xy)z = x(yz)$

*(G2) (***identities***) For each* $x \in \Gamma$ *we have* $(\epsilon(\alpha(x)), x) \in \Gamma_{(2)}; (x, \epsilon(\beta(x))) \in \Gamma_{(2)}$ *and* $\quad \epsilon(\alpha(x)) \cdot x = x \cdot \epsilon(\beta(x)) = x$

*(G3) (***inverses***) For each* $x \in \Gamma$ *we have* $(x, i(x)) \in \Gamma_{(2)}; (i(x), x) \in \Gamma_{(2)}$ *and* $\quad x \cdot i(x) = \epsilon(\alpha(x)), \quad i(x) \cdot x = \epsilon(\beta(x))$. .

We will denote sometimes a groupoid Γ over Γ_0 by $(\Gamma, \alpha, \beta; \Gamma_0)$ or $(\Gamma; \Gamma_0)$.

A groupoid Γ over Γ_0 such that Γ_0 is a subset of Γ is called Γ_0-**groupoid** or **Brandt groupoid**.

For each $u \in \Gamma_0$, the set $\Gamma_u = \alpha^{-1}(u)$ (resp. $\Gamma^u = \beta^{-1}(u)$), called the α-**fibre** (resp. β-**fibre**) of Γ over $u \in \Gamma_0$ and if $u, v \in \Gamma$, we will write $\Gamma_u^v = \Gamma_u \cap \Gamma^v$.

Definition 1.2 *A groupoid* Γ *over* Γ_0 *is said to be* **transitive** *if the map* $\alpha \times \beta : \Gamma \longrightarrow \Gamma_0 \times \Gamma_0$, *given by* $(\alpha \times \beta)(x) = (\alpha(x), \beta(x)), (\forall) x \in \Gamma$ *is surjective.*

We summarize some properties of groupoids obtained from the definitions.

Proposition 1.1 *Let* Γ *be a groupoid over* Γ_0. *The following assertions hold:*

(i) For all $x \in \Gamma$ *we have* $\beta(x^{-1}) = \alpha(x)$ *and* $\alpha(x^{-1}) = \beta(x)$

(ii) If $(x, y) \in \Gamma_{(2)}$ *then* $x^{-1}(xy) = y$ *and* $(xy)y^{-1} = x$.

(iii) If $(x, y) \in \Gamma_{(2)}$ *then* $\alpha(xy) = \alpha(x)$ *and* $\beta(xy) = \beta(y)$.

*(iv) (***cancellation law***) If* $xz_1 = xz_2$ *(resp.,* $z_1x = z_2x$*) then* $z_1 = z_2$.

(v) If $(x, y) \in \Gamma_{(2)}$ *then* $(y^{-1}, x^{-1}) \in \Gamma_{(2)}$ *and* $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

(vi) $(x^{-1})^{-1} = x, \quad (\forall) \quad x \in \Gamma$.

(vii) $\alpha(\epsilon(u)) = u$ *and* $\beta(\epsilon(u)) = u, \quad (\forall) \ u \in \Gamma_0$.

(viii) $\epsilon(u) \cdot \epsilon(u) = \epsilon(u)$ *and* $(\epsilon(u))^{-1} = \epsilon(u)$ *for each* $u \in \Gamma_0$.

Proof. (i) This assertion follows from the axiom (G.3). For example , for each $x \in \Gamma$ we have $(x^{-1}, x) \in \Gamma_{(2)}$. Then $\beta(x^{-1}) = \alpha(x)$.

(ii) Let $(x, y) \in \Gamma_{(2)}$. By the axiom (G.2), for $y \in \Gamma$ we have $\epsilon(\alpha(y))y = y$. Hence $\epsilon(\beta(x))y = y$, since $\beta(x) = \alpha(y)$. But, using (G.3), we have that $\epsilon(\beta(x)) = x^{-1}x$ and we obtain $(x^{-1}x)y = y$, i.e. the triple product $(x^{-1}x)y$ is defined. Applying now (G.1), imply that the triple $x^{-1}(xy)$ is defined and we have $(x^{-1}x)y = x^{-1}(xy)$. Therefore, $x^{-1}(xy) = y$.

(iii) Let $(x, y) \in \Gamma_{(2)}$. Then $x^{-1}(xy) = y$, by (ii). Hence, $(x^{-1}, xy) \in \Gamma_{(2)}$ and it follows that $\beta(x^{-1}) = \alpha(xy)$. We have $\alpha(xy) = \alpha(x)$, since $\beta(x^{-1}) = \alpha(x)$, by (i).

(iv) It is a consequence of (ii). Indeed, if $xz_1 = xz_2$, then $x^{-1}(xz_1) = x^{-1}(xz_2)$, i.e. $z_1 = z_2$.

(v) Let $(x, y) \in \Gamma_{(2)}$. By (ii), $\beta(y^{-1}) = \alpha(y)$ and $\alpha(x^{-1}) = \beta(x)$. Then, $\beta(y^{-1}) = \alpha(x^{-1})$, since $\beta(x) = \alpha(y)$. Hence , $(y^{-1}, x^{-1}) \in \Gamma_{(2)}$.

We have $(xy)(xy)^{-1} = \epsilon(\alpha(xy)) = \epsilon(\alpha(x))$, and $(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = x\epsilon(\alpha(y))x^{-1} = x\epsilon(\beta(x))x^{-1} = xx^{-1} = \epsilon(\alpha(x))$. Hence $(xy)(xy)^{-1} = (xy)(y^{-1}x^{-1})$ and by (iv) we obtain $(xy)^{-1} = y^{-1}x^{-1}$.

(vi) Applying (G.3) for the elements $x, x^{-1} \in \Gamma$ and (i), we have $x^{-1}x = \epsilon(\beta(x))$ and $x^{-1}(x^{-1})^{-1} = \epsilon(\alpha(x^{-1})) = \epsilon(\beta(x))$. Hence $x^{-1}x = x^{-1}(x^{-1})^{-1}$ and using (iv), we obtain $(x^{-1})^{-1} = x$.

(vii) Let $u \in \Gamma_0$. We denote $\epsilon(u) = x$. From (G.2) follows $\epsilon(\alpha(x))x = x$ and $\epsilon(\alpha(\epsilon(u))x = x$. We have $\epsilon(\alpha(\epsilon(u)))x = \epsilon(\alpha(x))x$, and by (iv), we obtain $\epsilon(\alpha(\epsilon(u))) = \epsilon(\alpha(x))$. Hence, $\alpha(\epsilon(u)) = \alpha(x)$, since ϵ is injective . Then, $\alpha(\epsilon(u)) = u$, since $\alpha(x) = u$.

(viii) Let $u \in \Gamma_0$. We denote $\epsilon(u) = x$. From (G.2) follows $\epsilon(\alpha(x))x = x$, i.e. $\epsilon(\alpha(\epsilon(u)))\epsilon(u) = \epsilon(u)$, i.e. $\epsilon(u)\epsilon(u) = \epsilon(u)$, since $\alpha(\epsilon(u)) = u$, cf. (vii).

We have $\epsilon(u)(\epsilon(u))^{-1} = \epsilon(\alpha(\epsilon(u))) = \epsilon(u)$ and $\epsilon(u)\epsilon(u) = \epsilon(u)$. Hence, $\epsilon(u)(\epsilon(u))^{-1} = \epsilon(u)\epsilon(u)$ and by (iv) we have $(\epsilon(u))^{-1} = \epsilon(u)$.

The element $\epsilon(\alpha(x))$ (resp. $\epsilon(\beta(x))$) is the **left unit** (resp., **right unit**) of $x \in \Gamma$, and the subset $\epsilon(\Gamma_0)$ of Γ is called the **unity set** of Γ.

Applying (i), (vi) and (vii) from Prop. 1.1, respectively , we obtain:

Remark 1.1 *Let* $(\Gamma, \alpha, \beta, \epsilon, i; \Gamma_0)$ *be a groupoid. Then the maps* $\alpha, \beta, \epsilon, i$ *satisfy the following relations :*

(i) $\alpha \circ i = \beta$ *and* $\beta \circ i = \alpha$.

(ii) $i \circ i = Id_\Gamma$.

(iii) $\alpha \circ \epsilon = \beta \circ \epsilon = Id_{\Gamma_0}$.

Proposition 1.2 *Let* Γ *be a groupoid over* Γ_0*. The following assertions hold:*

(i) For each $u \in \Gamma_0$*, the set* $\Gamma(u) = \alpha^{-1}(u) \cap \beta^{-1}(u)$ *is a group under the restriction of the partial multiplication, called the* **isotropy group at** u *of* Γ.

(ii) If $\alpha(x) = u$ *and* $\beta(x) = v$, *then the map* $\omega : \Gamma(u) \to \Gamma(v)$, $z \to \omega(z) = x^{-1}zx$ *is an isomorphism of groups.*

(iii) If Γ *is transitive, then the isotropy groups of* Γ *are isomorphes.*

Proof. (i) For any $x, y \in \Gamma(u)$ we have $\alpha(x) = \beta(x) = \alpha(y) = \beta(y) = u$. Hence ,the product xy is defined. We have $\alpha(xy) = \alpha(x) = u$, $\beta(xy) = \beta(y) = u$ and imply that $xy \in \Gamma(u)$. Therefore, the restriction of the partial multiplication defined on Γ is a binary operation on $\Gamma(u)$.

It is easy to verify that $\Gamma(u)$ is a group. The unity of $\Gamma(u)$ is the element $\epsilon(u)$, since by (G.2) we have $\epsilon(\alpha(x))x = x\epsilon(\beta(x)) = x$, i.e. $\epsilon(u)x = x\epsilon(u) = x$, $(\forall)x \in \Gamma(u)$.

(ii) Let $z \in \Gamma(u)$. Then $\alpha(z) = \beta(z) = u$. Then the map ω is well-defined, since $\alpha(x^{-1}zx) = \alpha(x^{-1}) = \beta(x) = v$ $\beta(x^{-1}zx) = \beta(x) = v$ and hence $x^{-1}zx \in \Gamma(v)$. It is easy to verify that $\omega(z_1 z_2) = \omega(z_1)\omega(z_2)$, $(\forall)$ $z_1, z_2 \in \Gamma(u)$.

Applying the cancellation law we obtain that the map ω is injective. If $y \in \Gamma(v)$ then there exists $z = xyx^{-1} \in \Gamma(u)$ such that $\omega(z) = y$, i.e. the map ω is surjective. Therefore, ω is a bijective morphism of groups. Hence, the groups $\Gamma(u)$ and $\Gamma(v)$ are isomorphes.

(iii) Since Γ is transitive, then the map $(\alpha \times \beta) : \Gamma \to \Gamma_0 \times \Gamma_0$ is surjective, i.e. for any pair $(u, v) \in \Gamma_0 \times \Gamma_0$ there exists $x \in \Gamma$ such that $\alpha(x) = u$ and $\beta(x) = v$. Applying now (ii) we obtain that the isotropy groups $\Gamma(u)$ and $\Gamma(v)$ are isomorphes, for all $u, v \in \Gamma_0$.

Definition 1.3 *By* **group bundle** *we mean a groupoid* Γ *over* Γ_0 *such that* $\alpha(x) = \beta(x)$ *for each* $x \in \Gamma$. *Moreover, a group bundle is the union of its isotropy groups* $\Gamma(u) = \alpha^{-1}(u), u \in \Gamma_0$ *(here two elements may be composed iff they lie in the same fiber.)*

If $(\Gamma, \alpha, \beta; \Gamma_0)$ is a groupoid, then $Is(\Gamma) = \{x \in \Gamma \mid \alpha(x) = \beta(x)\}$ is a group bundle called the **isotropy group bundle** associated to Γ.

Example 1.1. (a) Every group G is a groupoid with the base $G_0 = \{e\}$, where e is the unity of G.

(b) **Nul groupoid.** Any set B is a groupoid on itself with $\Gamma = \Gamma_0 = B$, $\alpha = \beta = \epsilon = id_B$ and every element is a unity. The multiplication is given by $x \cdot x = x$ for all $x \in B$.

(c) **Coarse groupoid associated to a set B.** If B is any non-empty set, then $B \times B$ is a groupoid over B with the rules: $\alpha(x, y) = x; \beta(x, y) = y; \epsilon(x) = (x, x)$, $i(x, y) = (y, x)$ and $\mu((x, y), (y', z)) = (x, z) \iff y = y'$.

For this groupoid, $\epsilon(\Gamma_0)$ is the diagonal Δ_B of the Cartesian product $B \times B$.If $u \in B$, and the isotropy group $\Gamma(u)$ at u is the nul group $\{(u, u)\}$.

(d) **Trivial groupoid on a set** B **with group** $\mathcal{G}$. Let B be any nonempty set and $\mathcal{G}$ be a multiplicative group with e as unity. $\Gamma = B \times B \times \mathcal{G}$ has a structure of transitive groupoid over B, wdere $\Gamma_0 = B$; $\alpha, \beta, \epsilon, i, \mu$ are defined by: $\alpha(a, b, x) = a$; $\beta(a, b, x) = b$; $\epsilon(b) = (b, b, e)$; $i(a, b, x) = (b, a, x^{-1})$ and $\mu((a, b, x), (b', c, y)) = (a, c, xy) \iff b = b'$.
The unit set of this groupoid is $\{(b, b, e) \mid b \in B\}$ and the isotropy group is $\Gamma(b) = \{(b, b, x) \mid x \in \mathcal{G}\}$ which are identified with B resp. $\mathcal{G}$.

(e) **Action groupoid**. Let $G \times B \longrightarrow B$ be an action of the group G on the set B. Give on $G \times B$ a groupoid structure over B in the following way: α is the projection of the second factor of $G \times B$ and β is the action $G \times B \longrightarrow B$ itself, i.e. $\alpha(g, x) = x$, $\beta(g, x) = g \cdot x$, $(\forall) g \in G, x \in B$; $\epsilon(x) = (1, x)$, where 1 is the unity of G ; $i(g, x) = (g^{-1}, g \cdot x)$ and $\mu((g_2, x_2), (g_1, x_1)) = (g_2 g_1, x_1) \iff g_2 \cdot x_2 = x_1$.

(f) If $\{\mathcal{G}_i \mid i \in I\}$ is a disjoint family of groupoids, then the **disjoint union** $\mathcal{G} = \cup_{i \in I} \mathcal{G}_i$ is a groupoid with the base $\mathcal{G}_0 = \cup_{i \in I} \mathcal{G}_{i,0}$. Here, $\mathcal{G}_{(2)} = \cup_{i \in I} \mathcal{G}_{i(2)}$, i.e. two elements $x, y \in \mathcal{G}$ may be composed iff they lie in the same groupoid $\mathcal{G}_i$ and are composable in $\mathcal{G}_i$.

In particular, any disjoint union of groups is a group bundle.

Definition 1.4 *Let* $(\Gamma, \alpha, \beta, \epsilon, i, \mu; \Gamma_0)$ *and* $(\Gamma', \alpha', \beta', \epsilon', i', \mu'; \Gamma'_0)$ *be two groupoids. A* **morphism of groupoids** *or* **groupoid morphism** *is a pair* (f, f_0) *of maps* $f : \Gamma \longrightarrow \Gamma'$ *and* $f_0 : \Gamma_0 \longrightarrow \Gamma'_0$ *such that the following two conditions are satisfied:*
(1) $f(\mu(x, y)) = \mu'(f(x), f(y))$ *for every* $(x, y) \in \Gamma_{(2)}$
(2) $\alpha' \circ f = f_0 \circ \alpha$ *and* $\beta' \circ f = f_0 \circ \beta$.
If $\Gamma_0 = \Gamma'_0$ *and* $f_0 = Id_{\Gamma_0}$, *we say that* f *is a* Γ_0**- morphism**.

Note that the condition (1) ensure that $(f(x), f(y)) \in \Gamma'_{(2)}$, i.e. $\mu'(f(x), f(y))$ is defined whenever $\mu(x, y)$ is defined.

Proposition 1.3 *The groupoids morphisms preserve unities and inverses, i.e.*
(i) $f(\widetilde{u}) = \widetilde{f_0(u)}$, $(\forall)u \in \Gamma_0$,
(ii) $f(x^{-1}) = (f(x))^{-1}$, $(\forall)x \in \Gamma$

Proof. (i) Let $u \in \Gamma_0$ and we denote $x = \epsilon(u)$. By $(G.2)$, we have $(\epsilon(\alpha(x)), x) \in \Gamma_{(2)}$. Then, $(f(\epsilon(\alpha(x))), f(x)) \in \Gamma'_{(2)}$ and $f(\epsilon(\alpha(x))x) = f(\epsilon(\alpha(x)))f(x)$, since f is a groupoid morphism. Hence, $f(\epsilon(\alpha(x)))f(x) = f(x)$. Since $f(x) \in \Gamma'$, we have $\epsilon'(\alpha'(f(x)))f(x) = f(x)$, by the axiom (G.2). Then $f(\epsilon(\alpha(x)))f(x) = \epsilon'(\alpha'(f(x))))f(x)$ and applying the cancellation law we obtain $f(\epsilon(\alpha(x))) = \epsilon'(\alpha'(f(x)))$ i.e. $(f \circ \epsilon)(\alpha(x)) = \epsilon'((\alpha' \circ f)(x))$. But $\alpha' \circ f = f_0 \circ \alpha$. Then $(f \circ \epsilon)(\alpha(x)) = \epsilon'((f_0 \circ \alpha)(x))$, i.e. $(f \circ \epsilon)(u) = \epsilon'(f_0(u))$. Therefore, $f(\widetilde{u}) = \widetilde{f_0(u)}$.

(ii) Let $x \in \Gamma$. By (G.2), we have $(x, x^{-1}) \in \Gamma_{(2)}$ and $xx^{-1} = \epsilon(\alpha(x))$. Then $(f(x), f(x^{-1}) \in \Gamma'_{(2)}$ and $f(xx^{-1}) = f(x)f(x^{-1})$, since f is a groupoid morphism. Hence, $f(\epsilon(\alpha(x))) = f(x)f(x^{-1})$. But, cf.(i), $f(\epsilon(\alpha(x))) = \epsilon'(f_0(\alpha(x)))$, and we obtain $f(x)f(x^{-1}) = \epsilon'(f_0(\alpha(x)))$. On the other hand, since $f(x) \in \Gamma'$ we have $f(x)(f(x))^{-1} = \epsilon'(\alpha'(f(x)))$, and it follows that $f(x)(f(x))^{-1} = \epsilon'(f_0(\alpha(x)))$, since $\alpha' \circ f = f_0 \circ \alpha$. Therefore, $f(x)f(x^{-1}) = f(x)(f(x))^{-1}$, and by the cancellation law we obtain $f(x^{-1}) = (f(x))^{-1}$.

Using the assertions (i) and (ii) from Prop.1.3, respectively, we obtain:

Remark 1.2 *Let* $(\Gamma, \alpha, \beta, \epsilon, i; \Gamma_0)$, $(\Gamma', \alpha', \beta', \epsilon', i'; \Gamma'_0)$ *be two groupoids and a morphism of groupoids* $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$. *Then:*
(i) $f \circ \epsilon = \epsilon' \circ f_0$.
(ii) $f \circ i = i' \circ f$

Proposition 1.4 *A pair* $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ *is a groupoid morphism iff the following condition holds:*
(3) $(\forall)(x, y) \in \Gamma_{(2)} \implies (f(x), f(y)) \in \Gamma'_{(2)}$ *and* $f(\mu(x, y)) = \mu'(f(x), f(y))$

Proof. The condition (3) is a consequence of Definition 1.4. and Prop.1.3.
Conversely, let $f : \Gamma \longrightarrow \Gamma'$ which satisfy (3).

We define the map $f_0 : \Gamma_0 \longrightarrow \Gamma'_0$ by $f_0(u) = \alpha'(f(\epsilon(u))), \quad (\forall)u \in \Gamma_0$.

We prove that $\alpha' \circ f = f_0 \circ \alpha$ and $\beta' \circ f = f_0 \circ \beta$.

Indeed, since $(x, \epsilon(\beta(x))) \in \Gamma_{(2)}$ it follows that $(f(x), f(\epsilon(\beta(x)))) \in \Gamma'_{(2)}$ and $f(x) \cdot f(\epsilon(\beta(x)))) = f(x \cdot \epsilon(\beta(x))) = f(x)$; but $f(x) \cdot \epsilon'(\beta'(f(x))) = f(x)$; $\Longrightarrow \epsilon'(\beta'(f(x))) = f(\epsilon(\beta(x))) \Longrightarrow \alpha'(\epsilon'(\beta'(f(x)))) = \alpha'(f(\epsilon(\beta(x))))$ and applying Prop. 1.1. we obtain successively $\beta'(f(x)) = (f_0 \circ \alpha)(\epsilon(\beta(x))) \Longrightarrow \beta'(f(x)) = f_0(\beta(x))$ i.e. $\beta' \circ f = f_0 \circ \beta$. We also have $\beta'(f(\epsilon(u))) = (\beta' \circ f)(\epsilon(u)) = (f_0 \circ \beta)(\epsilon(u)) = f_0((\beta \circ \epsilon)(u)) = f_0(u)$, since $\beta \circ \epsilon = Id_{\Gamma_0}$. Similarly we prove now that $\alpha' \circ f = f_o \circ \alpha$.

Let $(f, f_0) : (\Gamma; \Gamma_0) \to (\Gamma'; \Gamma'_0)$ be a morphism of groupoids. The set $Ker f = \{x \in \Gamma \mid f(x) \in \epsilon(\Gamma_0)\}$ is called **the kernel** of f.

Definition 1.5 *A groupoid morphism $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is said to be an* **isomorphism** *of groupoids if there exists a groupoid morphism $(g, g_0) : (\Gamma'; \Gamma'_0) \to (\Gamma, \Gamma_0)$ with $(g, g_0)o(f, f_0) = (id_\Gamma, id_{\Gamma_0})$ and $(f, f_0)o(g, g_0) = (id_{\Gamma'}, id_{\Gamma'_0})$. Two groupoids $(\Gamma; \Gamma_0)$ and $(\Gamma'; \Gamma'_0)$ are said to be* **isomorphic** *if there exists an isomorphism $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$.*

Using Proposition 1.3, we obtain:

Proposition 1.5 *A groupoid morphism $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is an isomorphism iff the map $f : \Gamma \longrightarrow \Gamma'$ is bijective.*

Example 1.2. (a) If $(\Gamma, \alpha, \beta, \epsilon; \Gamma_0)$ is a groupoid then the pair of identity maps $(Id_\Gamma, Id_{\Gamma_0})$ is a groupoid morphism.

(b) If $(f, f_0) : (\Gamma, \Gamma_0) \to (\Gamma'; \Gamma'_0)$ and $(g, g_0) : (\Gamma', \Gamma'_0) \to (\Gamma'', \Gamma''_0)$ are groupoid morphisms, then the composition $(g, g_0)o(f, f_0) : (\Gamma, \Gamma_0) \to (\Gamma'', \Gamma''_0)$, defined by $(g, g_0)o(f, f_0) = (gof, g_0of_0)$, is a groupoid morphism.

(c) Let $(\Gamma, \alpha, \beta, \epsilon; \Gamma_0)$ be a groupoid and $(\Gamma_0 \times \Gamma_0, \alpha', \beta', \epsilon'; \Gamma_0)$ the coarse groupoid associated to Γ_0. Then $\alpha \times \beta : \Gamma \longrightarrow \Gamma_0 \times \Gamma_0$, $(\alpha \times \beta)(x) = (\alpha(x), \beta(x))$ is a Γ_0-morphism of the groupoid Γ into the coarse groupoid $\Gamma_0 \times \Gamma_0$.

(d) Let $(\Gamma, \alpha, \beta, \epsilon, i, \mu; \Gamma_0)$ be a groupoid over Γ_0 and X a set with the same cardinal as Γ_0, i.e. there exists a bijection map φ from Γ_0 to X. Then Γ has a canonical structure of a groupoid over X, that is $(\Gamma, \alpha', \beta', \epsilon', i', \mu'; X)$ is a groupoid over X where $\alpha' = \varphi o \alpha; \beta' = \varphi o \beta; \epsilon' = \epsilon o \varphi^{-1}; i' = \varphi o i; \mu' = \mu$. Moreover, $(id_\Gamma, \varphi) : (\Gamma; \Gamma_0) \to (\Gamma; X)$ is an isomorphism of groupoids.

The category of groupoids denoted by $\mathcal{G}$ has as its objects all groupoids $(\Gamma; \Gamma_0)$ and as morphisms from $(\Gamma; \Gamma_0)$ to $(\Gamma'; \Gamma'_0)$ the set of all morphisms of groupoids.

The familly of maps : $\Gamma \longrightarrow Is(\Gamma)$ and $(u : \Gamma \longrightarrow \Gamma') \longrightarrow (\widehat{u} : Is(\Gamma) \longrightarrow Is(\Gamma'))$, where $\widehat{u}$ is the restriction of u on $Is(\Gamma)$, defines a functor $Is : \mathcal{G} \longrightarrow \mathcal{GB}$, called the **isotropy functor**, where $\mathcal{GB}$ is the category of group bundles.

For each set X, the subcategory of groupoids over X, denoted $\mathcal{G}(X)$, has as objects groupoids for which the base has the same cardinal as X and X-morphims of groupoids as its morphisms.

If $(\Gamma,\alpha,\beta,\epsilon;\Gamma_0)$ and $(\Gamma',\alpha',\beta',\epsilon';\Gamma_0')$ are groupoids, then $(\Gamma\times\Gamma',\alpha\times\alpha',\beta\times\beta',\epsilon\times\epsilon';\Gamma_0\times\Gamma_0')$ is a groupoid, called the **direct product** of $(\Gamma;\Gamma_0)$ and $(\Gamma';\Gamma_0')$.

Let be given two groupoids $(\Gamma,\alpha,\beta,\epsilon;\Gamma_0),(\Gamma',\alpha',\beta',\epsilon';\Gamma_0)$ over the same base Γ_0, and the set:

$$\Gamma\oplus\Gamma'=\{(x,x')\in\Gamma\times\Gamma' \mid \quad \alpha(x)=\alpha'(x');\beta(x)=\beta'(x')\}$$

has a natural structure of the groupoid with the following rules:

$$(x,x')\cdot(y,y')=(x\cdot y,x'\cdot y') \quad \Longleftrightarrow \quad (x,y)\in\Gamma_{(2)} \ \ and \ \ (x',y')\in\Gamma'_{(2)}$$

$\alpha_{\oplus}(x,x')=\alpha(x);\beta_{\oplus}(x,x')=\beta(x);\epsilon_{\oplus}:\Gamma_0\to\Gamma\oplus\Gamma',\epsilon_{\oplus}(u)=(\epsilon(u),\epsilon'(u'))$, $i_{\oplus}:\Gamma\oplus\Gamma'\longrightarrow\Gamma\oplus\Gamma'$, $i_{\oplus}(x,x')=(i(x),i'(x'))$. Then $(\Gamma\oplus\Gamma';\Gamma_0)$ is a groupoid over Γ_0. It is the direct product of Γ and Γ' in the category $\mathcal{G}(\Gamma_0)$, called **Whitney's sum** of two groupoids Γ and Γ'.

If $(\Gamma\oplus\Gamma';\Gamma_0)$ is the Whitney sum of the groupoids Γ and Γ' over Γ_0 then the projections maps $p:\Gamma\oplus\Gamma'\to\Gamma$ and $p':\Gamma\oplus\Gamma'\to\Gamma'$, defined by $p(x,x')=x$ and $p'(x,x')=x'$, are Γ_0-morphisms of groupoids.

Theorem 1.1 *The triple $(\Gamma\oplus\Gamma',p,p')$ is the direct product of Γ and Γ' in the category $\mathcal{G}(\Gamma_0)$, i.e. the triple $(\Gamma\oplus\Gamma',p,p')$ verifies the universal property: for all triple (Γ_1,u,u') composed by the groupoid $(\Gamma_1,\widetilde{\alpha},\widetilde{\beta},\widetilde{\epsilon};\Gamma_0)$ and two Γ_0-morphisms of groupoids $\Gamma'\xleftarrow{u'}\Gamma_1\xrightarrow{u}\Gamma$, there exists a unique Γ_0-morphism of groupoids $f:\Gamma_1\to\Gamma\oplus\Gamma'$ such that the following diagram:*

$$\begin{array}{ccccc}\Gamma' & \xleftarrow{p'} & \Gamma\oplus\Gamma' & \xrightarrow{p} & \Gamma\\ & {}^{u'}\nwarrow & \uparrow f & \nearrow u & \\ & & \Gamma_1 & & \end{array}$$

is commutative.

Proof. We define the map $f:\Gamma_1\to\Gamma\oplus\Gamma'$ by taking $f(x)=(u(x),u'(x))$ for all $x\in\Gamma_1$.

We can easily verify that f is a unique Γ_0-morphism of groupoids such that $pof=u$ and $p'of=u'$. .

We have that $\oplus:\mathcal{G}(X)\times\mathcal{G}(X)\to\mathcal{G}(X)$ is a functor, where $\mathcal{G}(X)$ is the category of groupoids over the same base X.

For this, we consider the groupoids $(\Gamma_i,\alpha_i,\beta_i,\epsilon_i;X)$ and $(\Gamma_i',\alpha_i',\beta_i',\epsilon_i';X)$ for $i=1,2$. If $u_i:(\Gamma_i;X)\to(\Gamma_i';X)$ for $i=1,2$ are X-morphisms of groupoids, then $u_1\oplus u_2:(\Gamma_1\oplus\Gamma_2;X)\to(\Gamma_1'\oplus\Gamma_2';X)$ given by the relation: $(u_1\oplus u_2)(x_1,x_2)=(u_1(x_1),u_2(x_2)), \quad \forall \ (x_1,x_2)\in\Gamma_1\oplus\Gamma_2$ is an X-morphism.

Clearly, the relation: $1_{\Gamma_1}\oplus 1_{\Gamma_2}=1_{\Gamma_1\oplus\Gamma_2}$ holds.

If $v_i:(\Gamma_i',X)\to(\Gamma_i'';X)$ for $i=1,2$, then we have:
$(v_1\oplus v_2)o(v_1\oplus u_2)=(v_1ou_1)\oplus(v_2ou_2)$. Consequently, $\oplus$ is a functor.

Proposition 1.6 *The direct product (resp., the Whitney sum) of two transitive groupoids is also a transitive groupoid.*

Proof. Straightforward.

2. The induced groupoid

Let $(\Gamma;\Gamma_0)$ be a groupoid and let $h: X \longrightarrow \Gamma_0$ be a given map. Then the set: $h^*(\Gamma) = \{(x,y,a) \in X \times X \times \Gamma \ \ | \ \ h(x) = \alpha(a), h(y) = \beta(a)\}$ has a canonical structure of groupoid over X with respect to the following rules: $\alpha^*(x,y,a) = x; \beta^*(x,y,a) = y; \epsilon^*(x) = (x,x,\epsilon(h(x))); i^*(x,y,a) = (y,x,i(a))$, and
$\mu^*((x,y,a),(y',z,b)) = (x,z,\mu(a,b))$ iff $y = y'$ and $(a,b) \in \Gamma_{(2)}$.

The groupoid $(h^*(\Gamma), \alpha^*, \beta^*, \epsilon^*, \mu^*; X)$ is called the **induced groupoid** of Γ **under** h; it is denoted sometimes by $h^*(\Gamma)$.

If $h^*(\Gamma)$ is the induced groupoid of Γ under $h: X \longrightarrow \Gamma_0$ then $h^*_\Gamma: f^*(\Gamma) \longrightarrow \Gamma$ defined by $h^*_\Gamma(x,y,a) = a$ together with h define a groupoid morphism $(h^*_\Gamma, h): (h^*(\Gamma); X) \longrightarrow (\Gamma;\Gamma_0)$, called the **canonical morphism of an induced groupoid**.

Theorem 2.1 *The pair* $(h^*_\Gamma, h): (h^*(\Gamma); X) \longrightarrow (\Gamma;\Gamma_0)$ *verify the universal property:*

for every groupoid morphism $(u,h): (\Gamma'; X) \longrightarrow (\Gamma;\Gamma_0)$ *there exists a unique* X*-morphism of groupoids* $v: \Gamma' \longrightarrow h^*(\Gamma)$ *such that the diagram:*

$$\begin{array}{ccc} \Gamma' & \stackrel{u}{\longrightarrow} & \Gamma \\ & & \\ (\exists)v\searrow & & \nearrow h^*_\Gamma \\ & h^*(\Gamma) & \end{array}$$

is commutative, i.e. $h^*_\Gamma \circ v = u$.

Proof. Let $(\Gamma', \alpha', \beta', \epsilon'; X)$ be a groupoid over X and $(u,h): (\Gamma'; X) \longrightarrow (\Gamma;\Gamma_0)$ be a groupoid morphism. We define $v: \Gamma' \longrightarrow f^*(\Gamma)$ by $v(a') = (\alpha'(a'), \beta'(a'), u(a'))$ for all $a' \in \Gamma'$.

We have that $\alpha^* \circ v = \alpha'$ and $\beta^* \circ v = \beta'$. It is easy to check that v is a X-morphism of groupoids. Clearly, we have $h^*_\Gamma \circ v = u$, and we prove by a standard manner that v is unique.

If $u: (\Gamma; X) \longrightarrow (\Gamma'; X)$ is an X-morphism of groupoids over X and if $f: Y \longrightarrow X$ is a map, then there exists a Y-morphism of groupoids over Y, $f^*(u): f^*(\Gamma) \longrightarrow f^*(\Gamma')$ defined by the relation:
$f^*(u)(y_1,y_2,a) = (y_1,y_2,u(a)) \in f^*(\Gamma'), \quad (\forall) \quad (y_1,y_2,a) \in f^*(\Gamma)$

It is enough to establish that $f^*(u)$ is well-defined and $f^*(u)$ is a Y-morphism of groupoids.

Clearly, we have $f^*(id_\Gamma) = id_{f^*(\Gamma)}$ and if $v : (\Gamma'; X) \longrightarrow (\Gamma''; X)$ is a second X-morphism of groupoids, then $f^*(vou) : f^*(\Gamma) \longrightarrow f^*(\Gamma'')$ defined by:
$f^*(vou)(y_1, y_2, a) = (y_1, y_2, (vou)(a)), \quad (\forall)(y_1, y_2, a) \in f^*(\Gamma)$
is a Y-morphism of groupoids such that $\quad f^*(v \circ u) = f^*(v) \circ f^*(u)$.

Therefore, we have the next proposition:

Proposition 2.1 *For each map $f : Y \longrightarrow X$, the family of maps*

$$(\Gamma \overset{\alpha}{\underset{\beta}{\longrightarrow}} X) \longrightarrow (f^*(\Gamma) \overset{\alpha^*}{\underset{\beta^*}{\longrightarrow}} Y)$$

and

$$(u : \Gamma \longrightarrow \Gamma') \longmapsto (f^*(u) : f^*(\Gamma) \longrightarrow f^*(\Gamma'))$$

defines a functor $f^* : \mathcal{G}(X) \longrightarrow \mathcal{G}(Y)$ of the category $\mathcal{G}(X)$ in the category $\mathcal{G}(Y.)$.

Finally, we have the following transitivity relation.

Proposition 2.2 *Let $v : Z \longrightarrow Y$ and $u : Y \longrightarrow X$ be two maps and let $(\Gamma, \alpha, \beta, \epsilon; X)$ be a groupoid over X.Then the groupoids $v^*(u^*(\Gamma))$ and $(uov)^*(\Gamma)$ are Z-isomorphic.*

Proof. We have:
$u^*(\Gamma) = \{(y_1, y_2, a) \in Y \times Y \times \Gamma \quad | \quad u(y_1) = \alpha(a), u(y_2) = \beta(a)\}$;
$v^*(u^*(\Gamma)) = \{(z_1, z_2, (y_1, y_2, a)) \in Z \times Z \times u^*(\Gamma) \quad | \quad v(z_1) = y_1; v(z_2) = y_2\}$;
$(uov)^*(\Gamma) = \{(z_1, z_2, a) \in Z \times Z \times \Gamma \quad | \quad (u \circ v)(z_1) = \alpha(a), (u \circ v)(z_2) = \beta(a)\}$

We take the maps $\varphi : (u \circ v)^*(\Gamma) \longrightarrow v^*(u^*(\Gamma))$ and $\psi : v^*(u^*(\Gamma)) \longrightarrow (u \circ v)^*(\Gamma)$ given by: $\varphi(z_1, z_2, a) = (z_1, z_2, (v(z_1), v(z_2), a))$ and $\psi(z_1, z_2, (y_1, y_2, a)) = (z_1, z_2, a)$.

It is easy to verify that φ and ψ are Z-morphisms of groupoids such that $\psi \circ \varphi = Id_{(u \circ v)^*(\Gamma)}$ and $\varphi \circ \psi = Id_{v^*(u^*(\Gamma))}$. We obtain that φ is a Z-isomorphism of groupoids.

Proposition 2.3 *Let $(\Gamma; X)$ be a groupoid over X. Then the groupoid $Id_X^*(\Gamma)$ and Γ are X-isomorphic.*

Proof. We have $Id_X^*(\Gamma) = \{(x_1, x_2, a) \in X \times X \times \Gamma \quad | \quad \alpha(a) = x_1, \beta(a) = x_2\}$ and we prove that $u : \Gamma \longrightarrow Id_X^*(\Gamma), u(a) = (\alpha(a), \beta(a), a)$ is an X- isomorphism.

Let $(\Gamma, \alpha, \beta, \epsilon; X)$ be a groupoid over X and let Y be a subset of X with inclusion map $j : Y \hookrightarrow X$. We consider the set $\Gamma_1 = \alpha^{-1}(Y) \cap \beta^{-1}(Y)$.

We can easily prove that $(\Gamma_1, \alpha_1, \beta_1, \epsilon_1; Y)$ where $\alpha_1 = \alpha|_{\Gamma_1}, \beta_1 = \beta|_{\Gamma_1}, \epsilon_1 = \epsilon|_Y$, is a groupoid over Y, denoted $(\Gamma|_Y; Y)$ and it is be called **the restriction of** Γ **to** Y.

Proposition 2.4 *(i) The induced groupoid $c^*(\Gamma)$ of a groupoid $(\Gamma; X)$ over the constant map $c : Y \longrightarrow X, c(y) = x_0, (\forall)y \in Y$ is Y-isomorphic with the trivial groupoid $Y \times Y \times \Gamma(x_0)$ where $\Gamma(x_0)$ is the isotropy group of Γ at x_0.*

(ii) The induced groupoid $j^*(\Gamma)$ of a groupoid $(\Gamma; X)$ over the inclusion map $j : Y \hookrightarrow X$, where Y is a subset of X is Y-isomorphic with the restriction $\Gamma|_Y$ of Γ to Y.

(iii) Let $(\Gamma, \alpha = \beta; X)$ be a group bundle over X and let $f : Y \longrightarrow X$ be an injective map. Then the induced groupoid $f^*(\Gamma)$ is a group bundle.

Proof. (i) We have: $c^*(\Gamma) = \{(y_1, y_2, a) \in Y \times Y \times \Gamma \ \ | \ \ \beta(a) = \alpha(a) = x_0\}$ and we observe that for each $(y_1, y_2, a) \in c^*(\Gamma)$ imply that $a \in \Gamma(x_0)$. Clearly, $c^*(\Gamma)$ and $Y \times Y \times \Gamma(x_0)$ are Y-isomorphic.

(ii) We have $j^*(\Gamma) = \{(y_1, y_2, a) \in Y \times Y \times \Gamma \ \ | \ \ y_1 = \alpha(a), y_2 = \beta(a)\}$ and it is enough to establish that the groupoids $j^*(\Gamma)$ and $\Gamma|_Y$ are Y-isomorphic.

(iii) Since $(\Gamma, \alpha = \beta; X)$ is a group bundle, it follows that $f^*(\Gamma) = \{(y_1, y_2, a) \in Y \times Y \times \Gamma \ \ | \ \ \alpha(a) = f(y_1) = f(y_2)\}$. The maps $\alpha^*, \beta^* : f^*(\Gamma) \longrightarrow Y$ are defined by $\alpha^*(y_1, y_2, a) = y_1$ and $\beta^*(y_1, y_2, a) = y_2$. Since f is injective we can prove that $\alpha^* = \beta^*$ and therefore $f^*(\Gamma)$ is a group bundle.

Proposition 2.5 *The induced groupoid $f^*(\Gamma)$ of a transitive groupoid Γ under a map $f : Y \longrightarrow X$ is a transitive groupoid over Y.*

Proof. Since the map $\alpha \times \beta : \Gamma \longrightarrow X \times X$, $(\alpha \times \beta)(a) = (\alpha(a), \beta(a))$ is surjective, it follows that there exists an element $a \in \Gamma$ such that $(\alpha \times \beta)(a) = (f(y_1), f(y_2))$ whenever $(y_1, y_2) \in Y \times Y$, i.e. $\alpha(a) = f(y_1)$ and $\beta(a) = f(y_2)$. Consequently, $(y_1, y_2, a) \in f^*(\Gamma)$ and $(\alpha^* \times \beta^*)(y_1, y_2, a) = (\alpha^*(y_1, y_2, a), \beta^*(y_1, y_2, a)) = (y_1, y_2)$, i.e. $\alpha^* \times \beta^* : f^*(\Gamma) \longrightarrow Y \times Y$ is surjective. Hence $f^*(\Gamma)$ is a transitive groupoid.

3. Special morphisms of groupoids

Definition 3.1 *A groupoid morphism $(\varphi, \varphi_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma', \Gamma'_0)$ is called a* **pullback** *if (φ, φ_0) verify the* **universal property** *$(\mathcal{PU})$, where:*
$(\mathcal{PU})$ for every groupoid $(\Gamma_1; \Gamma_0)$ and every groupoid morphism $(\psi, \psi_0 = \varphi_0) : (\Gamma_1; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ there exists a unique Γ_0-morphism $\overline{\psi} : \Gamma_1 \longrightarrow \Gamma$ such that $\varphi \circ \overline{\psi} = \psi$; in other words, every groupoid morphism $(\psi, \psi_0 = \varphi_0) : (\Gamma_1; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ can be factored uniquely into $\Gamma_1 \xrightarrow{\overline{\psi}} \Gamma \xrightarrow{\varphi} \Gamma'$ so that the following diagram:

$$\begin{array}{ccc} & \Gamma_1 \xrightarrow{\psi} \Gamma' & \\ (\exists)!\overline{\psi} & \searrow \quad \nearrow \varphi & \\ & \Gamma & \end{array}$$

is commutative.

Using Theorem 2.1, we get immediately:

Proposition 3.1 *The canonical morphism (h^*_Γ, h) of induced groupoid $h^*(\Gamma)$ of Γ by $h : X \longrightarrow \Gamma_0$ is a pullback.*

If $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is a groupoid morphism, then for every $u, v \in \Gamma_0$ we have: $f(\Gamma_u) \subseteq \Gamma'_{f_0(u)}$; $f(\Gamma^v) \subseteq (\Gamma')^{f_0(v)}$ and $f(\Gamma^v_u) \subseteq (\Gamma')^{f_0(v)}_{f_0(u)}$.

Then the restriction of f to $\Gamma_u, \Gamma^v, \Gamma^v_u$ respectively, defines the maps $\Gamma_u \longrightarrow \Gamma'_{f_0(u)}$; $\Gamma^v \longrightarrow (\Gamma')^{f_0(v)}$ and $\Gamma^v_u \longrightarrow (\Gamma')^{f_0(v)}_{f_0(u)}$, denoted by f_u, f^v and f^v_u.

We say that $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ has **discrete kernel**, if $Ker f = \epsilon(\Gamma_0)$.

Definition 3.2 *A groupoid morphism $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is to be called:*

(i) **base injective** *(resp.,* **base surjective**, **base bijective***) if f_0 is injective (resp., surjective, bijective).*

(ii) **fibrewise injective** *(resp.,* **surjective**, **bijective***) if $f_u : \Gamma_u \longrightarrow \Gamma_{f_0(u)}$ is injective (resp., surjective, bijective), for all $u \in \Gamma_0$.*

(iii) **piecewise injective** *(resp.,* **surjective**, **bijective***) if $f^v_u : \Gamma^v_u \longrightarrow (\Gamma')^{f_0(v)}_{f_0(u)}$ is injective (resp., surjective, bijective), for all $u, v \in \Gamma_0$.*

Proposition 3.2 *(i) The canonical morphism (f^*_Γ, f) of the induced groupoid $f^*(\Gamma)$ of Γ by $f : X \longrightarrow \Gamma_0$ is piecewise bijective.*

(ii) If $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is base surjective and piecewise surjective, then f is surjective.

(iii) A groupoid morphism $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ has **discrete kernel**, *iff $Ker f^u_u = \{\epsilon(u)\}$, for all $u \in \Gamma_0$, i.e. the group morphism $f^u_u : \Gamma(u) \longrightarrow \Gamma'(f_0(u))$ has a trivial kernel, for all $u \in \Gamma_0$.*

Proof. (i) We prove that $(f^*_\Gamma)^y_x : (\alpha^*)^{-1}(x) \cap (\beta^*)^{-1}(y) \longrightarrow \alpha^{-1}(f(x)) \cap \beta^{-1}(f(y))$ is bijective. Indeed, let $(x_1, y_1, a_1); (x_2, y_2, a_2) \in (\alpha^*)^{-1}(x) \cap (\beta^*)^{-1}(y)$ so that $(f^*_\Gamma)^y_x(x_1, y_1, a_1) = (f^*_\Gamma)^y_x(x_2, y_2, a_2) \implies \alpha^*(x_1, y_1, a_1) = \alpha^*(x_2, y_2, a_2) = x$, $\beta^*(x_1, y_1, a_1) = \beta^*(x_2, y_2, a_2) = y$ and $a_1 = a_2$; $\implies (x_1, y_1, a_1) = (x_2, y_2, a_2)$. Hence $(f^*_\Gamma)^y_x$ is injective.

Let $b \in \alpha^{-1}(f(x)) \cap \beta^{-1}(f(y)) \implies \alpha(b) = f(x)$ and $\beta(b) = f(y) \implies (x, y, b) \in f^*(\Gamma)$ and $(x, y, b) \in (\alpha^*)^{-1}(x) \cap (\beta^*)^{-1}(y)$. We have $f^*(\Gamma)(x, y, b) = b \implies (f^*_\Gamma)^y_x$ is surjective.

(ii) Let $y' \in \Gamma'$. We take $\alpha'(y') = u'$, $\beta'(y') = v'$. Then there exist $u, v \in \Gamma_0$ such that $u' = f_0(u), v' = f_0(v)$, since f_0 is surjective. But $y' \in (\Gamma')^{v'}_{u'}$ and applying the fact that $f^v_u : \Gamma^v_u \longrightarrow (\Gamma')^{v'}_{u'}$ it follows that there exists $x \in \Gamma^v_u$ such that $f^v_u(x) = y'$, and we deduce that f is surjective, since $f(x) = y'$.

(iii) The proof of this is straightforward.

Theorem 3.1 *Let* $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ *be a groupoid morphism. Then:*
(i) (f,f_0) *is a pullback iff it is piecewise bijective.*
(ii) f *is injective iff it is base injective and piecewise injective.*
(iii) (f,f_0) *is fibrewise injective iff it has a discrete kernel.*

Proof. (i) We suppose that (f,f_0) is a pullback, and we consider the induced groupoid $f_0^*(\Gamma')$ of Γ' by $f_0:\Gamma_0\longrightarrow\Gamma'_0$.

Since $(f_0)^*_{\Gamma'}$ is a pullback (cf. Proposition 3.2), then considering the given groupoid morphism $f:\Gamma\longrightarrow\Gamma'$, this can be factored uniquely into: $f=(f_0)^*_{\Gamma'}\circ\varphi$, where $\varphi:\Gamma\longrightarrow f_0^*(\Gamma')$ is a groupoid morphism over Γ_0.

Also, for groupoid morphism $f:\Gamma\longrightarrow\Gamma'$ which is a pullback, the groupoid morphism $(f_0)^*_{\Gamma'}:f_0^*(\Gamma')\longrightarrow\Gamma'$ can be factored uniquely into: $(f_0)^*_{\Gamma'}=f\circ\overline{\varphi}$, where $\overline{\varphi}:f_0^*(\Gamma')\longrightarrow\Gamma$ is a groupoid morphism over Γ_0.

From $f=(f_0)^*_{\Gamma'}\circ\varphi$, and $(f_0)^*_{\Gamma'}=f\circ\overline{\varphi}$, it follows that $f=f\circ(\overline{\varphi}\circ\varphi)=f\circ Id$, and $(f_0)^*_{\Gamma'}=(f_0)^*_{\Gamma'}\circ(\varphi\circ\overline{\varphi})=(f_0)^*_{\Gamma'}\circ Id$.

Therefore, $\overline{\varphi}\circ\varphi=Id$, and $\varphi\circ\overline{\varphi}=Id$, since f and $(f_0)^*_{\Gamma'}$ are pullbacks.

Hence,$\varphi:\Gamma\longrightarrow f_0^*(\Gamma')$ is an isomorphism of groupoids such that $f=(f_0)^*_{\Gamma'}\circ\varphi$. Since $(f_0)^*_{\Gamma'}$ is piecewise bijective and φ is bijective, it follows that f is piecewise bijective.

Conversely, suppose that (f,f_0) is piecewise bijective. Let $(\Gamma_1,\alpha_1,\beta_1;\Gamma_0)$ be a groupoid and $(\psi,\psi_0)=f_0:(\Gamma_1;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ be a groupoid morphism.

For each $u,v\in\Gamma_0$, define $\overline{\psi}^v_u:(\Gamma_1)^v_u\longrightarrow\Gamma^v_u$ by $\overline{\psi}^v_u=(f^v_u)^{-1}\circ\psi^v_u$.

We have that $f^v_u\circ\overline{\psi}^v_u=\psi^v_u,\quad(\forall)u,v\in\Gamma_0$.

We consider $\overline{\psi}:\Gamma_1\longrightarrow\Gamma$ defined by $\overline{\psi}(x_1)=(f^{\beta_1(x_1)}_{\alpha_1(x_1)})^{-1}\circ\psi^{\beta_1(x_1)}_{\alpha_1(x_1)},\quad(\forall)x_1\in\Gamma_1$.

It is easy to check that $\overline{\psi}$ is a groupoid morphism uniquelly determined such that $\psi=f\circ\overline{\psi}$. Hence, f is a pullback.

(ii) We suppose that f is injective. From $f\circ\epsilon=\epsilon'\circ f_0$, it follows that $f\circ\epsilon$ is injective. Hence, $\epsilon'\circ f_0$ is injective and therefore f_0 is injective.

Conversely, suppose that f_0 and f^v_u are injective, for all $u,v\in\Gamma_0$. We prove that f is injective.

Let $x,y\in\Gamma$ such that $f(x)=f(y)$. From $f(x)=f(y)\implies(\alpha'\circ f)(x)=(\alpha'\circ f)(y),\quad(\beta'\circ f)(x)=(\beta'\circ f)(y),\implies(f_0\circ\alpha)(x)=(f_0\circ\alpha)(y),\quad(f_0\circ\beta)(x)=(f_0\circ\beta)(y)\implies\alpha(x)=\alpha(y),\beta(x)=\beta(y)$. If we denote $\alpha(x)=u,\beta(y)=v$, then $x,y\in\Gamma^v_u$ and $f^v_u(x)=f^v_u(y)$. It follows $x=y$, since f^v_u is injective.

(iii) We suppose that (f,f_0) is fibrewise injective. Then $f_u:\Gamma_u\longrightarrow\Gamma'_{f_0(u)}$ is injective. We prove that $Kerf\subseteq\epsilon(\Gamma_0)$. Indeed, let $x\in Kerf$ and we denote $u=\alpha(x)$ and $u'=f_0(u)$. Then $f(x)\in\epsilon'(\Gamma'_0)\implies f(x)=\epsilon'(u')$ with $u'\in\Gamma'_0$. Also, we have $f(\epsilon(u))=\epsilon'(f_0(u))=\epsilon'(u')\implies f(x)=f(\epsilon(u))\implies f_u(x)=f_u(\epsilon(x))\implies x=\epsilon(u)$, since f_u is injective. Therefore, $x\in\epsilon(\Gamma_0)$. Hence, $Kerf=\epsilon(\Gamma_0)$ and we deduce that $Kerf$ is discrete.

Conversely, we suppose that f has discrete kernel. Let $x,y\in\Gamma_u$ such that $f_u(x)=f_u(y)$, then $\alpha(x)=\alpha(y)=u$ and $f(x)=f(y)$. It follows $f(x)\cdot$

$(f(y))^{-1} = \epsilon'(u')$, with $u' \in \Gamma'_0 \implies f(x \cdot y^{-1}) = \epsilon'(u') \implies x \cdot y^{-1} \in Kerf = \epsilon(\Gamma_0) \implies x \cdot y^{-1} = \epsilon(u)$ with $u \in \Gamma_0 \implies x = \epsilon(u) \cdot y \implies \epsilon(u) \cdot x = \epsilon(u) \cdot \epsilon(u) \cdot y \implies \epsilon(u) \cdot x = \epsilon(u) \cdot y \implies x = y$.

Proposition 3.3 *A fibrewise surjective morphism $f : \Gamma \longrightarrow \Gamma'$ is a fibrewise bijective morphism iff $Kerf$ is discrete.*

Proof. This follows from Th. 3.1.(iii).

Let us give some rules for deriving new fibrewise surjective or fibrewise bijective morphisms from the old ones.

Theorem 3.2 *Let $(G, \alpha, \beta; G_0), (H, \alpha', \beta'; H_0)$ and $(K, \alpha'', \beta''; K_0)$ be three groupoids. Let $(f, f_0) : (G; G_0) \longrightarrow (H; H_0)$ and $(g, g_0) : (H; H_0) \longrightarrow (K; K_0)$ be groupoid morphisms. Then:*

(i) if f, g are fibrewise surjective (resp., bijective) morphisms, then so is $g \circ f$;

(ii) if $g \circ f$ and f are fibrewise surjective morphism and $f_0 : G_0 \longrightarrow H_0$ is surjective, then g is a fibrewise surjective morphism;

(iii) if $g \circ f$ and g are fibrewise bijective morphisms, then so is f.

Proof. (i) Straightforward.

(ii) We prove that $g_{u'} : (\alpha')^{-1}(u') \longrightarrow (\alpha'')^{-1}(g_0(u'))$ is surjective. For this, let $y'' \in (\alpha'')^{-1}(g_0(u'))$, then $(\alpha'')(y'') = g_0(u')$ with $u' \in H_0$. Since f_0 is surjective, it follows that there exists $u \in G_0$ such that $f_0(u) = u'$. Then $(g \circ f)_u : \alpha^{-1}(u) \longrightarrow (\alpha'')^{-1}((g \circ f)_0(u))$ is surjective. We have $(\alpha'')(y'') = (g_0 \circ f_0)(u))$ and therefore $(\exists) x \in \alpha^{-1}(u)$ such that $\alpha(x) = u$ and $(g \circ f)(u) = y''$. If we consider $x' = f(x)$, we have $x' \in (\alpha')^{-1}(u')$, since $\alpha'(x') = \alpha'(f(x)) = f_0(\alpha(x)) = f_0(u) = u'$. From $(g \circ f)(x) = y''$, we obtain that $g_{u'}(x') = y''$. Hence, $g_{u'}$ is surjective.

(iii) Let $x, y \in \alpha'(u)$ such that $f_u(x) = f_u(y)$. Then $f(x) = f(y) \implies (g \circ f)(x) = (g \circ f)(y) \implies (g \circ f)_u(x) = (g \circ f)_u(y) \implies x = y$, since $(g \circ f)_u$ is injective. Hence, f is fibrewise injective.

Clearly, $(g \circ f)_u = g_{f_0(u)} \circ f_u$ and $(g \circ f)_0 = g_0 \circ f_0$.

Let $y' \in (\alpha')^{-1}(v)$, where $v = f_0(u), \implies g_v(y') \in (\alpha'')^{-1}(g_0(v)) \implies g_{f_0(u)}(y') \in (\alpha'')^{-1}(g_0 \circ f_0)(u))$. For $g_{f_0(u)}(y') \in (\alpha'')^{-1}(g_0 \circ f_0)(u))$ there exists $x \in \alpha^{-1}(u)$ such that $(g \circ f)(x) = g_{f_0(u)}(y')$, since $(g \circ f)_u$ is surjective.
It follows that $g_{f_0(u)}(f_u(x)) = g_{f_0(u)}(y')$ and we obtain $f_u(x) = y'$, since $g_{f_0(u)}$ is injective. Therefore, f_u is surjective. Hence f is fibrewise surjective.

Corollary 3.1 *Let be given the following commutative diagram of groupoid morphisms such that g is a pullback:*

$$\begin{array}{ccc} X & \xrightarrow{f} & G \\ p\downarrow & & \downarrow\overline{p} \\ Y & \xrightarrow{g} & H. \end{array}$$

If p is a fibrewise surjective morphism, then also is $\overline{p}$.

Proof. The proof of this is a simple consequence of Th. 3.2.

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