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A NEW ALGORITHM FOR THE FOUR COUNTERFEIT COINS PROBLEM

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Abstract. We consider the problem of determining the minimum number of weighings which suffice to find the counterfeit (heavier) coins in a set of n coins given a balance scale and the information that there are exactly four heavier coins present. A sequential algorithm is constructed for which the maximum number of steps differs by at most one from the information-theoretical lower bound.

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1 Introduction

Let X be a set of n coins indistinguishable except for the fact that exactly m of them are slightly heavier than the rest. All heavier (counterfeit) coins are supposed to be of equal weight and so are all good coins. We denote the set of counterfeit coins by SC. Given a balance scale, we want to find an optimal weighing algorithm, i.e. an algorithm which minimizes the maximum number of steps (weighings) which are required to determine SC. We assume that the difference in weight between a normal and a counterfeit coin is so small that we can gain no information by balancing two subsets of different cardinalities, i.e. the larger of two numerically unequal subsets is always heavier.

The step (A, B) will mean balancing A against B, where A and B are two disjoint subsets of X of the same cardinality. There are three possible outcomes of the step:

1) $A \approx B$ which means A and B are of the same weight, i.e. they contain the same number of counterfeit coins.

2) $A > B$ which means A is heavier than B, i.e. A contains more counterfeit coins than B.

3) $A < B$ which means B is heavier than A.

Let $c_m(n)$ be the minimum number of weighings required to find all counterfeit coins (if there are exactly m of them in the set of n coins). There is a

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simple lower bound for $c_m(n)$ usually referred to as the information-theoretical lower bound

$$c_m(n) \geq \left\lceil \log_3 \binom{n}{m} \right\rceil . \tag{1}$$

The case $m = 1$ is a simple one. It is well known (see [1, 5]) that

$$c_1(n) = \lceil \log_3 n \rceil . \tag{2}$$

The case $m = 2$ turned out to be much more complicated. R. Tošić [8] proved the following:

$$c_2(n) \leq \left\lceil \log_3 \binom{n}{2} \right\rceil + 1. \tag{3}$$

For this case, it is still an open question whether the information-theoretical lower bound is achievable for every n.

The following statements proved in [2] will be of use in the sequel:

$$n \leq 4 \cdot 3^k \Rightarrow c_2(n) \leq 2k + 2 \tag{4}$$

$$n \leq 20 \cdot 3^k \Rightarrow c_2(n) \leq 2k + 5 \tag{5}$$

The case $m = 3$ has been studied in [3, 4, 11]. I. Bošnjak [3] obtained the following result:

$$c_3(n) \leq \left\lceil \log_3 \binom{n}{3} \right\rceil + 1.$$

A slight improvement has been made in [4] by proving the following statements:

$$n \leq 10 \cdot 3^k \Rightarrow c_3(n) \leq 3k + 5 \tag{6}$$

$$n \leq 15 \cdot 3^k \Rightarrow c_3(n) \leq 3k + 6 \tag{7}$$

$$n \leq 20 \cdot 3^k \Rightarrow c_3(n) \leq 3k + 7. \tag{8}$$

Suppose now that $A_1, A_2, \ldots, A_k$ are pairwise disjoint subsets of X such that $|A_i| = n_i$, $i = 1, 2, \ldots, k$, and we have the information that each of the sets A_i contains exactly m_i counterfeit coins. By $c_{m_1,m_2,\ldots,m_k}(n_1, n_2, \ldots, n_k)$ we denote the minimum number of weighings required to find all counterfeit coins in this case. In the sequel, we will use the following result from [3]:

$$c_{2,1}(3^k, 2 \cdot 3^{k-1}) = 3k - 1. \tag{9}$$

L. Pyber [7] investigated the general case. He proposed an algorithm for the situation when m is an upper bound for the number of counterfeit coins and proved that

$$c_m(n) \leq \left\lceil \log_3 \binom{n}{m} \right\rceil + 15m.$$

2. The Results

R. Tošić [10] studied the case $m = 4$. He obtained the following results:

$$c_4(2 \cdot 3^k) \leq 4k + 2$$

$$c_4(4 \cdot 3^k) \leq 4k + 4.$$

R.C. Guy and R.J. Nowakowski [6] gave a survey of coin-weighing problems in which they included another result concerning this case

$$c_4(3^k) \leq 4k - 1.$$

To improve these results, we will prove the following three theorems:

Theorem 1.

$$n \leq 4 \cdot 3^k \Rightarrow c_4(n) \leq 4k + 3, \qquad k = 0, 1, 2 \dots \tag{10}$$

$$n \leq 20 \cdot 3^{k-1} \Rightarrow c_4(n) \leq 4k + 5, \qquad k = 0, 1, 2 \dots . \tag{11}$$

Proof. The proof is by induction. It is obvious that (10) and (11) are true for k=0. Also, it is easy to verify that $c_4(7) = 4$. Suppose now that $k > 0$ and the theorem is true for all $l < k$.

(i) Let $n \leq 4 \cdot 3^k$ and $|A| = |B| = \lfloor n/2 \rfloor$. The first step is (A, B).

(a) If $A \approx B$, then $|A \cap SC| = |B \cap SC| = 2$ and according to (5), we can determine SC using at most $4k + 2$ additional weighings.

(b) If $A > B$, the second step is (B_1, B_2), where $B_i \subseteq B$ and $|B_i| = \lfloor |B|/2 \rfloor$, $i = 1, 2$.

(b.1) If $B_1 \approx B_2$, then $SC \cap (B_1 \cup B_2) = \phi$ and according to the induction hypothesis we can find counterfeit coins using $4k + 1$ additional steps, since $c_4(2 \cdot 3^k + 1) \leq 4k + 1$.

(b.2) If $B_1 > B_2$, then $|SC \cap (X \setminus B)| = 3$ and according to (8) and (2), we need not more than $c_3(2 \cdot 3^k) + c_1(3^k) = 4k + 1$ additional weighings to determine SC.

(b.3) The case $B_1 < B_2$ is analogous to (b.2).

(c) The case $A < B$ is analogous to (b).

(ii) Let $n \leq 20 \cdot 3^{k-1}$ and $|A| = |B| = \lfloor n/2 \rfloor$. The first step is (A, B).

(a) If $A \approx B$, then $|A \cap SC| = |B \cap SC| = 2$ and according to (4) we can determine SC using at most $4k + 4$ additional weighings.

(b) If $A > B$, the second step is (B_1, B_2), where $B_i \subseteq B$ and $|B_i| = \lfloor |B|/2 \rfloor$, $i = 1, 2$.

(b.1) If $B_1 \approx B_2$, then $SC \cap (B_1 \cup B_2) = \phi$ and according to (i), we can find irregular coins using $4k + 3$ additional steps.

(b.2) If $B_1 > B_2$, then $|SC \cap (X \setminus B)| = 3$ and we can determine SC using not more than $c_3(10 \cdot 3^{k-1}) + c_1(5 \cdot 3^{k-1}) = 4k + 3$ additional weighings.

(b.3) The case $B_1 < B_2$ is analogous to (b.2).

(c) The case $A < B$ is analogous to (b). □

Theorem 2.

$$(12) \qquad 12\cdot 3^k < n \le 16\cdot 3^k \Rightarrow c_4(n) \le 4k+8, \qquad k=0,1,2,\dots$$

Proof. Let $|A|=|B|=|C|=|D|=3\cdot 3^k$, $E=X\setminus(A\cup B\cup C\cup D)$, $E=E_1\cup E_2$, $|E_i|\le 2\cdot 3^k$ for $i=1,2$.
The first two steps are (A,B) and (C,D).

(a) If $A\approx B$ and $C\approx D$, the third step is (B,C).

(a.1) If $B\approx C$, the next step is (E,F) where F is any subset of $X\setminus E$ such that $|F|=|E|$.

(a.1.1) If $E>F$, then $SC\subseteq E$ and we can apply (10).

(a.1.2) If $E\approx F$ or $E<F$, then $|SC\cap A|=|SC\cap B|=|SC\cap C|=|SC\cap D|=1$ and we can determine SC using $4(k+1)$ additional steps.

(a.2) If $B>C$, we weigh E against B adding good coins from C if necessary.

(a.2.1) If $E>B$, then $|SC\cap A|=|SC\cap B|=1$ and $|SC\cap E|=2$. Therefore, according to (2) and (4), $4k+4$ additional weighings will suffice to identify counterfeit coins.

(a.2.2) If $E<B$, then $|SC\cap A|=|SC\cap B|=2$ and according to (4), we can determine SC using $4k+4$ additional steps.

(a.2.3) The case $E\approx B$ is impossible.

(a.3) The case $B<C$ is analogous to (a.2).

(b) If $A>B$ and $C\approx D$, the third step is (B,C).

(b.1) If $B\approx C$, we first weigh A against E (using good coins from B,C or D).

(b.1.1) If $A\approx E$, then $|SC\cap A|=|SC\cap E|=2$. Therefore, $4k+4$ additional weighings will suffice to determine SC.

(b.1.2) If $A<E$, then $|SC\cap A|=1$ and $|SC\cap E|=3$ and according to (2) and (7), we can determine SC using $4k+4$ additional steps.

(b.1.3) If $A>E$, the fifth step is (E_1,E_2) (we again use the coins which have been already proved to be good).

(b.1.3.1) If $E_1\approx E_2$, all counterfeit coins are in A and we can apply (10).

(b.1.3.2) If $E_1\not\approx E_2$, then $|SC\cap A|=3$ and according to (6) and (2), we can determine SC using at most $4k+3$ additional weighings.

(b.2) If $B<C$, the fourth step is (E_1,E_2).

(b.2.1) If $E_1\approx E_2$, then $|SC\cap A|=2$ and $|SC\cap C|=|SC\cap D|=1$. Therefore, we can find counterfeit coins using $c_2(3\cdot 3^k)+2c_1(3\cdot 3^k)=4k+4$ additional weighings.

(b.2.2) If $E_1\not\approx E_2$, then $|SC\cap A|=|SC\cap C|=|SC\cap D|=1$ and we need at most $3c_1(3\cdot 3^k)+c_1(2\cdot 3^k)=4k+4$ additional steps to find counterfeit coins.

(b.3) If $B>C$ the fourth step is (E_1,E_2).

(b.3.1) If $E_1\approx E_2$, then $|SC\cap A|=3$ and $|SC\cap B|=1$. Therefore, we can determine SC using $4k+3$ additional weighings.

(b.3.2) If $E_1\not\approx E_2$, then $|SC\cap A|=2$ and $|SC\cap B|=1$ and we can determine SC using at most $c_2(3\cdot 3^k)+c_1(3\cdot 3^k)+c_1(2\cdot 3^k)=4k+4$ additional steps.

(c) If $A > B$ and $C > D$, the third step is (A, C).
(c.1) If $A \approx C$, we first weigh A against E.
(c.1.1) The case $A > E$ is quite similar to (a.2.2).
(c.1.2) The case $A < E$ is quite similar to (a.2.1)
(c.1.3) The case $A \approx E$ is impossible.
(c.2) If $A > C$, we first weigh B against E.
(c.2.1) The case $B \approx E$ is analogous to (b.3.1)
(c.2.2) The case $B > E$ is analogous to (b.2.1)
(c.2.3) If $B < E$, the fifth step is (E_1, E_2). Regardless of the result of the weighing, $c_{2,1}(3 \cdot 3^k, 2 \cdot 3^k) + c_1(3 \cdot 3^k) = 4k + 3$ additional steps will suffice to determine SC, according to (9) and (2).
(c.3) The case $A < C$ is analogous to (c.2). □

Theorem 3.

(13) $$n \leq 9 \cdot 3^k \Rightarrow c_4(n) \leq 4k + 6, \qquad k = 0, 1, 2, \ldots$$

Proof. Let A, B, C be disjoint subsets of X such that $|A| = |B| = 2t \leq 4 \cdot 3^k$, for some $t \in N$, $C = X \setminus (A \cup B)$, $|C| \leq 3^k$.
The first step is (A, B).

(a) If $A \approx B$, the second step is $(A, B' \cup C)$, where B' is any subset of B such that $|B' \cup C| = |A|$.

(a.1) If $A < B' \cup C$, the third step is (A_1, A_2), where $A_i \subseteq A$, $|A_i| = t$, for $i = 1, 2$.

(a.1.1) If $A_1 \approx A_2$, then $SC \subseteq C$ and we can obviously find counterfeit coins using less than $4k + 3$ additional weighings.

(a.1.2) If $A_1 \not\approx A_2$, then $|SC \cap B| = 1$ and $|SC \cap C| = 2$ and we can find counterfeit coins using not more than $c_{1,1}(2 \cdot 3^k, 4 \cdot 3^k) + c_2(3^k) = 4k+2$ additional steps, since $c_{1,1}(2, 4) = 2$.

(a.2) If $A \approx B' \cup C$ or $A > B' \cup C$, then $|A \cap SC| = |B \cap SC| = 2$ and according to (4), we can determine SC using at most $4k+4$ additional weighings.

(b) If $A > B$, the second step is (B_1, B_2), where $B_i \subseteq B$, $|B_i| = t$, for $i = 1, 2$.

(b.1) If $B_1 \approx B_2$, then $SC \subseteq A \cup C$ and according to (12), we can determine SC using at most $4k + 4$ additional steps.

(b.2) If $B_1 \not\approx B_2$, then $|SC \cap (A \cup C)| = 3$ and according to (7) and (2), we can determine SC using not more than $4k + 4$ additional steps.

(c) The case $A < B$ is quite similar to the case $A > B$. □

Let a_t be the sequence of integers defined in the following way: $a_{4k+6} = 9 \cdot 3^k$, $a_{4k+7} = 12 \cdot 3^k$, $a_{4k+8} = 16 \cdot 3^k$ and $a_{4k+9} = 20 \cdot 3^k$, $k = 0, 1, 2, \ldots$. By $TB(n)$ we will denote the information-theoretical lower bound (1), for $m = 4$. Then the following statement holds:

Lemma 1 *There exists $t_0 \in N$ such that:*

$$t \geq t_0 \Rightarrow TB(a_t + 1) \geq t \tag{14}$$

Proof. It is easy to verify that

$$TB(a_t + 1) \geq t \Rightarrow TB(a_{t+4} + 1) \geq t + 4.$$

Since $TB(a_t + 1) \geq t$ for $t = 8, 10, 11, 17$, the proof is completed. □

Theorem 4

$$c_4(n) \leq \left\lceil \log_3 \binom{n}{4} \right\rceil + 1. \tag{15}$$

Proof. Let $t, n \in N$, $TB(a_t + 1) \geq t$ and $a_t < n \leq a_{t+1}$. Then $c_4(n) \leq t + 1 \leq TB(n) + 1$. According to Lemma 1, there are only finitely many n's for which $c_4(n) > TB(n) + 1$ might be true. In fact, careful analysis shows that the only possibilities are $n = 8, 10, 13, 21, 61$. Algorithms which prove $c_4(8) \leq 5$, $c_4(10) \leq 6$, $c_4(13) \leq 7$, $c_4(21) \leq 9$ and $c_4(61) \leq 13$, could be constructed by slightly modifying the algorithms from Theorems 1,2 and 3. The full description of the modified algorithms will be omitted here. □

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