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THE HOMOGENEOUS LIFT $\overset{*}{\mathbf{G}}$ ON THE COTANGENT BUNDLE

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Abstract. R. Miron ([3]) by means of the Sasaki lift $\overset{\circ}{G}$ introduced a new lift G which is 0-homogeneous on $\widetilde{TM} = TM \backslash \{0\}$. Some geometrical properties are studied using the almost complex structure F which preserves the properties of homogenity. In this paper, we similary studied the case of the cotangent bundle $\widetilde{T^*M} = T^*M \backslash \{0\}$ with a 0-homogeneous lift $\overset{*}{\mathbf{G}}$, using ([5]).

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1. Introduction

Let (T^*M, π^*, M) be a cotangent bundle, where M is a C^∞-differentiable, real n-dimensional manifold and the vertical distribution V on T^*M (V is the kernel of the submersion $\pi^* : T^*M \to M$), which is the integrable distribution. If M is a paracompact manifold there exists a C^∞- distribution H on T^*M which is supplementary to the vertical distribution V, such as the Whitney sum $TT^*M = HT^*M \oplus VT^*M$ holds. Also, H is called a nonlinear connection N on T^*M.

If (U, φ) is a local chart on M and (x^i) being the coordinates of the point $p \in M$, $p \in \varphi^{-1}(x) \in U$ then a point $u \in \pi^{*-1}(U)$, $\pi^*(u) = p$ has the coordinates (x^i, τ_i), $(i = \overline{1,n})$. The natural basis of the module $\mathcal{X}(T^*M)$ is given by $(\partial_i = \frac{\partial}{\partial x^i}, \partial^r = \frac{\partial}{\partial \tau_r})$. Given a nonlinear connection N on T^*M ([1]), there exist a single system of functions $N_{ia}(x,\tau)$ such that $\delta_k = \partial_k + N_{ka}(x,\tau)\partial^a$ and (δ_k, ∂^a) is a local basis of $\mathcal{X}(T^*M)$, which is called the adapted basis to N. We have the dual basis $(dx^i, \delta\tau_a = d\tau_a - N_{ka}(x,\tau)dx^k)$. For $X \in \mathcal{X}(T^*M)$ is obtained a unique decomposing $X = hX + vX$, $hX \in H$, $vX \in V$ and for $\omega \in \mathcal{X}^*(T^*M)$ we have $\omega = h\omega + v\omega$ where $(h\omega)(X) = \omega(hX)$, $(v\omega)(X) = \omega(vX)$. In the adapted basis (δ_k, ∂^a) we have $X = X^i\delta_i + X_a\partial^a$ and $\omega = \omega_i dx^i + \omega^a \delta\tau_a$.

The 1-form $\tau = \tau_i dx^i$ is a horizontal 1-form field ($h\tau = \tau$) on T^*M, which is

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called the fundamental 1-form on T^*M. The Liouville 1-form $\tau^v = \tau_r \partial^r$ is a vertical 1-form ($v\tau^v = \tau^v$). The field of 2- form Ω given by

$$(1) \qquad \Omega(X,Y) = -v[hX, vY], \quad \forall X, Y \in \mathcal{X}(T^*M)$$

is called the *curvature of the nonlinear connection* N. If $\Omega(\delta_j, \delta_k) = \Omega_{jk(a)}\partial^a$ then we have

$$(2) \qquad \Omega_{jk(a)}(x,\tau) = -R_{jk(a)}(x,\tau), \quad R_{jk(a)} = \delta_j N_{ka} - \delta_k N_{ja}.$$

Evidently, H is integrable if and only if $\Omega = 0$. The almost symplectic structure θ to N is given by $\theta = \delta\tau_r \wedge dx^r$ ([1]). If

$$(3) \qquad \overline{\tau} = \frac{1}{2}\tau_{kr} dx^k \wedge dx^r, \quad \tau_{kr} = N_{kr}(x,\tau) - N_{rk}(x,\tau)$$

then we obtain the exterior differential

$$(4) \quad d\tau = \theta + \overline{\tau};\ d\theta = -d\overline{\tau} = -\frac{1}{6}\sum_{(jkr)} R_{jk(r)} dx^j \wedge dx^k \wedge dx^r - \partial^s \tau_{ij} \delta\tau_s \wedge dx^i \wedge dx^j.$$

Let N be a fixed nonlinear connection on T^*M and $\overset{*}{G}$ a pseudo-Riemannian structure on T^*M, with the property $\overset{*}{G} = h\,\overset{*}{G} + v\,\overset{*}{G}$. In the adapted basis we have

$$(5) \qquad \overset{*}{G} = g_{ij}(x,\tau) dx^i \otimes dx^j + \overline{g}^{rs}(x,\tau)\delta\tau_r \otimes \delta\tau_s.$$

We consider an almost complex structure $\overset{*}{F}$ on T^*M, $\overset{*}{F}: \mathcal{X}(T^*M) \to \mathcal{X}(T^*M)$ given by ([1])

$$(6) \qquad \overset{*}{F}(\delta_k) = g_{kr}\partial^r, \quad \overset{*}{F}(\partial^r) = -g^{rs}\delta_s, \quad \overset{*}{F}{}^2 = -I.$$

If $\overset{*}{G}(FX, FY) = \overset{*}{G}(X,Y), \forall X, Y \in \mathcal{X}(T^*M)$ then $\overline{g}^{rs}(x,\tau) = g^{rs}(x,\tau)$, where $g_{ik}g^{ks} = \delta_i^s$ and

$$(7) \qquad \overset{*}{G} = g_{ij}(x,\tau) dx^i \otimes dx^j + g^{rs}(x,\tau)\delta\tau_r \otimes \delta\tau_s.$$

The structure $(T^*M, \overset{*}{G}, \overset{*}{F})$ is called *almost Hermitian structure*. We have

$$(8) \qquad \theta(X,Y) = \overset{*}{G}(X, \overset{*}{F}Y)$$

and it results that θ is the almost symplectic structure associated with $(\overset{*}{G}, \overset{*}{F})$. The space $(M, g^{rs}(x,\tau)) = \overset{*}{H}{}^n$ is called a *generalized Hamilton space* ([1]).

Definition 1. *([5]) The tensor field $\overset{*}{\Omega}$ defined by*

(9) $$\overset{*}{\Omega}(X,Y) = v\mathcal{N}(hX,hY), \quad \forall X,Y \in \mathcal{X}(T^*M)$$

is called self-curvature of the nonlinear connection N, where $\mathcal{N}$ is the Nijenhuis tensor of the almost complex structure $\overset{*}{F}$

$$\mathcal{N}(X,Y) = [\overset{*}{F}X, \overset{*}{F}Y] - \overset{*}{F}[\overset{*}{F}X,Y] - \overset{*}{F}[X,\overset{*}{F}Y] + \overset{*}{F}{}^2[X,Y], \ \forall X,Y \in \mathcal{X}(T^*M).$$

Definition 2. *([5]) The tensor field $\omega = \overset{*}{\Omega} - \Omega$ is called the non-holonomy distorsion of the space $(T^*M, \overset{*}{G}, \overset{*}{F})$ relative to $\overset{*}{H}{}^n$.*

Definition 3. *([5]) The tensor field $\overset{*}{t}$ defined by*

(10) $$\overset{*}{t}(X,Y) = h\mathcal{N}(vX,vY), \quad \forall X,Y \in \mathcal{X}(T^*M)$$

is called the self-torsion of nonlinear connection N.

Remark 1. *([5]) The almost complex structure $\overset{*}{F}$ is a complex structure if and only if $\overset{*}{\Omega} = 0$, $\overset{*}{t} = 0$. Then $(T^*M, \overset{*}{G}, \overset{*}{F})$ is a Hermitian space.*

2. The case of Riemannian structure.

Let $(M, g_{ij}(x))$ be a Riemannian space and $(T^*M, \overset{*}{G}, \overset{*}{F})$ its cotangent bundle and $g^{rs}(x)$ with $g_{ik}(x)g^{ks}(x) = \delta_i^s$.
We consider

(11) $$\overset{c}{N}_{kr}(x,\tau) \overset{def}{=} \tau_s \Gamma^s_{rk}(x),$$

where $\Gamma^s_{rk}(x)$ are the Christoffel symbols of g. Evidently, $\{\overset{c}{N}_{kr}(x,\tau)\}$ are the coefficients of nonlinear connection on $\widetilde{T^*M} = T^*M \setminus \{0\}$ which is 1-homogeneous on the fibres. Using $\overset{c}{N}_{kr}$ we consider $\delta_k = \partial_k + \overset{c}{N}_{kr}(x,\tau)\partial^r$; $\delta\tau_k = d\tau_k - \overset{c}{N}_{ik}(x,\tau)dx^i$.
We get

(12) $$h\overset{*}{G} = g_{ij}(x)dx^i \otimes dx^j, \quad v\overset{*}{G} = g^{rs}(x)\delta\tau_r \otimes \delta\tau_s,$$

(13) $$\overset{*}{G} = h\overset{*}{G} + v\overset{*}{G}, \quad \overset{*}{G} = g_{ij}(x)dx^i \otimes dx^j + g^{rs}(x)\delta\tau_r \otimes \delta\tau_s.$$

If $\overset{*}{h}_t: (x,\tau) \to (x, t\tau)$, $\forall t \in \mathbf{R}$ ($\overset{*}{h}_t$ is a homothety) we have

(14) $$\left(\overset{*}{G} \circ \overset{*}{h}_t\right)(x,\tau) = g_{ij}(x)dx^i \otimes dx^j + t^2 g^{rs}(x)\delta\tau_r \otimes \delta\tau_s \neq \overset{*}{G}(x,\tau).$$

Proposition 1. *$\overset{*}{G}$ is a globally defined Riemannian metric on $\widetilde{T^*M}$ and is not homogeneous on the fibres of T^*M.*

The space $(M, g^{rs}(x,\tau) = g^{rs}(x))$ is a particular Hamilton space.

We consider $\overset{*}{F}$ with $g^{rs}(x,\tau) = g^{rs}(x)$. We have:

Proposition 2. *$\overset{*}{F}$ depends only on g and is globally defined on T^*M.*

Proposition 3. *The almost complex structure $\overset{*}{F}$ is integrable (or complex structure) if and only if $\Omega = 0$.*

Proof. From $\partial^r g_{sk}(x) = 0$ we obtain $\omega = 0$, $\overset{*}{\Omega} = \Omega$. From (10) we get

$$\overset{*}{t}{}^{(r)(s)i} = g^{sk}\partial_k(g^{ri}) - g^{rk}\partial_k(g^{si}) + g^{sk}\partial^r(\overset{c}{N}_{kj})g^{ji} - g^{rk}\partial^s(\overset{c}{N}_{kj})g^{ij}.$$

But $\overset{c}{N}_{kj}(x,\tau) = \tau_s\Gamma^s_{jk}(x)$ and $g^{sk}g^{ji}g^{rm}\partial_k g_{jm} = -g^{sk}\partial_k g^{ri}$ then $\overset{*}{t}{}^{(r)(s)i} = 0$. □

Since $\Omega_{jk(r)} = -R_{jk(r)}$ and

$$R_{jk(r)} = \tau_s r^s_{rkj} \tag{15}$$

where r^s_{rkj} is the curvature tensor of Levi-Civita connection, we get:

Proposition 4. *The almost complex structure $\overset{*}{F}$ is integrable if and only if the space (M, g) is locally flat.*

Remark 2. *If $n = 2$, then the surface (M, g) is locally isometric with a plane.*

Proposition 5. *The space $(T^*M, \overset{*}{G}, \overset{*}{F})$ is an almost Kählerian space. The space $(\widetilde{T^*M}, \overset{*}{G}, \overset{*}{F})$ is a Kählerian space if and only if (M, g) is locally flat.*

Proof. Since $\tau_{jr} = \overset{c}{N}_{jr}(x,\tau) - \overset{c}{N}_{rj}(x,\tau) = \tau_s(\Gamma^s_{rj} - \Gamma^s_{jr}) = 0$ and $\sum_{(jkr)} R_{jk(r)} = 0$ we get $d\theta = 0$. □

The proposition is similar to Miron's results given for the tangent bundle $(\widetilde{TM}, G, F)$.

3. The homogeneous lift $\overset{*}{\mathbf{G}}$ of a Riemannian metric

We consider

$$H(x,\tau) = g^{rs}(x)\tau_r\tau_s. \tag{16}$$

Evidently, H is 2-homogeneous on the fibres of the cotangent bundle $\widetilde{T^*M}$.

If $\overset{*}{\mathbf{G}}$ is defined by

$$\overset{*}{\mathbf{G}} = g_{ij}(x)dx^i \otimes dx^j + \frac{r^2}{H} g^{rs}(x)\delta\tau_r \otimes \delta\tau_s \tag{17}$$

where $r > 0$ is a constant, then we get:

Theorem 1. *The following properties hold:*
1° The pair $(\widetilde{T^*M}, \overset{*}{\mathbf{G}})$ *is a Riemannian space depending only on the metric* g.
2° $\overset{*}{\mathbf{G}}$ *is* 0*-homogeneous on the fibres of* $\widetilde{T^*M}$.
3° The distributions N *and* V *are ortogonal with respect to* $\overset{*}{\mathbf{G}}$

$$\overset{*}{\mathbf{G}}(hX, vY) = 0, \quad \forall X, Y \in \mathcal{X}(T^*M).$$

Let $\overset{*}{\mathbf{F}}$ be the linear mapping $\overset{*}{\mathbf{F}}: \mathcal{X}(T^*M) \to \mathcal{X}(T^*M)$ given by

$$\overset{*}{\mathbf{F}}(\delta_k) = \frac{\sqrt{H}}{r} \overset{*}{F}(\delta_k), \quad \overset{*}{\mathbf{F}}(\partial^r) = \frac{r}{\sqrt{H}} \overset{*}{F}(\partial^r). \tag{18}$$

Theorem 2 $\overset{*}{\mathbf{F}}$ *has the following properties:*
1° $\overset{*}{\mathbf{F}}$ *is an almost complex structure on* $\widetilde{T^*M}$.
2° $\overset{*}{\mathbf{F}}$ *depends only on the metric* g.
3° $\overset{*}{\mathbf{F}}$ *is homogeneous on the fibres of* $\widetilde{T^*M}$.

Proof. We have $\overset{*}{\mathbf{F}^2}(\delta_j) = \overset{*}{\mathbf{F}}(\frac{\sqrt{H}}{r} g_{jk}\partial^k) = -\frac{\sqrt{H}}{r} g_{jk} \frac{r}{\sqrt{H}} g^{ki}\delta_i = -\delta_j$ and $\overset{*}{\mathbf{F}^2}(\partial^k) = \overset{*}{\mathbf{F}}(-\frac{r}{\sqrt{H}} g^{ki}\delta_i) = -\frac{r}{\sqrt{H}} g^{ki} \frac{\sqrt{H}}{r} g_{is}\partial^s = -\partial^k$. □

Theorem 3. *If we consider*

$$\overset{*}{\theta} \overset{def}{=} \frac{r}{\sqrt{H}}\theta \tag{19}$$

we get:
1° $\left(\overset{*}{\mathbf{G}}, \overset{*}{\mathbf{F}}\right)$ *is an almost Hermitian structure on* $\widetilde{T^*M}$.
2° $\overset{*}{\theta}$ *is the associated almost symplectic structure.*

Proof. 1° Follows from the equations $\overset{*}{\mathbf{G}}(\overset{*}{\mathbf{F}} X, \overset{*}{\mathbf{F}} Y) = \overset{*}{\mathbf{G}}(X, Y)$.
2° $\overset{*}{\theta}(X, Y) = \overset{*}{\mathbf{G}}(X, \overset{*}{\mathbf{F}} Y)$. □

Proposition 6. $\overset{*}{\theta}$ *cannot be an integrable structure.*

Proof. $d\overset{*}{\theta} = r(d\frac{1}{\sqrt{H}}) \wedge \theta \neq 0$. □

Let $\mathbb{N}$ be the Nijenhuis tensor of the homogeneous structure $\overset{*}{\mathbf{F}}$.

Proposition 7. *In the adapted basis we have the unique decomposition*

$$\begin{cases} \mathbb{N}(\delta_j, \delta_k) = \mathbb{N}^i_{jk}\delta_i + \mathbb{N}_{jk(r)}\partial^r, \\ \mathbb{N}(\delta_j, \partial^r) = \mathbb{N}^{(r)i}_{j}\delta_i + \mathbb{N}^{(r)}_{j(k)}\partial^k, \\ \mathbb{N}(\partial^s, \partial^r) = \mathbb{N}^{(s)(r)i}\delta_i + \mathbb{N}^{(s)(r)}_{(k)}\partial^k, \end{cases}$$

with

$$(20)\qquad \begin{cases} \mathbb{N}_{kj(i)} = \frac{H^2}{r^2}\mathbb{N}_j^{(r)s} g_{rk} g_{is} = -\frac{H}{r^2}\mathbb{N}_{(i)}^{(r)(s)} g_{sj} g_{rk}, \\ \mathbb{N}^i_{\alpha\beta} = \mathbb{N}^{(k)}_{\alpha(j)} g^{ij} g_{k\beta} = -\frac{H}{r^2}\mathbb{N}^{(r)(s)i} g_{r\alpha} g_{s\beta}. \end{cases}$$

Proposition 8. *We have the following relations*

$$(21)\qquad \begin{cases} \mathbb{N}_{kj(s)} = \frac{1}{r^2}(\tau_j \delta_k^l - \tau_k \delta_j^l) g_{ls} - R_{kj(s)}, \quad \mathbb{N}^i_{kj} = 0, \\ \mathbb{N}_j^{(k)s} = \frac{1}{H}(g^{ks} g_{jr}\tau^r - g_{jr} g^{rs}\tau^k) - \frac{r^2}{H} g^{kr} g^{is} R_{rj(i)}, \quad \mathbb{N}^{(k)}_{j(r)} = 0, \\ \mathbb{N}^{(k)(j)}_{(s)} = \frac{r^2}{H} g^{ji} g^{kr} R_{ri(s)} + \frac{1}{H}(\delta_s^j \tau^k - \delta_s^k \tau^j), \quad N^{(r)(s)i} = 0. \end{cases}$$

where $\tau^r = g^{rs}\tau_s$.

Proof. Follows from $N^{(r)(s)i} = \overset{*}{t}{}^{(r)(s)i} = 0$ and $\delta_k(H) = 0$.□

Theorem 4. $\overset{*}{\mathbf{F}}$ *is a complex structure if and only if*

$$(22)\qquad R_{kj(s)} = \frac{1}{r^2}\left(\tau_j \delta_k^l - \tau_k \delta_j^l\right) g_{ls}(x).$$

From (15) and (22) we obtain

$$(23)\qquad r^s_{rkj} = \frac{1}{r^2}\left(g_{rk}\delta^s_j - g_{rj}\delta^s_k\right)$$

Theorem 5. *The almost complex structure* $\overset{*}{\mathbf{F}}$ *is a complex structure on* $\widetilde{T^*M}$ *if and only if the Riemannian space* (M, g) *is of constant curvature* $K = \frac{1}{r^2}$.

Remark 3. *For* $n = 2$*,* (M, g) *is locally isometric with a sphere of radius* r*.*

Corollary 1. *The almost Hermitian structure* $\left(\overset{*}{\mathbf{G}}, \overset{*}{\mathbf{F}}\right)$ *is a Hermitian structure on* $\widetilde{T^*M}$ *if and only if the space* (M, g) *is of constant curvature.*

From (19) we get:

Corollary 2. *The structure* $\left(\overset{*}{\mathbf{G}}, \overset{*}{\mathbf{F}}\right)$ *on* $\widetilde{T^*M}$ *cannot be an almost Kählerian structure.*

From (23) we have

$$(24)\qquad r_{ij} = \frac{n-1}{r^2} g_{ij} = (n-1)K g_{ij}, \quad (n > 1)$$

where r_{rk} is the Ricci tensor and

$$(25)\qquad \overline{r} = \frac{n(n-1)}{r^2} > 0; \quad \overline{r} = n(n-1)K.$$

($\overline{r}$ is the scalar curvature and $K = \frac{1}{r^2} > 0$ is the curvature of (M, g)).

Corollary 3. *If the structure $\left(\overset{*}{\mathbf{G}}, \overset{*}{\mathbf{F}}\right)$ is a Hermitian structure on $\widetilde{T^*M}$ then (M, g) is an Einstein space with positive scalar curvature.*

Since $r_{ij} = r_{ji}$ then from (24) we get:

Corollary 4. *If the almost complex structure $\overset{*}{\mathbf{F}}$ is a complex structure then $(M, r_{ij}(x))$ is a Riemannian space.*

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