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REDUCED PRODUCTS OF INFINITE FORCING SYSTEMS

Milan Z. Grulović[1]

Abstract. We consider some basic properties of reduced products of infinite forcing systems.

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1. Preliminaries

Throughout the article L is a first order language. The basic logical symbols will be $\neg$ (negation), $\wedge$ (conjunction) and $\exists$ (existential quantifier); the others are defined by the basic ones in the standard way. The choice of the basic logical symbols is irrelevant, but being accustomed to this one we keep on using it. A theory T of the language L is a consistent deductively closed set of sentences; hence, $T \vdash \varphi$ means $\varphi \in T$. The class of models of a theory T will be denoted by $\mu(T)$. In general, models (of the language L) will be denoted by $\mathbf{A}$, $\mathbf{B}$, ..., while their domains will be $A, B, \ldots$. We will be dealing mostly with reduced products, thus a word about their notation. If $\mathbf{A}_i$, $i \in I$, is a family of models and if D is a filter over I, the reduced product of the given family of models *modulo* D will be standardly denoted by $\prod_D \mathbf{A}_i$. The elemenets of the reduced product $\mathbf{A} = \prod_D \mathbf{A}_i$ will be $f_A^1, f_A^2, \ldots, g_A^1, g_A^2, \ldots$, where $f^1, f^2, \ldots, g^1, g^2, \ldots$ (the elements of $\prod_I A_i$) are their representatives. This is not in accordance with the standard terminology, but it will be helpful in defining statements.

By an infinite forcing system we understand simply a class of models of the same language with the inclusion relation together with the infinite forcing relation between the models of the class and the sentences defined in them.

2. Reduced products of infinite forcing systems

Infinite (or, in general, n-infinite – [4]) forcing relation does not suit completely (under the natural interpretation) the definition of forcing relation given in [7], even if we accept to deal with classes rather than with sets. However, this definition can be easily adapted to cover the case of infinite forcing. In any

[1]Institute of Mathematics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Yugoslavia

case we will not bother ourselves with it this time. Instead we are getting to the point.

Let $\{\Sigma_i \mid i \in I\}$ be a family of classes of models of the (same) language L, let D be a proper filter over the index set I and let Σ_D be a class of models whose elements are reduced products of models from the classes Σ_i, $i \in I$, modulo $D - \prod_D \mathbf{A}_i$. Futhermore, let $=_D$ and $\leq_D$ be, respectively, the equality and partial ordering relation of the class Σ_D defined by:

$\prod_D \mathbf{A}_i =_D \prod_D \mathbf{B}_i$ iff $\{i \in I \mid \mathbf{A}_i = \mathbf{B}_i\} \in D$,

$\prod_D \mathbf{A}_i \leq_D \prod_D \mathbf{B}_i$ iff $\{i \in I \mid \mathbf{A}_i \leq \mathbf{B}_i\} \in D$,

where $=$ and $\leq$ are the ordinary identity and inclusion relations, respectively, on the classes Σ_i, $i \in I$. In the sequel the set $\{i \in I \mid \mathbf{A}_i \leq \mathbf{B}_i\}$ will be denoted by $X_{\mathbf{A},\mathbf{B}}$.

Lemma 2.1 *If* $X \stackrel{\text{def}}{=} \{i \in I \mid \Sigma_i$ *is an inductive class*$\}$ *and if* U *is* λ^+-*complete ultrafilter (in the sense of definition given in* [2]*), then every increasing chain of the lenght* λ $\mathbf{A}_0 \leq_U \mathbf{A}_1 \leq_U \ldots \leq_U \mathbf{A}_\alpha \leq_U \ldots$, $\alpha < \lambda$, *has a supremum.*

Proof. Let $\mathbf{A}_\alpha = \prod_U \mathbf{A}_i^\alpha$ and for each pair $\alpha < \beta$ $(< \lambda)$ let $X_{\alpha,\beta} = \{i \in I \mid A_i^\alpha \leq \mathbf{A}_i^\beta\}$ $(\in U)$. For each $j \in Y \stackrel{\text{def}}{=} X \cap \bigcap_{0 \leq \alpha < \beta < \lambda} X_{\alpha,\beta}$ let $\mathbf{B}_j = \bigcup_{\gamma<\lambda} A_j^\gamma$ and for each $j \in Y^c$ let us choose an arbitrary model $\mathbf{B}_j$ (from the class Σ_j). Obviously, $\mathbf{B} = \prod_U \mathbf{B}_i$ is an upper bound of the given chain. But if $\mathbf{C} = \prod_U \mathbf{C}_i$ is an upper bound of that chain too and if $Z_\alpha \stackrel{\text{def}}{=} \{i \in I \mid A_i^\alpha \leq \mathbf{C}_i\}$ $(\in U)$, $\alpha < \lambda$, then for any $j \in Y \cap \bigcap_{\alpha<\lambda} Z_\alpha$ we have $\mathbf{B}_j = \bigcup_{\gamma<\lambda} \mathbf{A}_j^\gamma \leq \mathbf{C}_j$; hence $\mathbf{B} \leq_U \mathbf{C}$. □

Definition 2.2 *Let* D *be a (proper) filter,* $\mathbf{A}, \mathbf{B} \in \Sigma_D$ *and* $\mathbf{A} \leq_D \mathbf{B}$. *We will write* $\mathbf{A} \preceq_D \mathbf{B}$ *and say that* $\mathbf{A}$ *is* D-*elementary less than* $\mathbf{B}$ *iff for each formula* $\phi(v_1, \ldots, v_k)$, $k \geq 0$, *holds:*

$$\mathbf{A} \models \phi(f_A^1, \ldots, f_A^k) \quad \text{iff} \quad \mathbf{B} \models \phi(g_B^1, \ldots, g_B^k),$$

whenever $X_j = \{i \in I \mid f^j(i) = g^j(i)\} \in D$ *for each* $j = 1, \ldots, k$.

Lemma 2.3 (a) *Let* U *be an ultrafilter and let* $\mathbf{A} = \prod_U \mathbf{A}_i$, $\mathbf{B} = \prod_U \mathbf{B}_i$. *If* $X \stackrel{\text{def}}{=} \{i \in I \mid \mathbf{A}_i \preceq \mathbf{B}_i\} \in U$, *then* $\mathbf{A} \preceq_U \mathbf{B}$;

(b) *If* U *is* λ^+-*complete ultrafilter and if for a given chain* $\mathbf{A}_0 \preceq_U \mathbf{A}_1 \preceq_U \ldots \preceq_U \mathbf{A}_\alpha \preceq_U \ldots$, $\alpha < \lambda$ *(where* $\mathbf{A}_\alpha = \prod_U \mathbf{A}_i^\alpha$*) holds:* $X_{\alpha,\beta} \stackrel{\text{def}}{=} \{i \in I \mid \mathbf{A}_i^\alpha \preceq_U \mathbf{A}_i^\beta\} \in U$ *for each* $\alpha < \beta$ $(< \lambda)$, *then the supremum of the chain is* U-*elementary greater than each* $\mathbf{A}_\alpha$.

Proof. (b) Let $\mathbf{B} = \prod_U \mathbf{B}_i$ be the supremum of the chain. Of course, we can assume that for each $i \in \bigcap_{\alpha<\beta<\lambda} X_{\alpha,\beta}$ $\mathbf{B}_i = \bigcup_{\gamma<\lambda} \mathbf{A}_i^\gamma$. Now, if the elements $f_{A_\alpha}^1, \ldots, f_{A_\alpha}^k$ and $g_B^1, \ldots, g_B^k$ are such that $X_j \stackrel{\text{def}}{=} \{i \in I \mid f^j(i) = g^j(i)\} \in U$,

$j = 1, \ldots, k$, and if, for instance, $\mathbf{B} \models \phi[g_B^1, \ldots, g_B^k]$, that is $Y \stackrel{\text{def}}{=} \{i \in I \mid \mathbf{B}_i \models \phi[g^1(i), \ldots, g^k(i)]\} \in U$, then for each $i \in \bigcap_{j=1}^k X_j \cap \bigcap_{\alpha<\beta<\lambda} X_{\alpha,\beta} \cap Y$ we have $\mathbf{A}_i^\alpha \models \phi[f^1(i), \ldots, f^k(i)]$; thus $\mathbf{A}_\alpha \models \phi[f_{A_\alpha}^1, \ldots, f_{A_\alpha}^k]$. □

Definition 2.4 *The relation* $\mathbf{A}$ *D-infinitely forces* $\phi(f_A^1, \ldots, f_A^n)$, *denoted by* $\mathbf{A} \Vdash_D \phi(f_A^1, \ldots, f_A^n)$, *between a model* $\mathbf{A} = \prod_D \mathbf{A}_i$ $(\in \Sigma_D)$ *and a sentence* $\phi(f_A^1, \ldots, f_A^n)$ *of the language* $L(A)$ *is defined inductively as follows:*

(1) *if* $\phi(f_A^1, \ldots, f_A^n)$ *is atomic, then* $\mathbf{A} \Vdash_D \phi(f_A^1, \ldots, f_A^n)$ *iff* $\{i \in I \mid \mathbf{A}_i \Vdash_i \phi(f^1(i), \ldots, f^n(i))\} \in D$, *where* $\Vdash_i$ *is the appropriate Robinson's infinite forcing relation "of the class* Σ_i*";*

(2) *if* $\phi \equiv \psi \wedge \theta$, *then* $\mathbf{A} \Vdash_D \phi$ *iff* $\mathbf{A} \Vdash_D \psi$ *and* $\mathbf{A} \Vdash_D \theta$*;*

(3) *if* $\phi \equiv \exists v\, \psi(v, f_A^1, \ldots, f_A^n)$, *then* $\mathbf{A} \Vdash_D \phi$ *iff there exists* $f_A \in A$ *such that* $\mathbf{A} \Vdash_D \psi(f_A, f_A^1, \ldots, f_A^n)$
and

(4) *if* $\phi(f_A^1, \ldots, f_A^n) \equiv \neg\psi(f_A^1, \ldots, f_A^n)$, *then* $\mathbf{A} \Vdash_D \phi$ *iff no* $\mathbf{B}$ *"greater" than* $\mathbf{A}$ $(\mathbf{A} \leq_D \mathbf{B})$ *D-infinitely forces* $\psi(g_B^1, \ldots, g_B^n)$, *where* $g_B^1, \ldots, g_B^n$ *are the elements of* $\mathbf{B}$ *such that* $X_k \stackrel{\text{def}}{=} \{i \in I \mid f^k(i) = g^k(i)\} \in D$, $k = 1, \ldots, n$.

The definition is correct, that is independent of the choice of the "representatives" both of models and of elements of these models. By this we mean: if $\prod_D \mathbf{A}_i = \mathbf{A} =_D \mathbf{B} = \prod_D \mathbf{B}_i$ and if the elements $f_A^1, \ldots, f_A^n$ $(\in A)$ and $g_B^1, \ldots, g_B^n$ $(\in B)$ are such that $X_j \stackrel{\text{def}}{=} \{i \in I \mid f^j(i) = g^j(i)\} \in D$ for each $j = 1, \ldots, n$, then $\mathbf{A} \Vdash_D \phi(f_A^1, \ldots, f_A^n)$ iff $\mathbf{B} \Vdash_D \phi(g_B^1, \ldots, g_B^n)$ (for any formula $\phi(v_1, \ldots, v_n)$, $n \geq 0$, of the language L). It is proved by a routine induction on the complexity of the formula ϕ (as usual, the complexity of a formula is determined by the number of logical connectives and quantifiers in it). The case "ϕ is atomic" is trivial. Just for illustration let us consider the case "$\phi \equiv \neg\psi(v_1, \ldots, v_n)$". Let us suppose that $\mathbf{A} \Vdash_D \phi(f_A^1, \ldots, f_A^n)$, while $\mathbf{B}$ does not D-infinitely forces $\phi(g_B^1, \ldots, g_B^n)$. Hence, for some $\mathbf{C} \in \Sigma_D$, $\mathbf{B} \leq_D \mathbf{C}$ and $\mathbf{C}$ D-infinitely forces $\psi(h_C^1, \ldots, h_C^n)$, where the elements $h_C^1, \ldots, h_C^n$ $(\in C)$ are such that $Y_j \stackrel{\text{def}}{=} \{i \in I \mid g^j(i) = h^j(i)\} \in D$, $j = 1, \ldots, n$. But $X_{\mathbf{A},\mathbf{C}} \supseteq \{i \in I \mid \mathbf{A}_i = \mathbf{B}_i\} \cap X_{\mathbf{B},\mathbf{C}} \in D$ and $\{i \in I \mid f^j(i) = h^j(i)\} \supseteq X_j \cap Y_j \in D$ $(j = 1, \ldots, n)$, contradictory to the assumption that $\mathbf{A}$ D-infinitely forces $\neg\psi(f_A^1, \ldots, f_A^n)$.

Lemma 2.5 *Let* $\mathbf{A}, \mathbf{B} \in \Sigma_D$ *and let* $\phi(f_A^1, \ldots, f_A^n), \psi$ *be sentences defined in* $\mathbf{A}$. *It holds:*

(1) $\mathbf{A}$ *cannot D-infinitely force both* ϕ *and* $\neg\phi$*;*

(2) *if* $\mathbf{A} \leq_D \mathbf{B}$ *and* $\mathbf{A} \Vdash_D \phi(f_A^1, \ldots, f_A^n)$, *then* $\mathbf{B} \Vdash_D \phi(g_B^1, \ldots, g_B^n)$ *for each* g_B^k, $k = 1, \ldots, n$, *such that* $\{i \in I \mid f^k(i) = g^k(i)\} \in D$.

(3) *if* $\mathbf{A} \Vdash_D \phi$, *then* $\mathbf{A} \Vdash_D \neg\neg\phi$*;*
$\mathbf{A} \Vdash_D \neg\phi$ *iff* $\mathbf{A} \Vdash_D \neg\neg\neg\phi$*;*

(4) *if* $\mathbf{A} \Vdash_D \phi$ *or* $\mathbf{A} \Vdash_D \psi$*, then* $\mathbf{A} \Vdash_D \neg(\neg\phi\wedge\neg\psi)$ *(that is* $\mathbf{A} \Vdash_D \phi\vee\psi$*);*

(5) *if* $\mathbf{A} \Vdash_D \neg\exists v\neg\psi(v)$*, then* $\mathbf{A} \Vdash_D \neg\neg\psi(f_A)$ *for each* $f_A \in M$*.*

Proof. Let us just remark that in the case of (4) we do not have the inverse implication. A counterexample is given in [4] for the case of one class but by someone's wish it could be easily adapted to be more appropriate for a "filter story". □

The next theorem (the version of Łos theorem for ultraproducts of infinite forcing systems) is in some way comparable with 1.8 from [3].

Theorem 2.6 *Let* U *be an ultrafilter over the index set* I*, let* $\mathbf{A} \in \Sigma_U$ *and let* $\phi(f_A^1,\ldots,f_A^n)$ *be a sentence defined in* $\mathbf{A}$*. It holds:*

$$\mathbf{A} \Vdash_U \phi(f_A^1,\ldots,f_A^n) \quad \textit{iff} \quad \{i\in I \mid \mathbf{A}_i \Vdash_i \phi(f^1(i),\ldots,f^n(i))\} \in U.$$

Proof. A routine induction by the complexity of the formula $\phi(v_1,\ldots,v_n)$. The case ϕ is atomic is a part of the definition. Let us still consider the case $\phi\equiv\neg\psi$.

Let $\mathbf{A} \Vdash_U \neg\psi(f_A^1,\ldots,f_A^n)$ and let us suppose that $X \stackrel{\text{def}}{=} \{i\in I \mid \mathbf{A}_i \Vdash_i \neg\psi(f^1(i),\ldots,f^n(i))\} \notin U$. For each $j\in X^c$ let $\mathbf{B}_j$ be an extension of the model $\mathbf{A}_j$ which infinitely forces $\psi(f^1(j),\ldots,f^n(j))$. If $\mathbf{C} = \prod_U \mathbf{C}_i$, where $\mathbf{C}_i = \begin{cases} \mathbf{A}_i & i\in X \\ \mathbf{B}_i & i\in X^c \end{cases}$, then $\mathbf{A} \le_U \mathbf{C}$ and, by inductive assumption, $\mathbf{C} \Vdash_U \psi(f_C^1,\ldots,f_C^n)$, a contradiction.

Let us assume now that $X \stackrel{\text{def}}{=} \{i\in I \mid \mathbf{A}_i \Vdash_i \neg\psi(f^1(i),\ldots,f^n(i))\} \in U$, but that $\mathbf{A}$ does not U-infinitely force $\neg\psi(f_A^1,\ldots,f_A^n)$. If $\mathbf{A} \le_U \mathbf{B} \Vdash_U \psi(g_B^1,\ldots,g_B^n)$, where $X_k = \{i\in I \mid f^k(i) = g^k(i)\} \in U$, $k = 1,\ldots,n$, and if $Y \stackrel{\text{def}}{=} \{i\in I \mid \mathbf{B}_i \Vdash_i \psi(g^1(i),\ldots,g^n(i))\}$ $(\in U)$, then $\emptyset = X_{\mathbf{A},\mathbf{B}} \cap X \cap \bigcap_{k=1}^n X_k \cap Y \in U$, a contradiction again. □

Naturally, if U is a principal ultrafilter nothing new is obtained. Namely, we have (compare with 1.5 from [3])

Corollary 2.7 *If* U *is a principal ultrafilter over the index set* I*, i. e.* $U = \{X\in P(I) \mid j\in X\}$ *for some* $j\in I$*, then the partial orderings* $\langle\Sigma_U,\le_U\rangle$ *and* $\langle\Sigma_j,\subseteq\rangle$ *are isomorphic and*

$$\mathbf{A} \Vdash_U \phi(f_A^1,\ldots,f_A^k) \quad \textit{iff} \quad \mathbf{A}_j \Vdash_j \phi(f^1(j),\ldots,f^k(j)),$$

where, of course, $\mathbf{A} = \prod_U \mathbf{A}_i$*.*

Hence, we will be only interested in ultraproducts (of infinite forcing systems) corresponding to nonprincipal ultrafilters.

3. Generic models

In accordance with the definition of infinitely generic models we introduce

Definition 3.1 *A model* $\mathbf{A}$ $(=\prod_D \mathbf{A}_i)$ *of the class* Σ_D *is* D*-infinitely generic iff for any sentence* $\phi(f_A^1,\ldots,f_A^n)$ *defined in* $\mathbf{A}$ *either* $\mathbf{A} \Vdash_D \phi(f_A^1,\ldots,f_A^n)$ *or* $\mathbf{A} \Vdash_D \neg\phi(f_A^1,\ldots,f_A^n)$.

Lemma 3.2 *If* U *is an ultrafilter over the index set* I, *then it holds:*

(a) *if* $X \stackrel{\text{def}}{=} \{i \in I \mid \mathbf{A}_i$ *is an infinitely generic model*$\} \in U$, *then* $\mathbf{A} = \prod_U \mathbf{A}_i$ *is* U*-infinitely generic;*

(b) *a model* $\mathbf{A} \in \Sigma_U$ *is* U*-infinitely generic iff for each sentence* $\phi(f_A^1,\ldots, f_A^k)$ *defined in* $\mathbf{A}$

$$\mathbf{A} \Vdash_U \phi(f_A^1,\ldots,f_A^k) \quad \textit{iff} \quad \mathbf{A} \models \phi(f_A^1,\ldots,f_A^k).$$

Proof. (a) Let $\phi(f_A^1,\ldots,f_A^n)$ be any sentence defined in $\mathbf{A}$ and let $Y \stackrel{\text{def}}{=} \{i \in X \mid \mathbf{A}_i \Vdash_i \phi(f^1(i),\ldots,f^n(i))\}$, $Z \stackrel{\text{def}}{=} \{i \in X \mid \mathbf{A}_i \Vdash_i \neg\phi(f^1(i),\ldots,f^n(i))\}$. Since $Y \cup Z = X$, one of the sets Y, Z is in U, and if, for instance, $Y \in U$, then by the previous theorem $\mathbf{A} \Vdash_U \phi(f_A^1,\ldots,f_A^n)$.

(b) Due to Łos theorem and the definition of infinite forcing we have for any model $\mathbf{B}$ from Σ_U and for any atomic formula $\psi(g_B^1,\ldots,g_B^r)$ defined in it:

$$\mathbf{B} \Vdash_U \psi(g_B^1,\ldots,g_B^r) \quad \text{iff} \quad \mathbf{B} \models \psi(g_B^1,\ldots,g_B^r).$$

The rest is the matter of the induction. □

Remark. The item (b) can be given in an apparently weaker form:

a model $\mathbf{A}$ *is* U*-infinitely generic iff for each sentence of the form* $\neg\phi(f_A^1,\ldots, f_A^k)$ *(defined in* $\mathbf{M}$*) holds:*

$$\mathbf{A} \Vdash_U \neg\phi(f_A^1,\ldots,f_A^k) \quad \text{iff} \quad \mathbf{A} \models \neg\phi(f_A^1,\ldots,f_A^k).$$

Corollary 3.3 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are* U*-infinitely generic and if* $\mathbf{A} \leq_U \mathbf{B}$, *then* $\mathbf{A} \preceq_U \mathbf{B}$.

Proof. A direct consequence of 2.5(2) and the previous lemma (b). □

Lemma 3.4 *Let* U *be an ultrafilter and let* $X \stackrel{\text{def}}{=} \{i \in I \mid \Sigma_i$ *be an inductive class* $\} \in U$. *Then every model from* Σ_U *is* $(U-)$*less than some* U*-infinitely generic model.*

Proof. It follows directly from the known fact that in an inductive class every model is contained in some infinitely generic model and 3.2(a). □

Corollary 3.5 *Let U be an ultrafilter and let $X \overset{\text{def}}{=} \{i \in I \mid \Sigma_i$ be an inductive class$\} \in U$. If a model $\mathbf{A} \in \Sigma_U$ is U-elementary less than each U-infinitely generic model $\mathbf{B}$ "greater" than $\mathbf{A}$ ($\mathbf{A} \leq_U \mathbf{B}$), then $\mathbf{A}$ is U-infinitely generic.*

Proof. We show by induction on the complexity of the formula $\phi(v_1, \ldots, v_k)$, $k \geq 0$, that for all $f_A^1, \ldots, f_A^k$ holds:

$$\mathbf{A} \Vdash_U \phi(f_A^1, \ldots, f_A^k) \quad \text{iff} \quad \mathbf{A} \models \phi(f_A^1, \ldots, f_A^k).$$

Surely, only the case $\phi \equiv \neg\psi(v_1, \ldots, v_k)$ is of interest. Let $\mathbf{A} \models \neg\psi(f_A^1, \ldots, f_A^k)$ and let us suppose that $\mathbf{A}$ does not U-infintely force $\neg\psi(f_A^1, \ldots, f_A^k)$. But if $\mathbf{B}$ is U-infinitely generic model such that $\mathbf{A} \leq_U \mathbf{B}$ and $\mathbf{B} \Vdash_U \psi(g_B^1, \ldots, g_B^k)$, where $X_j \overset{\text{def}}{=} \{i \in I \mid f^j(i) = g^j(i)\} \in U$, $j = 1, \ldots, k$ (the existence of such model is guaranteed by the previous lemma and 2.5(2)), then also $\mathbf{B} \models \phi[g_B^1, \ldots, g_B^k]$, contradictory to $\mathbf{A} \preceq_U \mathbf{B}$ (and the starting assumption). □

Let U be an ultrafilter over the index set I. At this momoment we can neither prove nor offer a counterexample to the following:

I a model $\mathbf{A} = \prod_U \mathbf{A}_i \in \Sigma_U$ is U-infinitely generic iff $\{i \in I \mid \mathbf{A}_i$ is an infinitely generic model of the class $\Sigma_i\} \in U$;

II $\mathbf{A} \preceq_U \mathbf{B}$ iff $\{i \in I \mid \mathbf{A}_i \preceq \mathbf{B}_i\} \in U$.

As for the first case we have already noted that implication $\Longleftarrow$ holds. The same implication equally well (and equally obvious) holds in the second case too. Obviously, on condition that $X \overset{\text{def}}{=} \{i \in I \mid \Sigma_i$ is inductive, generalized elementary class $\} \in U$, the second statement implies the first one.

In the next theorems we will assume that the statements I and II are fulfiled. But firstly

Definition 3.6 *Let D be a filter. A subclass $\mathcal{S} \subseteq \Sigma_D$ is D-model-consistent with Σ_D iff for any model $\mathbf{A} \in \Sigma_U$ there exists a model $\mathbf{B} \in \mathcal{S}$ such that $\mathbf{A} \leq_D \mathbf{B}$.*

Theorem 3.7 *If U is an ω^+-complete ultrafilter, $X \overset{\text{def}}{=} \{i \in I \mid \Sigma_i$ is inductive$\} \in U$ and if the conditions I and II are fulfiled, then the class of all U-infinitely generic models, in notation $\mathcal{L}_{\Sigma_U}$, is a unique subclass $\mathcal{S}$ of Σ_U satisfying:*

(1) *$\mathcal{S}$ is U-model-consistent with Σ_U;*

(2) *$\mathcal{S}$ is U-model complete, i.e. for all $\mathbf{A}, \mathbf{B} \in \mathcal{S}$ holds: $\mathbf{A} \leq_U \mathbf{B} \Longrightarrow \mathbf{A} \preceq_U \mathbf{B}$;*

(3) *$\mathcal{S}$ contains any other subclass of Σ_U which satisfies the conditions* (1) *and* (2).

Proof. We have already shown that $\mathcal{L}_{\Sigma_U}$ satisfies conditions (1) and (2). Let $\mathcal{D}$ be a subclass of Σ_U which satisfies these conditions too. We are to show that $\mathcal{D} \subseteq \mathcal{L}_{\Sigma_U}$. Let $\mathbf{A}_0 \in \mathcal{D}$, let us suppose that $\mathbf{A}_0$ does not U-infinitely force $\neg\phi(f^1_{A_0}, \ldots, f^k_{A_0})$ and let $\mathbf{A}_0 \leq_U \mathbf{A}_1 \in \mathcal{L}_{\Sigma_U}$ and $\mathbf{A}_1 \Vdash_U \phi(g^1_{A_1}, \ldots, g^k_{A_1})$, where $X_j \stackrel{\text{def}}{=} \{i \in I \mid f^j(i) = g^j(i)\} \in U$, $j = 1, \ldots, k$. Since both $\mathcal{L}_{\Sigma_U}$ and $\mathcal{D}$ are U-model-consistent with Σ_U we can construct a chain $\mathbf{A}_0 \leq_U \mathbf{A}_1 \leq_U \mathbf{A}_2 \leq_U \ldots \leq_U \mathbf{A}_n \leq_U \ldots$, where $\mathbf{A}_{2m} \in \mathcal{D}$, $\mathbf{A}_{2m+1} \in \mathcal{L}_{\Sigma_U}$, $m \geq 0$. Due to U-model completeness of the classes $\mathcal{D}$ and $\mathcal{L}_{\Sigma_U}$ Lemma 2.3 implies that the supremum of the given chain, let it be $\mathbf{A}_\omega$, is U-elementary greater than each $\mathbf{A}_i$. Thus $\mathbf{A}_0 \models \phi(f^1_{A_0}, \ldots, f^k_{A_0})$. In a similar way we can prove that from $\mathbf{A}_0 \Vdash_U \neg\phi(f^1_{A_0}, \ldots, f^k_{A_0})$ follows $\mathbf{A}_0 \models \neg\phi(f^1_{A_0}, \ldots, f^k_{A_0})$; whence, $\mathbf{A}_0$ is U-infinitely generic model. □

Theorem 3.8 *If U is an ω^+-complete ultrafilter, $X \stackrel{\text{def}}{=} \{i \in I \mid \Sigma_i$ is inductive$\}$ $\in U$ and if the conditions I and II are fulfilled, then the class of all U-infinitely generic models, in notation $\mathcal{L}_{\Sigma_U}$, is a unique subclass $\mathcal{S}$ of Σ_U satisfying:*

(1) *$\mathcal{S}$ is U-model-consistent with Σ_U;*

(2) *$\mathcal{S}$ is U-model complete, i.e. for all $\mathbf{A}, \mathbf{B} \in \mathcal{S}$ holhds: $\mathbf{A} \leq_U \mathbf{B} \Longrightarrow \mathbf{A} \preceq_U \mathbf{B}$;*

(3) *if a model $\mathbf{A}$ $(\in \Sigma_U)$ is U-elementary less than each model $\mathbf{B}$ from $\mathcal{S}$ "greater" than $\mathbf{A}$ $(\mathbf{A} \leq_U \mathbf{B})$, then $\mathbf{A}$ is in $\mathcal{S}$.*

Proof. We have proved that $\mathcal{L}_{\Sigma_U}$ satisfies the given conditions. If a subclass $\mathcal{D}$ satisfies the same conditions, it is shown in the same way as in the proof of the previous theorem that $\mathcal{L}_{\Sigma_U} \subseteq \mathcal{D}$ (and we already have the inverse inclusion).□

The next lemma is an immediate consequence of Łos theorem.

Lemma 3.9 *Let U be an ultrafilter over I and let $\Sigma_i = \mu(T_i)$, $i \in I$. Then $Th(\Sigma_U) \stackrel{\text{def}}{=} \{\phi \in SENT(L) \mid \mathbf{A} \models \phi$ for each $\mathbf{A} \in \Sigma_U\} = \prod_U T_i$.*

Proof. We recall that $\prod_U T_i \stackrel{\text{def}}{=} \{\psi \in SENT(L) \mid \{i \in I \mid T_i \vdash \psi\} \in U\}$ ([5]). The inclusion $\prod_U T_i \subseteq Th(\Sigma_U)$ is obvious. On the other hand if $\phi \notin \prod_U T_i$, i.e. if $X \stackrel{\text{def}}{=} \{i \in I \mid T_i \vdash \phi\} \notin U$, let for each $j \in X^c$ $\mathbf{A}_j$ be a model of $T_j \cup \{\neg\phi\}$. Now if we choose from each class Σ_k, $k \in X$, an arbitrary model, $\mathbf{A}_k$, then $\mathbf{A} = \prod_U \mathbf{A}_i \vdash \neg\phi$ and $\phi \notin Th(\Sigma_U)$. □

Theorem 3.10 *Let U be an ultrafilter and let $\Sigma_i = \mu(T_i \cap \Pi_1)$ for each $i \in I$. If we put $\Sigma^F_U \stackrel{\text{def}}{=} Th(\{\mathbf{A} \in \Sigma_U \mid \mathbf{A}$ is U-infinitely generic$\})$, then*

$$\Sigma^F_U = \prod\nolimits_U T^F_i,$$

where T^F_i is the infinite forcing companion of T_i, $i \in I$.

Proof. Let us suppose firstly $\phi \notin \prod_U T_i^F$ (that is $X \overset{\text{def}}{=} \{i \in I \mid T_i^F \vdash \phi\} \notin U$). For $j \in X^c$ let $\mathbf{A}_j$ be infinitely generic model satisfying $\neg\phi$. If we choose the other models $\mathbf{A}_k$, $k \in X$, arbitrary, then $\mathbf{A} = \prod_U \mathbf{A}_i$ is U-infinitely generic model and $\mathbf{A} \models \neg\phi$.

Let us assume now $\phi \notin \Sigma_U^F$ and let $\mathbf{A} = \prod_U \mathbf{A}_i$ be an U-infinitely generic model which satisfies $\neg\phi$; hence $X \overset{\text{def}}{=} \{i \in I \mid \mathbf{A}_i \models \neg\phi\} \in U$ and $\mathbf{A} \Vdash_U \neg\phi$. Let $Y \overset{\text{def}}{=} \{i \in I \mid T_i^F \vdash \phi\}$ and for $j \in X \cap Y$ let $\mathbf{B}_j$ be infinitely generic model extending $\mathbf{A}_j$; the other models $\mathbf{B}_k$, $k \in (X \cap Y)^c$ we choose arbitrarily. If Y were in U, it would imply that $\mathbf{B} = \prod_U \mathbf{B}_i$ is an U-infinitely generic model extending $\mathbf{A}$ ($\mathbf{A} \leq_U \mathbf{B}$) and U-infinitely forcing ϕ, a contradiction.□

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