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ON CONVEX OPERATOR FOR (p, q)-ANALYTIC FUNCTIONS

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Abstract. An analogue for multiplication of $(p, q)-$analytic functions, called convolution operator, is defined which retains many of the usual algebraic properties of the classical multiplier of functions of a continuous variable.

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1. Introduction

In 1993, the first author [7] introduced the concept $(p, q)-$analyticity by considering functions defined on the following geometric lattice:

$$K = \{(p^m x_0, q^n y_0) : \ \ m, n \in Z, \ \text{ the set of integers}, \\ 0 < p < 1, \ 0 < q < 1, \ (x_0, y_0) \text{ fixed}, x_0 > 0, \ y_0 > 0\}. \tag{1.1}$$

We recall here some of the definitions given in [7].

Definition 1. *The 'discrete plane' Q' with respect to some fixed point $z' = (x', y')$ in the first quadrant, is defined by the set of lattice points,*

$$Q' = \{(p^m x', q^n y') : \ \ m, n \in Z \ \ \ \textit{the set of integers}\}.$$

Definition 2. *Two lattice points $z_i, z_{i+1} \in Q'$ are said to be 'adjacent' if z_{i+1} is one of*

$$(px_i, y_i), \ (p^{-1}x_i, y_i), \ (x_i, qy_i) \ \ or \ \ (x_i, q^{-1}y_i).$$

Definition 3. *A ' discrete curve' C in Q' connecting z_0 to z_n is denoted by the sequence*

$$C \equiv \langle z_0, z_1, \ldots, z_n \rangle,$$

where z_i, z_{i+1}; $i = 0, 1, \ldots, (n-1)$ are adjacent points of Q'.

If the points are distinct ($z_i \neq z_j$; $i \neq j$) then the discrete curve C is said to be 'simple'.

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Definition 4. *A 'discrete closed curve' C in Q' is given by the sequence $\langle z_0, z_1, z_2, \dots, z_n\rangle$ where $\langle z_0, z_1, \dots, z_{n-1}\rangle$ is simple and $z_0 = z_n$.*

Denote by $\bar{C}$ the continuous closed curve formed by joining adjacent points of the discrete closed curve C. Then $\bar{C}$ encloses certain points of Q', denoted by Int(C).

Definition 5. *A 'finite discrete domain' B is defined as*

$$B = \{z \in Q' : \ \ z \in C \cup \ Int(C)\}.$$

Definition 6. *Functions defined on the points of a discrete domain B are said to be 'discrete functions'.*

Definition 7. *The $p-$difference and $q-$difference operators $D_{p,x}$ and $D_{q,y}$ are defined as follows:*

$$D_{p,x}[f(z)] = \frac{f(z) - f(px, y)}{(1-p)x} \tag{1.2}$$

$$D_{q,y}[f(z)] = \frac{f(z) - f(x, qy)}{(1-q)iy} \tag{1.3}$$

where f is a discrete function.

The two operators (1.2) and (1.3) involve a 'basic triad' of points denoted by

$$T(z) = \{(x, y), (px, y), \ (x, qy)\}. \tag{1.4}$$

Definition 8. *Let B be a discrete domain. Then a discrete function f is said to be '$(p, q)-$analytic' at $z \in B$ if*

$$D_{p,x}[f(z)] = D_{q,y}[f(z)]. \tag{1.5}$$

If in addition (1.5) holds

for every $z \in B$ such that $T(z) \subseteq B$ then f is said to be $(p, q)-$analytic in B. (1.6)

For simplicity, if (1.5) and (1.6) holds, the common operator D_q is used where

$$D_q \equiv D_{p,x} \equiv D_{q,y}. \tag{1.7}$$

Definition 9. *Since a discrete domain B is the union of basic sets S, so if the discrete domain B is given by*

$$B = \bigcup_{i=1}^{N} S(z_i)$$

then the 'subdomain' B_N is defined by

$$B_N = \{z_i : \ \ i = 1, 2, \dots, N\}. \tag{1.8}$$

In our earlier paper [10], a method is devised for the continuation into the 'discrete plane' Q' of functions defined on the axes. A continuation operator C_y is defined as

$$C_y \equiv \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j \tag{1.9}$$

and the function

$$f(z) = C_y[f(x,0)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [f(x,0)] \tag{1.10}$$

is called the '$(p,q)-$analytic continuation' of $f(x,0)$, into a $(p,q)-$analytic function f.

We also recall here some other definitions and notations given in [10] which will be subsequently used in the present paper.

Let

$$X \equiv \{(p^m x^i, 0) : \ m \in Z\}$$
$$Y \equiv \{(0, q^n y') : \ n \in Z\}$$

where (x', y') is the fixed point from which the lattice Q' is defined.

The 'extended discrete plane' $\bar{Q}$ is then defined as

$$\bar{Q} = Q' \cup X \cup Y. \tag{1.11}$$

The 'discrete rectangular domain' R' is defined by

$$R' = \{(p^m x', q^n y') : \ m = 0, 1, 2, \ldots, \ \ n = 0, 1, 2, \ldots\}. \tag{1.12}$$

If $X^+, Y^+$are defined by

$$\begin{aligned} X^+ &\equiv \{(p^m x', 0) : \ \ m = 0, 1, 2, \ldots\} \\ Y^+ &\equiv \{(0, q^n y') : \ \ n = 0, 1, 2, \ldots\} \end{aligned} \tag{1.13}$$

then the 'extended rectangular domain' $\bar{R}$ is defined as

$$\bar{R} = R' \cup X^+ \cup Y^+. \tag{1.14}$$

Discrete function can now be defined on X, Y. The values on the axes, of a discrete function f defined on R', are to be

$$f(x,0) = \lim_{n \to \infty} f(x, q^n y')$$
$$f(0,y) = \lim_{m \to \infty} f(p^m x', y); \quad (x,y) \in R'.$$

Alternatively, this can be expressed as

$$f(x,0) = \lim_{y \to 0} f(x,y); \ \ (x,y) \in R' \tag{1.15}$$

$$f(0,y)=\lim_{x\to 0} f(x,y);\quad (x,y)\in R' \tag{1.16}$$

The definition of $(p,q)-$analyticity is now extended to functions defined on the axes.

A function f is said to be $(p,q)-$analytic on X^+ if the limit in (1.15) exists for each x such that $(x,0)\in X^+$, and if

$$\lim_{y\to 0} D_y f(x,y)=D_x f(x,0);\quad (x,y)\in R'. \tag{1.17}$$

Similarly, f is $(p,q)-$analytic on Y^+ if the limit in (1.16) exists and

$$\lim_{x\to 0} D_x f(x,y)=D_y f(0,y);\ \ (x,y)\in R'. \tag{1.18}$$

(1.19) If f is $(p,q)-$analytic in R' and if (1.17), (1.18) hold, then f is said to be $(p,q)-$analytic in $\bar{R}$.

In two $(p,q)-$analytic functions are multiplied in the usual way, the resultant function is not, in general, a $(p,q)-$analytic function. Thus it is desirable to devise an alternative operation, analogous to multiplication, which retains many of the usual algebraic properties of the classical multiplier of functions of a continuous variable.

In the present paper an analogue for multiplication of $(p,q)-$analytic function is defined and its properties discussed.

2. Convolution

In two $(p,q)-$analytic functions are multiplied, and the resultant is $(p,q)-$ analytic, then the following theorem shows that one of the factors must be a $p-$periodic function in the $x-$component and $q-$periodic function in the $y-$component i.e.

$$W(z)=W(px,y)=W(x,qy). \tag{2.1}$$

Here the function $W(z)$ is given by

$$W(z)=\Phi_p(x)\Phi_q(iy) \tag{2.2}$$

where $\Phi_p(x)$ is Pincherle's $p-$periodic function defined by

$$\Phi_p(x)=x^{\alpha-\beta}\prod_{n=0}^{\infty}\left\{\frac{(1-xq^{\alpha+n})(1-x^{-1}q^{1-\alpha-n}}{(1-xq^{\beta+n})(1-x^{-1}q^{1-\beta-n})}\right\}$$

and $\Phi_q(iy)$ is Pincherle's $q-$periodic function.

Theorem 1. *If f, g and fg are $(p, q)-$analytic functions in Q', then either*

(i) $f(z) = f(px, y) = f(x, qy)$

or

(ii) $g(z) = g(px, y) = g(x, qy)$.

Proof.

$$\begin{aligned} D_{p,x}[f(z)g(z)] &= \frac{f(z)g(z) - f(px, y)g(px, y)}{(1-p)x} \\ &= \frac{f(z)g(z) - f(z)g(px, y) + f(z)g(px, y) - f(px, y)g(px, y)}{(1-p)x} \\ &= f(z)D_{p,x}[g(z)] + g(px, y)D_{p,x}[f(z)]. \end{aligned}$$

Similarly,

$$D_{q,y}[f(z)g(z)] = f(z)D_{q,y}[g(z)] + g(x, qy)D_{q,y}[f(z)].$$

Now f, g and fg are $(p, q)-$analytic by assumption and so combining the above two results

$$g(px, y)D_{p,x}[f(z)] = g(x, qy)D_{q,y}[f(z)]$$

or

$$g(px, y)D[f(z)] = g(x, qy)D[f(z)].$$

If $D[f(z)] \neq 0$, it follows that

$$g(px, y) = g(x, qy),$$

and by (2.2)

$$g(z) = g(px, y) = g(x, qy).$$

If $D[f(z)] = 0$,

$$f(z) = f(px, y) = f(x, qy).$$

This proved the theorem. □

An analogue for multiplication of $(p, q)-$analytic function is now defined.

Let R' denote a discrete rectangular domain and X^+ the corresponding points on the $x-$axis (see (1.12) and (1.13)). If the disctere functions f, g are $(p, q)-$analytic in R' and defined on X^+ then the convolution operator $*$ is defined by,

$$\begin{aligned} (f * g)(x) &\equiv C_y[f(x, 0)g(x, 0)] \\ &= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j}(iy)^j D^j_{p,x}[f(x, 0)g(x, 0)], \end{aligned} \tag{2.3}$$

for all $z \in R'$ such that the series converges.

The above definition is similar to a multiplicative operator introduced by Kurowski [11] for monodiffric functions. He defined the 'extension operator' E by the finite sum,

$$E[g(x)] = \sum_{j=0}^{y} i^j \binom{y}{j} \triangle_1^j [g(x)]$$

where x, y are integers and

$$\triangle_1 g(z) = g(z+1) - g(z).$$

He then defined a generalized product by

$$\begin{aligned}(f.g)(z) = & \quad E[f(x)g(x-1)] \\ = & \quad \sum_{j=0}^{y} i^j \binom{y}{j} \triangle_1^j [f(x)g(x-1)].\end{aligned}$$

Isaacs [5] defined a monodiffric analogue $z^{(n)}$ of the classical power z^n. The above product preserves the additive law,

$$z \cdot z^{(n)} = z^{(n+1)}, \quad n \geq 0; \;\; z^{(0)} = 1.$$

For general monodiffric functions the product is closed and distributive over addition. However it is neither commutative nor associative. These shortcomings led Berzsenyi [1,2] to consider a different approach. He defined several new analogues for multiplication of monodiffric functions by means of discrete contour integrals.

The convolution operator $*$ defined above is in many respect more suitable as an analogue of multiplication than the operator of Kurowski. It is shown in the next section that $*$ has most of the properties expected of a multiplicative operator and in fact it forms, with addition, an integral domain over a wide class of $(p, q)-$analytic functions.

3. Properties of the convolition operator $*$

In this section certain algebraic properties of the convolution operator $*$ are examined for discrete functions which are $(p, q)-$analytic in some region. For convenience it is assumed that the functions under consideration are $(p, q)-$ analytic in a discrete rectangular region $\bar{R}$ given by (1.14).

If in addition the $(p, q)-$analytic function f has a convergent representation,

$$f(z) = C_y[f(x,0)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [f(x,0)] \tag{3.1}$$

for all $z \in \bar{R}$, then f is said to belong to the class $\mathcal{A}$.

(3.2) If the series converges absolutely, f is said to belong to the class $\mathcal{B}$.

The following properties of $*$ can now be established.
(a) If $f, g \in \mathcal{A}$ then the two functions are $(p,q)-$analytic on $\bar{R}$ and so by (1.19) $f(x,0)$ and $g(x,0)$ exist for $z \in \bar{R}$. Hence by (2.1),

$$\begin{aligned}(f*g)(z) = & \sum_{j=0}^{\infty}\frac{(1-q)^j}{(1-q)_j}(iy)^j D^j_{p,x}[f(x,0)g(x,0)] \\ = & \sum_{j=0}^{\infty}\frac{(1-q)^j}{(1-q)_j}(iy)^j D^j_{p,x}[g(x,0)f(x,0)] \\ = & (g*f)(z),\end{aligned}$$

and so the operator $*$ is commutative.
(b) If $f, g, h \in \mathcal{A}$ then a distributive law can be proved as follows,

$$\begin{aligned}[f*(g+h)](z) = & C_y\{f(x,0)[g(x,0)+h(x,0)]\} \\ = & C_y\{f(x,0)g(x,0)+f(x,0)h(x,0)\}\end{aligned}$$

It is proved in [10] that C_y is a linear operator and so,

$$\begin{aligned}[f*(g+h)](z) = & C_y(f(x,0)g(x,0)) + C_y(f(x,0)h(x,0)) \\ = & (f*g)(z) + (f*h)(z),\end{aligned}$$

provided $f*g$ and $f*h$ exist (i.e. the series representations converge).
(c) If $f, g, h \in \mathcal{A}$, then assuming the existence of the operator C_y (i.e. convergence of the series),

$$\begin{aligned}[(f*g)*h](z) & = C_y[\{(f*g)(x,0)\}h(x,0)] \\ & = C_y[\{f(x,0)g(x,0)\}h(x,0)] \\ & = C_y[f(x,0)\{g(x,0)h(x,0)\}] \\ & = C_y[f(x,0)(g*h)(x,0)] \\ & = [f*(g*h)](z).\end{aligned}$$

Hence provided that the series defining C_y converges in the above, the operator $*$ satisfies an associative law.
(d) The operator $*$ has no non-ziro divisors of zero as shown in the following:
If $f, g \in \mathcal{A}$ such that
$(f*g)(z) = 0$ for all $z \in \bar{R}$, then,

$$\sum_{j=0}^{\infty}\frac{(1-q)^j}{(1-q)_j}(iy)^j\ D^j_{p,x}[f(x,0)g(x,0)] = 0$$

or

$$\sum_{j=0}^{\infty} a_j(x) y^j = 0; \quad z \in \bar{R},$$

where

$$a_j(x) = \frac{(1-q)^j}{(1-q)_j} i^j D_{p,x}^j [f(x,0)g(x,0)].$$

By the identity theorem for power series it follows that

$$a_j(x) = 0, \quad j = 0, 1, 2, \ldots,$$

and so,

$$a_0(x) = f(x,0)g(x,0) = 0.$$

Suppose that $f(x,0) = 0$, then

$$\begin{aligned} f(z) = & \quad C_y[f(x,0)] = C_y(0) \\ = & \quad \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [0] \\ = & \quad 0. \end{aligned}$$

Hence if $(f * g)(z) = 0$ for all $z \in \bar{R}$ then either $f(z) = 0$ or $g(z) = 0$, and so for the operator $*$ there are no non-zero divisors of zero.
(e) If $I(z)$ is the identity function ($I(z) = 1$ for all $z \in \bar{R}$), and if $f \in \mathcal{A}$, then

$$\begin{aligned} (f * I)(z) = & \quad C_y(f(x,0)I(x,0)) \\ = & \quad C_y(f(x,0)) \\ = & \quad f(z). \end{aligned}$$

Similarly, if $k(z)$ is the constant function $k(z) = k, z \in \bar{R}$, then,

$$(f * k)(z) = kf(z)$$

and if $k_1(z)$, $k_2(z)$ are two constant functions,

$$(k_1 * k_2)(z) = k_1 k_2.$$

In the above properties it was assumed that the $(p,q)-$analytic functions f and g were of class $\mathcal{A}$, which by (3.1) implies the existence of $C_y[f(x,0)]$ and $C_y[g(x,0)]$ in $\bar{R}$. However, this condition does not necessarily guarantee the existence of $(f * g)(z)$ (i.e. the convergence of the series representation).

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