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LAGRANGE GEOMETRY VIA COMPLEX LAGRANGE GEOMETRY

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Abstract. Asking that the metric of a complex Finsler space should be strong convex, Abate and Patrizio ([1]) associate to the real tangent bundle a real Finsler metric for which they analyze the relation between Cartan (real) connection of the obtained space and the real image of Chern-Finsler complex connection.

Following the same ideas, in the present paper we shall deal with the more general case of a complex Lagrange space (M, L).

As distinct from these authors, we shall associate to the Hermitian metric $g_{i\bar{j}}(z,\eta)$ of a complex Lagrangian L its real representation $\overset{R}{g}_{ab}$ (x, y). The obtained real space $(M, \overset{R}{g}_{ab})$ is a generalized Lagrange space ([10]). Furthermore, the possibility of its reduction to one real Lagrange space, in particular the Finsler one, is studied.

A comparative analysis of the elements of Lagrange geometry ([10]): nonlinear connection, $N-$linear connection, metric canonical connection, and so on, and their corresponding real image from the complex Lagrange geometry ([11]) is made.

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1. Introduction

The study of complex Lagrange geometry was initiated by us starting with the paper [11].

A complex Lagrange space is the pair (M, L), where M is a complex manifold and $L(z, \eta)$ is a real Lagrangian differentiable function on the holomorphic bundle $T'M$, which determines a nondegenerate metric $g_{i\bar{j}} = \partial^2 L/\partial\eta^i\partial\bar{\eta}^j$.

This geometry generalizes that of the known complex Finsler space ([1],[2],[5], [6],[7],[13],[14]), where, in addition, the homogeneity condition of complex Lagrangian in respect to η is required.

In this paper, a complex Lagrange space determines a real structure of Hermitian manifold on the tangent real bundle. Everywhere, the indices $i, j, k, ...$ run in the interval $\overline{1,n}$, and $a, b, c, ...$ run in $\overline{1,2n}$. We shall assume that the

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reader is familiar with the geometry of $T'M$, the holomorphic tangent bundle, and with the Lagrange geometry ([10]).

Let M be a complex manifold, $dim_C M = n$, (U, z^k) the local coordinates in a local chart, $z^k = x^k + ix^{n+k}$. M is also a real manifold, $dim_R M = 2n$, (U, x^a) is a real chart, and is endowed with the complex structure $\overset{R}{J}$, $\overset{R}{J}{}^2 = -I$, acting on $T_{R,x}M$ by $\overset{R}{J}(\frac{\partial}{\partial x^a}) = \frac{\partial}{\partial x^{n+k}}$, $\overset{R}{J}(\frac{\partial}{\partial x^{n+k}}) = -\frac{\partial}{\partial x^k}$.

Let us consider the well-known Poincaire operators:

$$\frac{\partial}{\partial z^k} = \frac{1}{2}(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial x^{n+k}}); \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial x^{n+k}}) \tag{1.1}$$

from which clearly results that:

$$\frac{\partial}{\partial x^k} = \frac{\partial}{\partial z^k} + \frac{\partial}{\partial \bar{z}^k}; \quad \frac{\partial}{\partial x^{n+k}} = i(\frac{\partial}{\partial z^k} - \frac{\partial}{\partial \bar{z}^k}) \tag{1.2}$$

The complex structure $\overset{R}{J}$ is extended to the complexification $T_C M$ of the tangent bundle, obtaining the complex structure $J(X + iY) = \overset{R}{J}(X) + i\,\overset{R}{J}(Y)$, $J^2 = -I$, behaving on Poincare operators as follows: $J(\frac{\partial}{\partial z^k}) = i\frac{\partial}{\partial z^k}$; $J(\frac{\partial}{\partial \bar{z}^k}) = -i\frac{\partial}{\partial \bar{z}^k}$. The eigenspaces of J determines two subbundles of $T_C M$ denoted by $T'M$, the $(1,0)-$type vectors, and respectively $T''M = \overline{T'M}$, the $(0,1)-$type vectors and $T_C M = T'M \oplus T''M$.

The bundle $T'M$ is holomorphic and as a manifold it is the geometric support of the complex Lagrange geometry.

The bundle $T'M$ is isomorphic with the real tangent bundle $T_R M$ by the map that acts on the corresponding tangent space as follows ([1]):

$$\overset{R}{\circ} : X \to X^R = X + \overline{X} \tag{1.3}$$

with the inverse:

$$\overset{C}{\circ} : X \to X^C = \frac{1}{2}(X - iJX) \tag{1.4}$$

Locally, if $X = \eta^k \frac{\partial}{\partial z^k}$, with $\eta^k = y^k + iy^{n+k}$, then $X^R = y^k \frac{\partial}{\partial x^k} + y^{n+k}\frac{\partial}{\partial x^{n+k}}$, and conversely, if $X = y^a \frac{\partial}{\partial x^a}$, then $X^C = (y^k + iy^{n+k})\frac{\partial}{\partial z^k}$.

Let us consider $\pi : T'M \to M$ the holomorphic bundle, $u = (z^k, \eta^k) \in T'M$ and $p : T_R M \to M$ the tangent real bundle, $u = (x^a, y^a) \in T_R M$. Now, taking $T'M$ as a base manifold, arguing as before, we obtain: $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. The bundle $\pi_T : T'(T'M) \to T'M$ is holomorphic and $Ker\ \pi_T = V(T'M)$ is called the vertical bundle, a local base in $V_u(T'M)$ being $\{\frac{\partial}{\partial \eta^k}\}$. Through conjugation, a local base $\{\frac{\partial}{\partial \bar{\eta}^k}\}$ in $\overline{V_u(T'M)}$ is obtained. Let us denote by $V_C(T'M) = V(T'M) \oplus \overline{V(T'M)}$ the vertical complexified bundle.

Analogously, we can consider the real vertical bundle $V(T_R M) = Ker\ p_T$, where $p_T : T_R(T_R M) \to T_R M$ is the tangent map. A local base in $V_u(T_R M)$

is indeed $\{\frac{\partial}{\partial y^a}\}$ and because the vertical bundle is isomorphic with the tangent bundle $T_R M$, we have:

$$\frac{\partial}{\partial \eta^k} = \frac{1}{2}(\frac{\partial}{\partial y^k} - i\frac{\partial}{\partial y^{n+k}}); \qquad \frac{\partial}{\partial \bar{\eta}^k} = \frac{1}{2}(\frac{\partial}{\partial y^k} + i\frac{\partial}{\partial y^{n+k}}) \tag{1.5}$$

and conversely,

$$\frac{\partial}{\partial y^k} = (\frac{\partial}{\partial \eta^k})^R = \frac{\partial}{\partial \eta^k} + \frac{\partial}{\partial \bar{\eta}^k}; \quad \frac{\partial}{\partial y^{n+k}} = (i\frac{\partial}{\partial \eta^k})^R = i(\frac{\partial}{\partial \eta^k} - \frac{\partial}{\partial \bar{\eta}^k}) \tag{1.6}$$

Therefore, if $V = V^k \frac{\partial}{\partial \eta^k}$ is a vertical complex field then $V^R = U^a \frac{\partial}{\partial y^a}$, with $U^k = ReV^k$ and $U^{n+k} = ImV^k$, is a real vertical field, and conversely. Hence, $(V_C(T'M))^R = V(T_R M)$. We denote the same here by $\overset{R}{\circ}$ the isomorphism of passing to real on $T'M$.

2. The induced real nonlinear connection

As is known, in the study of tangent bundles it is very useful to use the notion of a nonlinear connection that determines the adapted base in which the study is "linearized": many of the computations are made similarly as on the base manifold M.

A nonlinear connection can be given by a splitting in an exact sequence that determines a supplementary subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$, called the horizontal subbundle. This determines the distribution $N : u = (z^k, \eta^k) \to H_u(T'M)$, called the complex nonlinear connection, shortly (*c.n.c.*). A local base on $H_u(T'M)$ is $\{\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - N_j^k \frac{\partial}{\partial \eta^k}\}$, where N_j^k are the coefficients of (*c.n.c.*) and they are transforms at the local change of charts after the rule:

$$N'^i_k \frac{\partial z'^k}{\partial z^j} = \frac{\partial z'^i}{\partial z^k} N_j^k - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^k \tag{2.1}$$

and then the base $\{\frac{\delta}{\delta z^i}\}$, called the *adapted* base of N_i^j (*c.n.c.*), satisfies the following rule of transformation:

$$\frac{\delta}{\delta z^i} = \frac{\partial z'^j}{\partial z^i} \frac{\delta}{\delta z'^j} \tag{2.2}$$

Through conjugation is obtained an adapted base $\{\frac{\delta}{\delta z^k}, \frac{\partial}{\partial \eta^k}, \frac{\delta}{\delta \bar{z}^k}, \frac{\partial}{\partial \bar{\eta}^k}\}$ on $T_C(T'M)$, shortly denoted by $\{\delta_k\ , \partial_k\ , \delta_{\bar{k}}\ , \partial_{\bar{k}}\}$. The dual adapted base is denoted by $\{dz^k\ , \delta\eta^k\ , d\bar{z}^k\ , \delta\bar{\eta}^k\}$.

From (2.2) it results that there exists an isomorphism ([1]) $\overset{C}{\theta}: V_C(T'M) \to H_C(T'M)$, locally given by $\overset{C}{\theta}(\frac{\partial}{\partial \eta^i}) = \frac{\delta}{\delta z^i}$ and $\overset{C}{\theta}(\frac{\partial}{\partial \bar{\eta}^i}) = \frac{\delta}{\delta \bar{z}^i}$, where $H_C(T'M) = H(T'M) \oplus \overline{H(T'M)}$ is the complexified horizontal bundle.

As a complex manifold $T'M$ has the natural complex structure, still denoted by J, and locally given by:

$J(\partial/\partial z^k) = i\partial/\partial z^k$; $J(\partial/\partial \eta^k) = i\partial/\partial \eta^k$; $J(\partial/\overline{z}^k) = -i\partial/\partial\overline{z}^k$

$J(\partial/\partial\overline{\eta}^k) = -i\partial/\partial\overline{\eta}^k$.

Since, $J(\delta_k) = i\delta_k$ and $J(\delta_{\bar{k}}) = -i\delta_{\bar{k}}$ we deduce that $J(H(T'M)) = iH(T'M)$ and $J(\overline{H(T'M)}) = -i\overline{H(T'M)}$.

The same reasonings can be made on $T_R(T_RM)$. A real nonlinear connection, shortly (*r.n.c.*), is given by the splitting $T_R(T_RM) = H(T_RM) \oplus V(T_RM)$.

The $H(T_RM)$ bundle is for the moment not unique, being only supplementary to $V(T_RM)$. We shall fix $H(T_RM)$ acting analogously to [1]:

We see that T_RM and $T'M$ bundles are isomorphic by $\overset{C}{\circ}$ and $\overset{R}{\circ}$. The same isomorphism is between $T_R(T_RM)$ and $T'(T'M)$. On the other hand, the complex horizontal lift, locally expressed by $l^h_C(\frac{\partial}{\partial z^k}) = \frac{\delta}{\delta z^k}$ determines an isomorphism between $T'M$ and $H(T'M)$. Then the image of map $l^h_R = \overset{R}{\circ} \cdot l^h_C \cdot \overset{C}{\circ} : T_RM \to T_R(T_RM)$ defines a real horizontal lift. Let us consider now the local base $\frac{\delta}{\delta x^a} = l^h_R(\frac{\partial}{\partial x^a})$ in $H(T_RM) = l^h_R(T_RM)$ which determines in turn an (*r.n.c.*).

As it is known that the local expression of a real horizontal lift is ([10]): $\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - \overset{R}{N^b_a} \frac{\partial}{\partial y^b}$, where $\overset{R}{N^b_a}$ are the coefficients of (*r.n.c.*). Taking into account the local expression of a complex vertical field, we can deduce that:

$$\overset{R}{N^h_k} = ReN^h_k \ ; \ \overset{R}{N^{n+h}_k} = ImN^h_k \tag{2.3}$$

and therefore : $N^h_k = \overset{R}{N^h_k} + i\overset{R}{N^{n+h}_k}$ and $\delta_k = \frac{\partial}{\partial z^k} - N^h_k\partial_h$.

Note that the map $\overset{R}{*} = \overset{R}{\theta} \cdot \overset{R}{\circ} \cdot \overset{C}{\theta}{}^{-1} : H_C(T'M) \to H(T_RM)$ is an isomorphism, where $\overset{R}{\theta} : V(T_RM) \to H(T_RM)$ is the corresponding real isomorphism to $\overset{C}{\theta}$. Because the horizontal bundle $H_C(T'M)$ is J invariant, applying the operator $\overset{R}{*}$ it follows that $H(T_RM)$ is $\overset{R}{J}$ invariant. Hence, $\frac{\delta}{\delta x^{n+k}}$ corresponds to $i\frac{\delta}{\delta z^k}$ and in consequence $\overset{R}{N^a_{n+k}}$ corresponds to iN^h_k. So, we deduce that:

$$\overset{R}{N^h_{n+k}} = -ImN^h_k \ ; \ \overset{R}{N^{n+h}_{n+k}} = ReN^h_k \tag{2.4}$$

Thus we obtain an (*r.n.c.*) , $\overset{R}{N^a_b}$, on T_RM, determined by the given (*c.n.c.*) N^h_k on $T'M$, whose coefficients are in fact the real representation of the complex matrix N^h_k.

Now, taking into account the action of the $\overset{R}{*}$ operator on the adapted base $\{\delta_k, \partial_k\}$, we get in addition to (1.6) that:

$$\frac{\delta}{\delta x^k} = (\frac{\delta}{\delta z^k})^R \ ; \ \frac{\delta}{\delta x^{n+k}} = (i\frac{\delta}{\delta z^k})^R \tag{2.5}$$

and the inverse through $\overset{C}{*} = \overset{C}{\theta} \cdot \overset{R}{\circ} \cdot \overset{R}{\theta}^{-1}$.

For the next computation it will be useful to have the following consequences of (1.6) and (2.5):

$$\begin{aligned}(\frac{\partial}{\partial \bar{\eta}^k})^R &= \frac{\partial}{\partial y^k} \ ; \ (\frac{\delta}{\delta \bar{z}^k})^R = \frac{\delta}{\delta x^k} \ \textit{and} \\ \frac{\delta}{\delta x^k} &= \frac{\delta}{\delta z^k} + \frac{\delta}{\delta \bar{z}^k} \ ; \ \frac{\delta}{\delta x^{n+k}} = i(\frac{\delta}{\delta z^k} - \frac{\delta}{\delta \bar{z}^k})\end{aligned}$$

Further, let us consider (M, L) a complex Lagrange space ([11]), where $L : T'M \to R$ is a Lagrangian function such that $g_{j\bar{k}} = \partial^2 L / \partial \eta^j \partial \bar{\eta}^k$ is a nondegenerate metric on $T'M$. At each point $u = (z^k, \eta^k) \in T'M$, $L(u) = L(z^k, \eta^k)$ is a differentiable function. Because $z^k = x^k + ix^{n+k}$ and $\eta^k = y^k + iy^{n+k}$, it follows that $L(u) = L^R(x^a, y^a)$ is a real differentiable function.

Let as note that, in general, $\overset{R}{g}_{ab} = \frac{1}{2} \partial^2 L^R / \partial y^a \partial y^b$ might be degenerate and hence the pair (M, L^R) is not always a real Lagrange space. In the special case when $(g_{j\bar{k}})$ determines a nondegenerate matrix $(\overset{R}{g}_{ab})$, (A-P) ([1]) calls the metric $g_{j\bar{k}}$ as being *strongly convex.*

Moreover, let us note that if $L(z, \lambda\eta) = |\lambda|^2 L(z, \eta)$, then $L^R(x, \lambda y) = |\lambda|^2 L^R(x, y)$ and conversely. So, if (M, L) is a strong convex Finsler complex space then (M, L^R) is a Finsler real one.

From now on we shall act in a manner different from [1]. We shall consider the real metric structure determined by the real representation of $g_{j\bar{k}}$.

Proposition 2.1. *Let $g_{j\bar{k}}(z, \eta)$ be the Hermitian metric of a complex Lagrange space (M, L). Then the pair $(M, \overset{R}{g}_{ab}(x, y))$, where :*

$$\overset{R}{g}_{jk} = Re\ g_{j\bar{k}} = \frac{1}{2}(g_{j\bar{k}} + g_{k\bar{j}}) \ ; \ \overset{R}{g}_{n+jk} = -Im\ g_{j\bar{k}} \tag{2.6}$$

$$\overset{R}{g}_{jn+k} = Im\ g_{j\bar{k}} = \frac{-i}{2}(g_{j\bar{k}} - g_{k\bar{j}}) \ ; \ \overset{R}{g}_{n+jn+k} = Re\ g_{j\bar{k}}$$

determines a (real) generalized Lagrange space([10]).

For proof it suffices to remark that $\overset{R}{g}_{ab} = \overset{R}{g}_{ba}$ and $det \left(\overset{R}{g}_{ab} \right) \neq 0$

Thanks to (2.6) and $g_{j\bar{k}} g^{\bar{k}l} = \delta^l_j$ it results that $\overset{R}{g}^{ab}$, the inverse of $\overset{R}{g}_{ab}$, is the real representation of $g^{\bar{k}l}$, i.e. $\overset{R}{g}^{jk} = Re\ g^{\bar{j}k}$; $\overset{R}{g}^{jn+k} = Im\ g^{\bar{j}k}$; $\overset{R}{g}^{n+jn+k} = Re\ g^{\bar{j}k}$; and $\overset{R}{g}^{n+jk} = -Im\ g^{\bar{j}k}$.

Now, considering a fixed $(c.n.c.) N^h_k$ and $\{dz^k \ , \delta\eta^k \ , d\bar{z}^k \ , \delta\bar{\eta}^k\}$ the dual adapted base determined by it, then ([11]):

$$G = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j \tag{2.7}$$

gives a Hermitian metric on $T'M$ with respect to the complex structure J and both to the almost Hermitian structure J_N, locally given by $J_N(\delta_k) = \partial_k$; $J_N(\partial_k) = -\delta_k$; $J_N(\delta_{\bar{k}}) = \partial_{\bar{k}}$; $J_N(\partial_{\bar{k}}) = -\delta_{\bar{k}}$ and globally defined.

Replacing $g_{i\bar{j}} = Re\ g_{i\bar{j}} + iIm\ g_{i\bar{j}}$; $dz^j = dx^j + idx^{n+j}$ and $\delta\eta^j = \delta y^j + i\delta y^{n+j}$ in (2.7), it results that:

Proposition 2.2. *The structure*

$$\overset{R}{G} = Re\ G = \overset{R}{g}_{ab}\ dx^a \otimes dx^b + \overset{R}{g}_{ab}\ \delta y^a \otimes \delta y^b \tag{2.8}$$

is a Hermitian metric on $T_R M$ with respect to the complex structure $\overset{R}{J}$, and an almost Hermitian metric with respect to $J_{\underset{N}{R}}$ structure, $(J_{\underset{N}{R}})^2 = -I$, locally given by $J_{\underset{N}{R}}(\frac{\delta}{\delta x^a}) = \frac{\partial}{\partial y^a}$ and $J_{\underset{N}{R}}(\frac{\partial}{\partial y^a}) = -\frac{\delta}{\delta x^a}$.

The integrability of J_N and $J_{\underset{N}{R}}$ structures depends only on the vanishing of torsion of (*c.n.c.*) and respectively (*r.n.c.*).

Let as note that $\overset{R}{\tilde{G}} = Im\ G$ also defines a metric structure on $T_R M$.

From the computation $g_{j\bar{k}} = \frac{\partial^2 \overset{R}{L}}{\partial \eta^j \partial \bar{\eta}^k} = \frac{1}{4}(\frac{\partial}{\partial y^k} + i\frac{\partial}{\partial y^{n+k}})(\frac{\partial \overset{R}{L}}{\partial y^j} - i\frac{\partial \overset{R}{L}}{\partial y^{n+j}}) = \frac{1}{4}(\frac{\partial^2 \overset{R}{L}}{\partial y^j \partial y^k} + \frac{\partial^2 \overset{R}{L}}{\partial y^{n+j} \partial y^{n+k}}) + \frac{i}{4}(\frac{\partial^2 \overset{R}{L}}{\partial y^j \partial y^{n+k}} - \frac{\partial^2 \overset{R}{L}}{\partial y^{n+j} \partial y^k})$ we deduce just the real representation of the matrix $g_{j\bar{k}}$:

$$\overset{R}{g}_{jk} = \frac{1}{4}(\frac{\partial^2 \overset{R}{L}}{\partial y^j \partial y^k} + \frac{\partial^2 \overset{R}{L}}{\partial y^{n+j} \partial y^{n+k}})\ ;\ \overset{R}{g}_{jn+k} = \frac{1}{4}(\frac{\partial^2 \overset{R}{L}}{\partial y^j \partial y^{n+k}} - \frac{\partial^2 \overset{R}{L}}{\partial y^{n+j} \partial y^k}) \tag{2.9}$$

Let us remark that, in general, the tensor $C_{abc} = \frac{1}{2}\{\frac{\partial \overset{R}{g}_{bc}}{\partial y^a} + \frac{\partial \overset{R}{g}_{ac}}{\partial y^b} - \frac{\partial \overset{R}{g}_{ab}}{\partial y^c}\}$ is not totally symmetric and therefore the generalized Lagrange space $(M, \overset{R}{g}_{ab})$ is not always reducible to a Largange space ([10]). Moreover, the space is neither weakly regular, hence the known procedures of Lagrange (real) geometry to obtain a (*r.n.c.*) cannot be applied, remaining in principle the method described above.

The question is, however, when the generalized Lagrange space is a Lagrange one, particularly a Finsler space. This means that the tensor $h_{ab} = \frac{1}{2}\frac{\partial^2 \overset{R}{L}}{\partial y^a \partial y^b}$ is nondegenerated. In particular, if L is a complex Finsler metric then $\overset{R}{L}$ becomes a real Finsler metric. A sufficient condition is when $h_{ab} = \overset{R}{g}_{ab}$, which in view of (2.9) is equivalent to:

$$\frac{\partial^2 \overset{R}{L}}{\partial y^j \partial y^k} = \frac{\partial^2 \overset{R}{L}}{\partial y^{n+j} \partial y^{n+k}}\ ;\ \frac{\partial^2 \overset{R}{L}}{\partial y^j \partial y^{n+k}} = -\frac{\partial^2 \overset{R}{L}}{\partial y^{n+j} \partial y^k} \tag{2.10}$$

and that happens if and only if :

$$g_{jk} = \frac{\partial^2 L}{\partial \eta^j \partial \eta^k} = 0 \ , \ \forall j,k = \overline{1,n} \tag{2.11}$$

Obviously, by conjugation from (2.11) it results also $g_{\bar{j}\bar{k}} = 0$.

Definition 2.1. *In the condition (2.11) we call the complex Lagrange space (M, L) being with pure Hermitian metric.*

In particular, the Finsler complex space with pure Hermitian metric is obtained.

Proposition 2.3. *The generalized Lagrange space $(M, \overset{R}{g}_{ab})$ associated to a complex Lagrange space with pure Hermitian metric is reductible to a real Lagrange space $(M, \overset{R}{L})$.*

As shown in [10], the variational method in the real Lagrange space $(M, \overset{R}{L})$ gives an $(r.n.c.)$:

$$\begin{aligned} \overset{0}{N}{}^a_b &= \frac{\partial G^a}{\partial y^b}, \ where \\ G^a &= \frac{1}{4} \overset{R}{g}{}^{ac} \left\{ \frac{\partial^2 \overset{R}{L}}{\partial y^c \partial x^d} y^d - \frac{\partial \overset{R}{L}}{\partial x^c} \right\} = \frac{1}{4} \overset{R}{g}{}^{ac} \ \Phi_c \end{aligned} \tag{2.12}$$

On the other hand, by the variational method a $(c.n.c.)$, called canonical, in a complex Lagrange space (M, L) is obtained ([11]):

$$\begin{aligned} \overset{c}{N}{}^k_j &= \frac{\partial H^k}{\partial \eta^j}, \ where \\ H^k &= \frac{1}{2} g^{\bar{m}k} \frac{\partial^2 L}{\partial z^h \partial \bar{\eta}^m} \eta^h \end{aligned} \tag{2.13}$$

Our next goal is to determine the circumstances when $\overset{cR}{N}{}^a_b$, the $(r.n.c.)$ induced by $\overset{c}{N}{}^k_j$, coincides with $\overset{0}{N}{}^a_b$ or, in an equivalent way, when the complex image $\overset{0C}{N}{}^k_j = \overset{0}{N}{}^k_j + i \overset{0}{N}{}^{n+k}_j$ of $\overset{0}{N}{}^a_b$ coincides with $\overset{c}{N}{}^k_j$.

For this reason we shall calculate the difference $d-$tensor of two $(c.n.c.)$:

$$2D^k_l = \left(\frac{\partial}{\partial \eta^l} + \frac{\partial}{\partial \bar{\eta}^l}\right)(G^k + iG^{n+k}) - \frac{\partial H^k}{\partial \eta^l} \tag{2.14}$$

First, we make the computation:

$4(G^k + iG^{n+k}) = Re\ g^{\bar{m}k}\Phi_m - Im\ g^{\bar{m}k}\Phi_{n+m} + i(Im\ g^{\bar{m}k}\Phi_m + Re\ g^{\bar{m}k}\Phi_{n+m}) = g^{\bar{m}k}(\Phi_m + i\Phi_{n+m})$.

Replacing Φ_m and Φ_{n+m} from (2.12) and recalling that $\frac{\partial}{\partial y^m} = \frac{\partial}{\partial \eta^m} + \frac{\partial}{\partial \bar{\eta}^m}$ and $y^k = \frac{1}{2}(\eta^k + \bar{\eta}^k)$; $\frac{\partial}{\partial y^{n+m}} = i(\frac{\partial}{\partial \eta^m} - \frac{\partial}{\partial \bar{\eta}^m})$ and $y^{n+k} = \frac{i}{2}(\eta^k - \bar{\eta}^k)$, a long but trivial computation gives:

$$\begin{aligned} 4D_l^k &= \frac{\partial}{\partial \eta^l}[g^{\bar{m}k}(\frac{\partial^2 L}{\partial \bar{\eta}^m \partial \bar{z}^p}\bar{\eta}^p - \frac{\partial L}{\partial \bar{z}^m})] + \\ &\quad \frac{\partial}{\partial \bar{\eta}^l}[g^{\bar{m}k}(\frac{\partial^2 L}{\partial \bar{\eta}^m \partial z^p}\eta^p + \frac{\partial^2 L}{\partial \bar{\eta}^m \partial \bar{z}^p}\bar{\eta}^p - \frac{\partial L}{\partial \bar{z}^m})] \end{aligned} \tag{2.15}$$

Hence, we have:

Proposition 2.4. *In the complex Lagrange space* (M, L) *the induced* $(r.n.c.)$ *of the* $\overset{c}{N}{}_j^k$ *(c.n.c.) from (2.13) coincides with the* $\overset{0}{N}{}_b^a$ *(r.n.c.) given by (2.12) if and only if* $D_l^k = 0$.

We recall here that a complex Lagrange space is called *local Minkowski* ([2],[3]) if there exist local charts in any $u = (z, \eta)$ such that the Lagrange function L depends only on the direction, i.e., $L = L(\eta, \bar{\eta})$.

The above Proposition and (2.15) yields:

Proposition 2.5. *If* (M, L) *is a complex Lagrange local Minkowski space there exist local charts in any* $u = (z, \eta) \in T'M$ *such that* $\overset{cR}{N}{}_b^a = \overset{0}{N}{}_b^a$.

More interesting results are obtained in the particular case of complex Finsler space, when $L(z, \lambda\eta) = |\lambda|^2 L(z, \eta)$ and the consequences from it ([1],[12]).

Then the formulas (2.12) lead to ([10]):

$$\begin{aligned} \overset{0F}{N}{}_b^a &= \frac{1}{2}\frac{\partial \gamma_{00}^a}{\partial y^b}, \text{ where } \gamma_{00}^a = \gamma_{bc}^a y^b y^c \text{ and} \\ \gamma_{bc}^a &= \frac{1}{2}\overset{R}{g}{}^{da}\{\frac{\partial \overset{R}{g}_{dc}}{\partial x^b} + \frac{\partial \overset{R}{g}_{bd}}{\partial x^c} - \frac{\partial \overset{R}{g}_{bc}}{\partial x^d}\} \end{aligned} \tag{2.16}$$

$\overset{0F}{N}{}_b^a$ is the well-known Cartan $(r.n.c.)$.

And the formulas (2.13) give the Cartan $(c.n.c.)$ ([11],[12]):

$$\begin{aligned} \overset{cF}{N}{}_j^k &= \frac{1}{2}\frac{\partial \Gamma_{00}^k}{\partial \eta^j}, \text{ where } \Gamma_{00}^k = \Gamma_{ij}^k \eta^i \eta^j \text{ with} \\ \Gamma_{ij}^k &= \frac{1}{2}g^{\bar{m}k}\{\frac{\partial g_{j\bar{m}}}{\partial z^i} + \frac{\partial g_{i\bar{m}}}{\partial z^j}\} \end{aligned} \tag{2.17}$$

Γ_{ij}^k being the first complex Christoffel symbol.

Now, acting as before, after a long computation of passing from real to complex, we obtain that $\overset{0F}{N}{}^a_b$ coincides with $\overset{cFR}{N}{}^a_b$, the real image of complex Cartan connection, if and only if the difference $d-$tensor D^k_l given by:

$$D^k_l = \frac{\partial}{\partial \eta^l}(\Gamma^k_{i\bar{j}}\eta^i\bar{\eta}^j) + \frac{\partial}{\partial \bar{\eta}^l}(\Gamma^k_{ij}\eta^i\eta^j + \Gamma^k_{i\bar{j}}\eta^i\bar{\eta}^j) \tag{2.18}$$

is vanishing, where $\Gamma^k_{i\bar{j}} = \frac{1}{2}g^{\bar{m}k}\{\frac{\partial g_{i\bar{m}}}{\partial \bar{z}^j} - \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^m}\}$ is the second Chritoffel symbol of the Levi-Civita connection on $T'M$.

Clearly, if $\Gamma^k_{i\bar{j}} = 0$, that is the Levi-Civita connection is of $(1,0)-$type, or equivalently the fact that $g_{i\bar{j}}$ is a Kähler metric, then the difference $d-$tensor is reduced to $D^k_l = \frac{\partial \Gamma^k_{ij}}{\partial \bar{\eta}^l}\eta^i\eta^j$. Therefore, we can state:

Proposition 2.6. *If $g_{i\bar{j}}$ is a Kähler metric of the complex Finsler space (M,L) and the coefficients Γ^k_{ij} of the Levi-Civita linear connection on $T'M$ are holomorphic functions, then $\overset{0F}{N}{}^a_b$ coincides with $\overset{cFR}{N}{}^a_b$.*

Also, let us note that if $g_{i\bar{j}}$ locally depends on $T'M$ only on z, i.e. $g_{i\bar{j}}(z)$ (the point is called normal cf. [3]), then the metric comes from a Hermitian metric on M. In the Kählerian situation such metric is called Hermitian-Kähler ([1], [12]). So, from the local expression of Γ^k_{ij} , we have:

Proposition 2.7. *If $g_{i\bar{j}}$ is a Hermitian-Kähler metric of the complex Finsler space (M,L), then $\overset{0F}{N}{}^a_b$ coincides with $\overset{cFR}{N}{}^a_b$.*

3. The induced $N-$real linear connection

Let us study now the real image of other geometric elements on T_RM induced from a *(c.n.c.)* on $T'M$.

Let $D : \chi(T'M)\times\chi(T'M) \to \chi(T'M)$ be a normal complex linear connection (shortly, $N-$ *(c.n.c.)*), that is a derivative law on $T'M$ which preserves the distributions $V(T'M)$, $H(T'M)$ and their conjugates. Locally, an $N-$*(c.l.c.)* is characterized by its coefficients $(L^i_{jk}\,; L^i_{j\bar{k}}\,; C^i_{jk}\,; C^i_{j\bar{k}})$, where:

$$D_{\delta_k}\delta_j = L^i_{jk}\delta_i\ ; \qquad D_{\partial_k}\partial_j = C^i_{jk}\partial_i$$

$$D_{\delta_{\bar{k}}}\delta_j = L^i_{j\bar{k}}\delta_i\ ; \qquad D_{\partial_{\bar{k}}}\partial_j = C^i_{j\bar{k}}\partial_i \tag{3.1}$$

As was proved in [11], D is an $N-$*(c.l.c.)* iff $DJ = DF = DF^* = 0$, where J is the complex structure on $T'M$, F is the natural tangent structure and F^* is the adjoint tangent structure of F with respect to the adapted base determined by a given *(c.n.c.)*N. Locally, F^* behaves on the adapted base as follow: $F^*(\delta_k) = 0$; $F^*(\partial_k) = \delta_k$; $F^*(\delta_{\bar{k}}) = 0$; $F^*(\partial_{\bar{k}}) = \delta_{\bar{k}}$, and globally defined.

Correspondingly, on $T_R M$ we have in addition to the complex structure $\overset{R}{J}$, the natural tangent structure $\overset{R}{F}$ defined by $\overset{R}{F}(X^R) = (FX)^R$, and its adjoint tangent structure $\overset{R}{F^*}$ in respect to an adapted base. Locally, their actions are given by:

$\overset{R}{F}(\frac{\delta}{\delta x^a}) = \frac{\partial}{\partial y^a}$; $\overset{R}{F}(\frac{\partial}{\partial y^a}) = 0$; $\overset{R}{F^*}(\frac{\delta}{\delta x^a}) = 0$; $\overset{R}{F^*}(\frac{\partial}{\partial y^a}) = \frac{\delta}{\delta x^a}$

Then a derivative law $\overset{R}{D}$ on $T_R M$ is an $\overset{R}{N}$ $-$real linear connection, shortly $\overset{R}{N}$ $-(r.l.c.)$, conformity to [10] iff $\overset{R}{D}\overset{R}{J}=\overset{R}{D}\overset{R}{F}=\overset{R}{D}\overset{R}{F^*}=0$.

Theorem 3.1. *If D is an $N-(c.l.c.)$ on $T'M$, then the following derivative law:*

$$\overset{R}{D}_A B = D_A B\ ,\ \ \forall\, A, B \in \chi(T_R M) \tag{3.2}$$

or in other words:

$$\overset{R}{D}_{X^R} Y^R = (D_X Y)^R + (D_X \bar{Y})^R\ ,\ \forall X, Y \in \chi(T'M) \tag{3.3}$$

is an $\overset{R}{N}$ $-(r.l.c.)$ on $T_R M$.

Proof. Let us remark that $\overset{R}{D}$ is a linear connection on $T'M$. Since $J(A+iB) = \overset{R}{J}(A) + i\overset{R}{J}(B)$ and taking into account the definitions of $\overset{R}{F}$ and $\overset{R}{F^*}$ structures it is verified that $\overset{R}{D}\overset{R}{J}=\overset{R}{D}\overset{R}{F}=\overset{R}{D}\overset{R}{F^*}=0$.

Moreover, if D is of $(1,0)-$type, because $D_{JX}Y = D_X JY$, it results that $\overset{R}{D}_{\overset{R}{J}A} B = \overset{R}{D}_A \overset{R}{J} B$.

If the $\overset{R}{N}$ $-(r.l.c.)$ $\overset{R}{D}$ is given, obviously then D is obtained by linearity.

Now, let us suppose that $\overset{R}{D}$ is given in the local base by its coefficients ([10]):

$$\overset{R}{D}_{\frac{\delta}{\delta x^c}} \frac{\delta}{\delta x^b} = \overset{R}{L}{}^a_{bc} \frac{\delta}{\delta x^a}\ ;\quad \overset{R}{D}_{\frac{\partial}{\partial y^c}} \frac{\delta}{\delta x^b} = \overset{R}{C}{}^a_{bc} \frac{\delta}{\delta x^a}$$

$$\overset{R}{D}_{\frac{\delta}{\delta x^c}} \frac{\partial}{\partial y^b} = \overset{R}{L}{}^a_{bc} \frac{\partial}{\partial y^a}\ ;\quad \overset{R}{D}_{\frac{\partial}{\partial y^c}} \frac{\partial}{\partial y^b} = \overset{R}{C}{}^a_{bc} \frac{\partial}{\partial y^a} \tag{3.4}$$

Then making the computations in (3.3), thanks to (1.6) and (2.5) formulas, we find the relation between the coefficients of induced $\overset{R}{N}$ $-(r.l.c.)$ $\overset{R}{D}$ and $N-(c.l.c)$ D :

$$\overset{R}{L}{}^i_{jk} = \overset{R}{L}{}^{n+i}_{n+jk} = Re\ (L^i_{jk} + L^i_{J\bar{k}}) \tag{3.5}$$

$$\overset{R}{L}{}^{n+i}_{jk} = -\overset{R}{L}{}^{i}_{n+jk} = Im\ (L^i_{jk} + L^i_{J\bar{k}})$$

$$\overset{R}{L}{}^{i}_{jn+k} = \overset{R}{L}{}^{n+i}_{n+jn+k} = Im\ (L^i_{j\bar{k}} - L^i_{jk})$$

$$\overset{R}{L}{}^{n+i}_{jn+k} = \overset{R}{L}{}^{i}_{n+jn+k} = Re\ (L^i_{jk} - L^i_{J\bar{k}})$$

Definition 3.1. *An* $\overset{R}{N}$ $-$*(r.l.c.)* $\overset{R}{D}$ *whose coefficients are connected by the relations:* $\overset{R}{L}{}^{i}_{jk} = \overset{R}{L}{}^{n+i}_{n+jk}$; $\overset{R}{L}{}^{n+i}_{jk} = -\overset{R}{L}{}^{i}_{n+jk}$; $\overset{R}{L}{}^{i}_{jn+k} = \overset{R}{L}{}^{n+i}_{n+jn+k}$; $\overset{R}{L}{}^{n+i}_{jn+k} = \overset{R}{L}{}^{i}_{n+jn+k}$ *will be called of Hermitian type.*

The definition is justified by the fact that for a fixed index c the coefficient $\overset{R}{L}{}^{a}_{bc}$ is the real representation of a Hermitian matrix.

Proposition 3.1. $\overset{R}{D}$ *is an* $\overset{R}{N}$ $-$*(r.l.c.) of Hermitian type if and only if* $\overset{R}{D}\overset{R}{J} = \overset{R}{J}\overset{R}{D}$.

If D is of $(1,0)-$type then formulas (3.5) are simplified, because $L^i_{j\bar{k}} = C^i_{j\bar{k}} = 0$.

The calculus of bracket gives that $\left[\overline{X}, \overline{Y}\right] = \overline{[X,Y]}$ and hence we have $\left[X^R, Y^R\right] = [X,Y]^R$. So, the curvature and the torsion of the induced $\overset{R}{N}$ $-(r.l.c.)$ $\overset{R}{D}$ are expressed as a function of the curvature and respectively the torsion of $N-(c.l.c.)D$ as follows:

$$\overset{R}{\mathbf{R}}\,(X^R, Y^R)Z^R = \mathbf{R}(\mathbf{X^R}, \mathbf{Y^R})\mathbf{Z^R}\ ;\ \overset{\mathbf{R}}{\mathbf{T}}\,(\mathbf{X^R}, \mathbf{Y^R}) = \mathbf{T}(\mathbf{X^R}, \mathbf{Y^R}) \tag{3.6}$$

The components of this curvature and torsion are directly obtained from (3.6) as a function of the real and imaginary parts of the complex curvatures and torsions.

As we have seen, the Hermitian metric G on $T'M$ is : $G = Re\ G + iIm\ G = \overset{R}{G} + i\,\overset{R}{\tilde{G}}$.

If D is a metrical $N-(c.l.c.)$, i.e. $(D_X G)(Y,Z) = XG(Y,Z) - G(D_X Y, Z) - G(Y, D_X Z) = 0$, replacing X, Y, Z to their real parts, we obtain that $\overset{R}{D}\overset{R}{G} = \overset{R}{D}\overset{R}{\tilde{G}} = 0$. Therefore, $\overset{R}{D}$ is metrical with respect to both real metric induced by G from $T'M$.

In a real case ([10]), and in the complex one ([11]), are known metrical $N-$linear connections. In a real Lagrange geometry one has a special meaning, the so-called real canonical, or Miron's metric $\overset{R}{N}$ $-(r.l.c.)$:

$$\overset{cR}{L}{}^{a}_{bc} = \frac{1}{2}\,\overset{R}{g}{}^{da}\,\left\{\frac{\delta\,\overset{R}{g}_{dc}}{\delta x^b} + \frac{\delta\,\overset{R}{g}_{bd}}{\delta x^c} - \frac{\delta\,\overset{R}{g}_{bc}}{\delta x^d}\right\} \tag{3.7}$$

$$\overset{cR}{C}{}^{a}_{bc}=\frac{1}{2}\overset{R}{g}{}^{da}\{\frac{\partial \overset{R}{g}_{dc}}{\partial y^{b}}+\frac{\partial \overset{R}{g}_{bd}}{\partial y^{c}}-\frac{\partial \overset{R}{g}_{bc}}{\partial y^{d}}\}$$

In the complex case we know the next metric $N-(c.l.c.)$:

-The complex canonical connection([11]):

$$\overset{cC}{L}{}^{i}_{jk}=\frac{1}{2}g^{\bar{l}i}(\frac{\delta g_{j\bar{l}}}{\delta z^{k}}+\frac{\delta g_{k\bar{l}}}{\delta z^{j}})\quad ;\quad \overset{cC}{C}{}^{i}_{jk}=\frac{1}{2}g^{\bar{l}i}(\frac{\partial g_{j\bar{l}}}{\partial \eta^{k}}+\frac{\partial g_{k\bar{l}}}{\partial \eta^{j}})=g^{\bar{l}i}\frac{\partial g_{j\bar{l}}}{\partial \eta^{k}}$$

$$\overset{cC}{L}{}^{\bar{i}}_{\bar{j}k}=\frac{1}{2}g^{\bar{i}l}(\frac{\delta g_{l\bar{j}}}{\delta z^{k}}-\frac{\delta g_{k\bar{j}}}{\delta z^{l}})\quad ;\quad \overset{cC}{C}{}^{\bar{i}}_{\bar{j}k}=\frac{1}{2}g^{\bar{i}l}(\frac{\partial g_{l\bar{j}}}{\partial \eta^{k}}-\frac{\partial g_{k\bar{j}}}{\partial \eta^{l}})=0 \tag{3.8}$$

-The Chern-Finsler complex connection ([1],[2]...), for the special Finsler case:

$$\overset{KF}{L}{}^{i}_{jk}=g^{\bar{l}i}\frac{\delta g_{j\bar{l}}}{\delta z^{k}}\ ;\ \overset{KF}{L}{}^{i}_{j\bar{k}}=\overset{KF}{C}{}^{i}_{j\bar{k}}=0\ ;\ \overset{KF}{C}{}^{i}_{jk}=\overset{cC}{C}{}^{i}_{jk} \tag{3.9}$$

(A comparative analysis of these is to be find in [12])

Let $\overset{R}{N}$ be the induced $(r.n.c.)$ of the $(c.n.c.)N$. Because $g_{j\bar{k}}=\overset{R}{g}_{jk}+i\overset{R}{g}_{jn+k}$, $g^{\bar{l}i}=\overset{R}{g}{}^{li}+i\overset{R}{g}{}^{ln+i}$ and $\frac{\delta}{\delta z^{k}}=\frac{1}{2}\{\frac{\delta}{\delta x^{k}}-i\frac{\delta}{\delta x^{n+k}}\}$, developing the computation in (3.8) and then suitably grouping of terms, after a straightforward computation we obtain that:

$$Re(\overset{cC}{L}{}^{i}_{jk})=\frac{1}{2}(\overset{cR}{L}{}^{i}_{jk}-\overset{cR}{L}{}^{i}_{n+jn+k}) \tag{3.10}$$

$$Im(\overset{cC}{L}{}^{i}_{jk})=\frac{1}{2}(\overset{cR}{L}{}^{n+i}_{jk}-\overset{cR}{L}{}^{n+i}_{n+jn+k})$$

$$Re(\overset{cC}{L}{}^{i}_{j\bar{k}})=\frac{1}{2}(\overset{cR}{L}{}^{i}_{jk}-\overset{cR}{L}{}^{n+i}_{jn+k})$$

$$Im(\overset{cC}{L}{}^{i}_{j\bar{k}})=\frac{1}{2}(\overset{cR}{L}{}^{n+i}_{jk}+\overset{cR}{L}{}^{i}_{jn+k})$$

and analogous formulas for the $\overset{cC}{C}{}^{i}_{jk}$ and $\overset{cC}{C}{}^{i}_{j\bar{k}}$, where in addition we have that $\overset{cR}{C}{}^{i}_{jk}=\overset{cR}{C}{}^{n+i}_{jn+k}$ and $\overset{cR}{C}{}^{n+i}_{jk}=-\overset{cR}{C}{}^{i}_{jn+k}$.

Now, corroborate the (3.10) and (3.5) formulas, we have the next correspondence between the real image of canonical $(c.l.c.)$ and canonical $(r.l.c.)$:

$$\overset{RcC}{L}{}^{i}_{jk}=\overset{cR}{L}{}^{i}_{jk}-\frac{1}{2}(\overset{cR}{L}{}^{i}_{n+jn+k}+\overset{cR}{L}{}^{n+i}_{jn+k})$$

$$\overset{RcC}{L}{}^{n+i}_{jk}=\overset{cR}{L}{}^{n+i}_{jk}-\frac{1}{2}(\overset{cR}{L}{}^{n+i}_{n+jn+k}-\overset{cR}{L}{}^{i}_{jn+k})$$

$$\overset{RcC}{L}{}^{i}_{jn+k}= \frac{1}{2}(\overset{cR}{L}{}^{i}_{jn+k} + \overset{cR}{L}{}^{n+i}_{n+jn+k})$$

$$\overset{RcC}{L}{}^{n+i}_{jn+k}= \frac{1}{2}(\overset{cR}{L}{}^{n+i}_{jn+k} - \overset{cR}{L}{}^{i}_{n+jn+k}) \tag{3.11}$$

and analogous formulas for the coefficients $\overset{RcC}{C}{}^{a}_{bc}$.

Proposition 3.2. *The real image* $\overset{RcC}{D}$ *of the canonical* $N-(c.l.c.)$ $\overset{cC}{D}$ *coincides with the real canonical (Miron's) connection* $\overset{cR}{D}$*,in respect to the adapted base of induced* $(r.n.c.)$ $\overset{R}{N}$*, if and only if* $\overset{cR}{D}$ *is of Hermitian type.*

Finally, let us consider again the complex structure $\overset{R}{J}$ on T_RM and the metric structure given by (2.8) for a fixed $(c.n.c.)$. It is easy to verify that $(T_RM, \overset{R}{J}, \overset{R}{G})$ is a Hermitian manifold in which $\overset{cR}{D}$ is a metrical $N-(r.l.c.)$. Next we assume in addition that $(T_RM, \overset{R}{J}, \overset{R}{G}, \overset{cR}{D})$ is a real Kähler space. Then, by the fact that $\overset{cR}{D}$ has zero torsion and because $\overset{cR}{D}$ coincides in this circumstances with $\overset{RcC}{D}$, it results that $\overset{cC}{D}$ has too zero torsion and hence, (M, L) is a complex Hermitian-Kähler space ([1],[12]). By the above Proposition and because the complex torsion vanishes if and only if the torsion of the induced $(r.l.c.)$ vanishes, the converse assertion is true.

So, really we have proved that:

Theorem 3.2. (M, L) *is a complex Hermitian-Kähler space if and only if* $(T_RM,$ *is a real Kähler manifold.*

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