Novi Sad J. Math.
Vol. 32, No. 2, 2002, 131-140

ON SOME CLASSES OF GOOD QUOTIENT RELATIONS

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Abstract. The notion of a good quotient relation has been introduced as an attempt to generalize the notion of a quotient algebra to relations on an algebra which are not necessarily congruence relations. In this paper we investigate some special classes of good relations for which the generalized versions of the well-known isomorphism theorems can be proved.

AMS Mathematics Subject Classification (2000): 08A30,08A99

Key words and phrases: power algebra, good relation, quasi-congruence, isomorphism theorems

1. Introduction

The notion of a *generalized quotient algebra* and the corresponding notion of a *good (quotient) relation* has been introduced in [6] and [7] as an attempt to generalize the notion of a quotient algebra to relations on an algebra which are not necessarily congruences. From Definition 1 it is easy to see that every non-trivial algebra has good relations which are not congruences.

Up to now, most of the results on good relations have been obtained in the context of power structures (see [4] for an overview on power structures). One of the reasons is that for any relation $R \subseteq A^2$, the generalized quotient set A/R is a subset of the power set $\mathcal{P}(A)$. A "power version" of the well-known homomorphism theorem was proved in [4]. This theorem was the motivation for introducing the notions of very good ([6],[7]), Hoare good and Smyth good relations ([5]). For these special good relations some versions of the "power" homomorphism theorems were proved in [5]. In [2] the relationships between Hoare good, Smyth good and very good relations were described. In [3], these sets of special good relations have been investigated with respect to the set-theoretical operations and various ways of powering.

In the present paper we introduce and study some special good relations for which the well-known isomorphism theorems can be proved. These sets of good relations are also studied from the lattice-theoretical point of view.

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2. Definitions and the isomorphism theorems

Definition 1. *([6], [7]) Let $\mathcal{A} = \langle A, F\rangle$ be an algebra and $R \subseteq A^2$.*

(1) For any $a \in A$ we define $a/R = \{b \mid bRa\}$. The corresponding **generalized quotient set** *is $A/R = \{a/R \mid a \in A\}$.*
(2) Relation $\varepsilon(R) \subseteq A^2$ is defined by:

$$(a, b) \in \varepsilon(R) \iff a/R = b/R.$$

(3) We call R a **good (quotient) relation** *on $\mathcal{A}$ if $\varepsilon(R)$ is a congruence on $\mathcal{A}$. The set of all good relations on $\mathcal{A}$ we denote by $G(\mathcal{A})$.*
(4) If R is a good quotient relation on $\mathcal{A}$, the corresponding **generalized quotient algebra** *$\mathcal{A}/R$ is*

$$\mathcal{A}/R = \langle A/R, \{\ulcorner f \urcorner \mid f \in F\}\rangle,$$

where operations $\ulcorner f \urcorner$ $(f \in F)$ are defined in the following way: if $ar(f) = n$, then for any $a_1, \ldots, a_n \in A$

$$\ulcorner f \urcorner(a_1/R, \ldots, a_n/R) = f(a_1, \ldots, a_n)/R.$$

Some examples of good relations are: partial orders, compatible quasiorders, structure preserving relations (see [4]), and quasi-congruences (see Definition 2).

In [3] the set $G(\mathcal{A})$ of all good relations on an algebra $\mathcal{A}$ is described and a necessary and sufficient condition is given for the partially ordered set $\mathcal{G}(\mathcal{A}) = \langle G(\mathcal{A}), \subseteq\rangle$ to be a lattice.

It can be easily proved that the following "extended" homomorphism theorem holds for good quotient relations.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type. Then $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$ if and only if there is a good relation R on $\mathcal{A}$ such that $\mathcal{B} \cong \mathcal{A}/R$.*

In [4] and [5] some "power" versions of the homomorphism theorem were proved (for congruences and for Hoare good, Smyth good and very good relations). In [1] some isomorphism theorems for the so-called regular power rings can be found. Unfortunately, the well-known isomorphism theorems cannot be generalized to the whole class of good relations.

Example 1. Let $A = \{0, 1, 2, 3, 4\}$, $R = \{(0, 2), (1, 3), (2, 4), (3, 1), (4, 0)\}$, $f(0) = 1$, $f(1) = 2$, $f(2) = 0$, $f(3) = 4$, $f(4) = 3$. Then R is a good relation on $\mathcal{A} = \langle A, f\rangle$, while for $B = \{0, 1, 2\}$, $R_{|B} = R \cap B^2$ is not a good relation on subalgebra $\mathcal{B}$ of $\mathcal{A}$.

Example 2. Let $\mathcal{A} = \langle A, F \rangle$ be an algebra with all relations on it being good (such algebras are described in [3]), and $a, b, c \in A$. If $R = \{(a,a), (a,b), (a,c)\}$, then (with the standard definition of a quotient relation) relation R/R is not well defined since $(a,c) \in R \Rightarrow (a/R, c/R) \in R/R$ and, on the other hand $(b,c) \notin R \Rightarrow (a/R, c/R) = (b/R, c/R) \notin R/R$.

In the sequel, we are searching for smaller classes of good relations for which some generalized versions of the isomorphism theorems could be proved. The following classes proved to be suitable.

Definition 2. *Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$.*

(1) ([5]) We call R a **quasi-equivalence** *on A if for all x,$y \in A$*

$$x/R = y/R \Leftrightarrow xRy \;\&\; yRx.$$

We denote the set of all quasi-equivalences on A by $QEqvA$.

(2) We call R a **quasi-congruence** *on $\mathcal{A}$ if R is a good quasi-equivalence. The set of all quasi-congruences on $\mathcal{A}$ we denote by $QCon\mathcal{A}$.*

Let us note that the expression "quasi-congruence" in [5] is used for compatible quasi-equivalences (which are also good relations). We will call these quasi-congruences *B-quasi-congruences* (Brink's quasi-congruences). The set of all B-quasi-congruences on $\mathcal{A}$ we will denote by $BQCon\mathcal{A}$.

Definition 3. *Let $\mathcal{A}$ be an algebra and $R \in QCon\mathcal{A}$. We call R a* **two-side quasi-congruence** *on $\mathcal{A}$ if for all x,y,$z \in A$*

$$xRy \;\&\; yRx \;\&\; xRz \Rightarrow yRz.$$

The set of all two-side quasi-congruences on $\mathcal{A}$ we denote by $TSQCon\mathcal{A}$.

Example 3. Every reflexive and transitive relation on A is a quasi-equivalence on A. Every reflexive and anti-symmetric relation on A is a two-side quasi-congruence on arbitrary algebra $\mathcal{A}$ with the carrier set A.

In general, $TSQCon\mathcal{A}$ is a proper subset of $QCon\mathcal{A}$ and incomparable with $BQCon\mathcal{A}$.

Example 4. Let $A = \{a,b,c\}$, $f(a) = f(b) = a$, $f(c) = c$, $\mathcal{A} = \langle A, f \rangle$, $R = \{(a,a), (b,a), (a,b), (b,b), (a,c), (c,c)\}$, $S = \{(a,a), (b,b), (b,c), (c,c)\}$.
Then $R \in BQCon\mathcal{A} \setminus TSQCon\mathcal{A}$ and $S \in TSQCon\mathcal{A} \setminus BQCon\mathcal{A}$.

Lemma 1. *Let $\mathcal{A} = \langle A, F \rangle$ be an algebra and $R \in QEqvA$. The following conditions are equivalent:*

(1) $R \in QCon\mathcal{A}$

(2) For any $f \in F$, if $ar(f) = n$ then for any $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$

$$(\forall i \leq n)(x_i R y_i \ \&\ y_i R x_i) \Rightarrow (f(x_1, \ldots, x_n) R f(y_1, \ldots, y_n) \ \&$$

$$f(y_1, \ldots, y_n) R f(x_1, \ldots, x_n))$$

Definition 4. *Let $\mathcal{A}$ be an algebra, B a subuniverse of $\mathcal{A}$, $R \subseteq A^2$. We define $B^R \subseteq A$ as*

$$B^R = \{a \in A \mid (\exists b \in B)\ a/R = b/R\}.$$

Lemma 2. *Let $\mathcal{A}$ be an algebra and $\mathcal{B}$ a subalgebra of $\mathcal{A}$. Then*
a) If $R \in QCon\mathcal{A}$ then $R_{|B} \in QCon\mathcal{B}$.
b) If $R \in G(\mathcal{A})$ then B^R is a subuniverse of $\mathcal{A}$.

If $R \in G(\mathcal{A})$ and $\mathcal{B} \leq \mathcal{A}$ then the subalgebra of $\mathcal{A}$ with the universe B^R we denote by $\mathcal{B}^R$.

Theorem 2. *Let $\mathcal{A}$ be an algebra, $\mathcal{B} \leq \mathcal{A}$ and $R \in QConA$. Then*

$$\mathcal{B}/R_{|B} \cong \mathcal{B}^R/R_{|B^R}.$$

Proof. According to Lemma 2, algebras $\mathcal{B}/R_{|B}$ and $\mathcal{B}^R/R_{|B^R}$ are well defined. Let $\Phi : B/R_{|B} \to B^R/R_{|B^R}$ be a mapping defined in the natural way:

$$\Phi(b/R_{|B}) = b/R_{|B^R}.$$

It is easy to prove that Φ is an isomorphism. □

Definition 5. *Let $\mathcal{A}$ be an algebra, $R, S \in TSQCon\mathcal{A}$, $R \subseteq S$. The relation $S/R \subseteq A/R \times A/R$ is defined as*

$$(a/R, b/R) \in S/R \Leftrightarrow (a, b) \in S.$$

Lemma 3. *Let $\mathcal{A}$ be an algebra, $R, S \in TSQCon\mathcal{A}$, $R \subseteq S$. Then $S/R \in TSQCon(\mathcal{A}/R)$.*

Theorem 3. *Let $\mathcal{A}$ be an algebra, $R, S \in TSQCon\mathcal{A}$, $R \subseteq S$. Then*

$$\mathcal{A}/R \Big/ S/R \cong \mathcal{A}/S.$$

Proof. According to Lemma 3, the algebra $\mathcal{A}/R \Big/ S/R$ is well defined. Let us define a mapping $\Phi : A/R \Big/ S/R \to A/S$ in the natural way:

$$\Phi(a/R \Big/ S/R) = a/S.$$

Then it is easy to prove that Φ is an isomorphism. □

3. The lattices of quasi-congruences

In [5] it is proved that for any algebra $\mathcal{A}$, the set $BQCon\mathcal{A}$ is closed under arbitrary intersections. Hence $\langle BQCon\mathcal{A}, \subseteq\rangle$ is a complete lattice. It is easy to see that the same is true for the sets $QEqvA$, $QCon\mathcal{A}$ and $TSQCon\mathcal{A}$. We can prove even more.

Theorem 4. *Let $\mathcal{A}$ be an algebra with the carrier set A. Then $\langle QEqvA, \subseteq\rangle$, $\langle QCon\mathcal{A}, \subseteq\rangle$, $\langle BQCon\mathcal{A}, \subseteq\rangle$, $\langle TSQCon\mathcal{A}, \subseteq\rangle$ are algebraic lattices.*

Proof. Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$. In all cases we will define an algebra $\mathcal{B}$ with the universe $A \times A$ such that $Sub\mathcal{B}$ (the set of all subuniverses of $\mathcal{B}$) will be equal to the corresponding set of quasi-equivalences. Since $\langle Sub\mathcal{B}, \subseteq\rangle$ is an algebraic lattice, the proposition holds.

- Case $\langle QEqvA, \subseteq\rangle$:
  It is not hard to see that $R \subseteq A^2$ is a quasi-equivalence if and only if R is reflexive and for all x,y,$z \in A$ it holds
  $$xRy \ \&\ yRx \ \&\ zRx \Rightarrow zRy.$$
  Hence the fundamental operations of $\mathcal{B}$ will be:
  - the nullary operations (a, a) for every $a \in A$;
  - a ternary operation $t : B^3 \to B$ defined by
  $$t((a_1, a_2), (b_1, b_2), (c_1, c_2)) = \begin{cases} (c_1, a_2) & \text{if } a_1 = b_2 = c_2 \ \&\ a_2 = b_1 \\ (a_1, a_2) & \text{else} \end{cases}$$
  Then $Sub\mathcal{B} = QEqvA$ and $\langle QEqvA, \subseteq\rangle$ is an algebraic lattice.

- Case $\langle BQCon\mathcal{A}, \subseteq\rangle$:
  The fundamental operations of the algebra $\mathcal{B}$ are all operations defined in the previous case and for all $f \in \mathcal{F}$ operations $f^{\mathcal{B}}$ defined in the following way: if $ar(f) = n$, then $ar(f^{\mathcal{B}}) = n$ and for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ it holds
  $$f^{\mathcal{B}}((a_1, b_1), \ldots, (a_n, b_n)) = (f^{\mathcal{A}}(a_1, \ldots, a_n), f^{\mathcal{A}}(b_1, \ldots, b_n)).$$
  Then $Sub\mathcal{B} = BQCon\mathcal{A}$ and $\langle BQCon\mathcal{A}, \subseteq\rangle$ is an algebraic lattice.

- Case $\langle QCon\mathcal{A}, \subseteq\rangle$:
  The fundamental operations of the algebra $\mathcal{B}$ are all operations defined in the case $QEqvA$ and for all $f \in \mathcal{F}$ operations $f_1^{\mathcal{B}}$, $f_2^{\mathcal{B}}$ defined in the following way: if $ar(f) = n$, then $ar(f_1^{\mathcal{B}}) = ar(f_2^{\mathcal{B}}) = 2n$ and for all $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots, d_n \in A$ it holds
  $$f_1^{\mathcal{B}}((a_1, b_1), \ldots, (a_n, b_n), (c_1, d_1), \ldots, (c_n, d_n)) =$$

$$\begin{cases} (f^{\mathcal{A}}(a_1,\dots,a_n), f^{\mathcal{A}}(b_1,\dots,b_n)) & \text{if } c_i = b_i \ \&\ d_i = a_i \ \text{ for all } i \le n \\ (a_1,b_1) & \text{else} \end{cases}$$

$$f_2^{\mathcal{B}}((a_1,b_1),\dots,(a_n,b_n),(c_1,d_1),\dots,(c_n,d_n)) =$$

$$\begin{cases} (f^{\mathcal{A}}(b_1,\dots,b_n), f^{\mathcal{A}}(a_1,\dots,a_n)) & \text{if } c_i = b_i \ \&\ d_i = a_i \ \text{ for all } i \le n \\ (a_1,b_1) & \text{else} \end{cases}$$

According to Lemma 1, $Sub\mathcal{B} = QCon\mathcal{A}$ and $\langle QCon\mathcal{A}, \subseteq\rangle$ is an algebraic lattice.

- Case $\langle TSQCon\mathcal{A}, \subseteq\rangle$:
The fundamental operations of the algebra $\mathcal{B}$ are all operations defined in the previous case and a ternary operation t' defined in the following way:

$$t'((a_1,a_2),(b_1,b_2),(c_1,c_2)) = \begin{cases} (a_2,c_2) & \text{if } b_1 = a_2 \ \&\ b_2 = c_1 = a_1 \\ (a_1,a_2) & \text{else} \end{cases}$$

Then $Sub\mathcal{B} = TSQCon\mathcal{A}$ and $\langle TSQCon\mathcal{A}, \subseteq\rangle$ is an algebraic lattice. $\square$

Let us note that $\langle G(\mathcal{A}), \subseteq\rangle$ is almost never a lattice (see [3]).

In [5] it is proved that $\langle Con\mathcal{A}, \subseteq\rangle$ is a sublattice of $BQCon\mathcal{A}$. We can prove that it is also a sublattice of $QEqvA$,$QCon\mathcal{A}$ and $TSQCon\mathcal{A}$.

Lemma 4. *([5])*

(1) For every R,S $\in QEqvA$ it holds

$$R \subseteq S \Rightarrow \varepsilon(R) \subseteq \varepsilon(S).$$

(2) For every $R \in QEqvA$ it holds $\varepsilon(R) \subseteq R$.

Theorem 5. *Let Σ be a set of quasi-congruences of some algebra $\mathcal{A}$ such that $Con\mathcal{A} \subseteq \Sigma$ and Σ is closed under (finite) intersections. If $\langle \Sigma, \subseteq\rangle$ is a lattice, then $\langle Con\mathcal{A}, \subseteq\rangle$ is a sublattice of $\langle \Sigma, \subseteq\rangle$.*

Proof. Since Σ is closed under intersections, the infimum in $\langle \Sigma, \subseteq\rangle$ coincides with the intersection, like in $Con\mathcal{A}$. Therefore, we only have to prove that supremums coincide in these two lattices.

Let $\rho, \sigma \in Con\mathcal{A}$ and $\theta = \sup_\Sigma(\rho,\sigma)$. Let us note that it is sufficient to prove that θ is a congruence relation, for this implies $\theta = \sup_{Con\mathcal{A}}(\rho,\sigma)$. According to Lemma 4(1), since $\rho \subseteq \theta$ and $\sigma \subseteq \theta$ and all these relations are quasi-equivalences, it holds $\varepsilon(\rho) \subseteq \varepsilon(\theta)$ and $\varepsilon(\sigma) \subseteq \varepsilon(\theta)$. But ρ,$\sigma \in Con\mathcal{A}$ which implies $\varepsilon(\rho) = \rho$ and $\varepsilon(\sigma) = \sigma$, i.e. $\rho \subseteq \varepsilon(\theta)$ and $\sigma \subseteq \varepsilon(\theta)$. Since $\theta \in QCon\mathcal{A} \subseteq G(\mathcal{A})$, we have $\varepsilon(\theta) \in Con\mathcal{A}$, which implies $\theta \subseteq \varepsilon(\theta)$. According to Lemma 4(2) it holds $\varepsilon(\theta) \subseteq \theta$, so $\varepsilon(\theta) = \theta$ which means $\theta \in Con\mathcal{A}$. $\square$

Corollary 1. *For any algebra $\mathcal{A}$, $\langle Con\mathcal{A}, \subseteq\rangle$ is a sublattice of the following lattices: $\langle QCon\mathcal{A}, \subseteq\rangle$, $\langle BQCon\mathcal{A}, \subseteq\rangle$, $\langle TSQCon\mathcal{A}, \subseteq\rangle$.*

Proof. Follows from Theorem 2. □

Corollary 2. *For any algebra $\mathcal{A} = \langle A, F\rangle$, $\langle Con\mathcal{A}, \subseteq\rangle$ is a sublattice of $\langle QEqvA, \subseteq\rangle$.*

Proof. Let $\mathcal{B}$ be an algebra with the carrier set A such that all fundamental operations of $\mathcal{B}$ are projections. Then $Con\mathcal{B} = EqvA$ and $BQCon\mathcal{B} = QEqvA$. Therefore, according to Corollary 1, $EqvA$ is a sublattice of $QEqvA$. Since $Con\mathcal{A}$ is a sublattice of $EqvA$ for any algebra $\mathcal{A} = \langle A, F\rangle$, this implies that $Con\mathcal{A}$ is a sublattice of $QEqvA$. □

The above two corollaries cover all the cases when one of the lattices of quasi-congruences is a sublattice of another.

Example 5. Let $A = \{a, b, c\}$, $f(a) = f(c) = c$,$f(b) = b$, $\mathcal{A} = \langle A, f\rangle$, $R = \{(a, b)\} \cup \Delta$, $S = \{(b, a)\} \cup \Delta$. Then R,S,$R \cup S \in QEqvA$ and $\sup(R, S) = R \cup S$. On the other hand, R,$S \in QCon\mathcal{A}$ but $R \cup S \notin QCon\mathcal{A}$, for $(a, b) \in R \cup S$, $(b, a) \in R \cup S$ but $(f(a), f(b)) \notin R \cup S$.

Example 6. Let $A = \{a, b, c, d, 1, 2, 3, 4\}$, $\mathcal{A} = \langle A, f\rangle$ where $f : A^2 \to A$ is defined in the following way:
$f(a, c) = 1$, $f(b, c) = 2$, $f(a, d) = 3$, $f(b, d) = 4$ and $f(x, y) = 1$ for all other $(x, y) \in A^2$.
If $R = \{(a, b), (1, 2), (3, 4)\} \cup \Delta$, $S = \{(c, d), (1, 3), (2, 4)\} \cup \Delta$, then $R, S \in BQCon\mathcal{A}$, $R \cup S \in QEqvA$, but $R \cup S \notin BQCon\mathcal{A}$ because $(a, b) \in R \cup S$, $(c, d) \in R \cup S$, but $(f(a, c), f(b, d)) = (1, 4) \notin R \cup S$.

Theorem 6.

1. *Lattices $QCon\mathcal{A}$,$BQCon\mathcal{A}$,$TSQCon\mathcal{A}$ are not necessarily sublattices of $QEqvA$.*
2. *Lattices $BQCon\mathcal{A}$ and $TSQCon\mathcal{A}$ are not necessarily sublattices of $QCon\mathcal{A}$.*

Proof.

1. Example 5 shows that $QCon\mathcal{A}$ does not have to be a sublattice of $QEqvA$. This example also shows that $TSQCon\mathcal{A}$ is not necessarily sublattice of $QEqvA$ for R,S,$R \cup S \in QEqvA$, R,$S \in TSQCon\mathcal{A}$, but $R \cup S \notin TSQCon\mathcal{A}$.
   Example 6 shows that $BQCon\mathcal{A}$ does not have to be a sublattice of $QEqvA$.

2. Example 6 also shows that $BQCon\mathcal{A}$ is not necessarily a sublattice of $QCon\mathcal{A}$ for R,$S \in BQCon\mathcal{A}$, $R \cup S \in QCon\mathcal{A}$, but $R \cup S \notin BQCon\mathcal{A}$.
   And finally, let $\mathcal{A}$ be an algebra from Example 6 and $R = \{(a, b), (1, 2), (3, 4)\} \cup \Delta$, $S = \{c, d), (1, 3), (2, 4), (2, 1)\} \cup \Delta$. Then R,$S \in TSQCon\mathcal{A}$, $R \cup S \in QCon\mathcal{A}$, but $R \cup S \notin TSQCon\mathcal{A}$ because $(1, 2) \in R \cup S$, $(2, 1) \in R \cup S$, $(1, 3) \in R \cup S$ and $(2, 3) \notin R \cup S$. □

There exists algebras $\mathcal{A} = \langle A, F\rangle$ such that $QEqvA = QCon\mathcal{A}$, or $QEqvA = BQCon\mathcal{A}$ and it is not difficult to describe them.

Theorem 7. *Let $\mathcal{A}$ be an algebra and $|A| \geq 3$. The following conditions are equivalent:*

(1) $QEqvA = QCon\mathcal{A}$,
(2) $QEqvA = BQCon\mathcal{A}$,
(3) $EqvA = Con\mathcal{A}$.

Proof.
(1) $\Rightarrow$ (3)
If $QEqvA = QCon\mathcal{A}$ then $QEqvA \subseteq G(\mathcal{A})$ and consequently $EqvA \subseteq G(\mathcal{A})$. This implies $\varepsilon(R)$ is a congruence for every $R \in EqvA$, and since $\varepsilon(R) = R$, we conclude $EqvA = Con\mathcal{A}$.
(3) $\Rightarrow$ (2)
Let $EqvA = Con\mathcal{A}$. It is known that all the fundamental operations of $\mathcal{A}$ have to be projections or constant operations. Then, every reflexive relation is compatible on $\mathcal{A}$, so $QEqvA = BQCon\mathcal{A}$.
(2) $\Rightarrow$ (1)
Obvious. □

If $|A| = 2$, conditions (1) and (3) hold for any algebra $\mathcal{A}$, but (2) is not always true.

Another problem that might be interesting is to describe algebras $\mathcal{A}$ such that $BQCon\mathcal{A} = Con\mathcal{A}$ (B-quasi-congruence-trivial algebras). Of course, one can easily notice that for any non-trivial algebra $\mathcal{A}$, $QCon\mathcal{A} \neq Con\mathcal{A}$ and $TSQCon\mathcal{A} \neq Con\mathcal{A}$.

Lemma 5. *Let $\mathcal{A}$ be an algebra which has a term $p(x, y, z)$ such that at least two of the following identities are satisfied on $\mathcal{A}$:*

$$p(x, x, y) \approx y, \;\; p(x, y, x) \approx y, \;\; p(y, x, x) \approx y.$$

Then $BQCon\mathcal{A} = Con\mathcal{A}$.

Proof. It is easy to see that every symmetric quasi-equivalence is an equivalence relation as well. Now, suppose, for example that $p(x, x, y) \approx p(x, y, x) \approx y$ holds on $\mathcal{A}$. Then, for every $R \in BQCon\mathcal{A}$ we have

$$(xRy \;\&\; xRx \;\&\; yRy) \Rightarrow p(x, x, y)Rp(y, x, y) \Rightarrow yRx.$$

Hence $R \in Con\mathcal{A}$. The proof is analogous in two remaining cases. □

Theorem 8. *Let $\mathcal{A}$ be an algebra which has a Mal'cev term (or a minority term). Then $BQCon\mathcal{A} = Con\mathcal{A}$.*

Proof. If $p(x,y,z)$ is a Mal'cev term on $\mathcal{A}$ then $\mathcal{A}$ satisfies identities $p(x,x,y) \approx y$, $p(x,y,y) \approx x$. Therefore, the conditions of Lemma 3 are satisfied. A term $m(x,y,z)$ is a minority term on $\mathcal{A}$ if $\mathcal{A}$ satisfies identities $m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx y$ and the theorem is a consequence of Lemma 3 again. □

Corollary 3. *Let V be a congruence permutable variety. Then for any $\mathcal{A} \in V$, $BQCon\mathcal{A} = Con\mathcal{A}$.*

Proof. Follows from Theorem 6 and the well-known theorem of Mal'cev on the congruence permutable varieties. □

The following example shows that there exist B-quasi-congruence-trivial algebras that are not congruence permutable.

Example 7. Let $A = \{a,b,c\}$, $f(a) = f(c) = b$, $f(b) = a$, $g(a) = g(b) = c$, $g(c) = a$, $\mathcal{A} = \langle A, \{f,g\}\rangle$. If $R \in BQCon\mathcal{A}$, the following is obvious:

$$(a,b) \in R \Leftrightarrow (b,a) \in R, \quad (a,c) \in R \Leftrightarrow (c,a) \in R.$$

Also, if $(b,c) \in R$ then

(1) $$(f(b), f(c)) = (a,b) \in R;$$

(2) $$(g(b), g(c)) = (c,a) \in R;$$

(3) $$(f^2(b), f^2(c)) = (b,a) \in R.$$

It follows directly from (1),(2),(3) and the definition of quasi-equivalence that $(c,b) \in R$. We can prove in a similar way that $(c,b) \in R \Rightarrow (b,c) \in R$. Thus every B-quasi-congruence on $\mathcal{A}$ is symmetric, which implies $\mathcal{A}$ is B-quasi-congruence-trivial.
Since $Con\mathcal{A}$ is a 4-element set, it is easy to check that $\mathcal{A}$ is not congruence permutable.

Example 8. Let $\mathcal{A} = \langle\{a,b\}, \vee, \wedge\rangle$ be a two-element chain. Then $R = \{(a,a), (a,b), (b,b)\}$ is a B-quasi-congruence which is not a congruence relation. Hence the variety of all lattices is an example of a congruence-distributive variety which is not B-quasi-congruence trivial.

The following question remains to be answered: is it true that for any algebraic lattice $\mathcal{L}$, there is an algebra $\mathcal{A}$ such that $\mathcal{L} \cong BQCon\mathcal{A}$?

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Received by the editors September 3, 1999.