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TAYLOR-TYPE EXPANSION OF THE k-TH DERIVATIVE OF THE DIRAC DELTA IN $u(x_1, ...x_n) - t$.

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Abstract. We obtain an expansion of Taylor style of the distribution $\delta^{(k)}(u(x_1, ...x_n) - t)$ where $u(x_1, ...x_n) \in C^\infty(R^n)$ without critical points and t is a real number. In particular, we obtain the expansion of the distribution $\delta^{(k)}(P + m^2)$(see ([3]), ([4]) and ([5])), where m is a positive real number and $P = P(x) = x_1^2 + x_2^2 + ...x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2$, $p + q = n$ dimension of the space.

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1. Introduction

Let ϕ_t denote a distribution of one variable t. Let $u \in C^\infty(R^n)$ be such that (n-1)-dimensional manifold $u(x_1, ...x_n) = 0$ has no critical point.

By $\phi_{u(x)}$ Leray(c.f. [2], p. 102) designates the distribution defined on R^n by

$$\left\langle \phi_{u(x)}, \varphi(x) \right\rangle = \langle \phi_t, \psi(t) \rangle \text{ ([2], page 102)} \tag{1}$$

where

$$\psi(t) = \int_{u(x)=t} \varphi(x) w_u(x,dx) \tag{2}$$

and $\varphi \in C_o^\infty(R^n)$ is the set of infinitely differentiable functions with compact support and w_u is a (n-1)-dimensional exterior differential form on u defined as

$$du \wedge w_u = dx_1 \wedge dx_2 \wedge ... \wedge dx_n. \tag{3}$$

By assumption, in the neighborhood of any point of the surface we can introduce a local coordinate system $u_1, u_2, ...u_n$ such that one of the coordinates, say u_j is $u(x_1, ...x_n)$ and such that the transformation from x_i to $u_i, i = 1, 2, ...$ is given by infinitely differentiable function with the positive Jacobian $D\binom{x}{u}$ ([1],p. 220).

If, in particular, in the neighborhood of the given point

$$\frac{\partial u}{\partial x_1} > 0, \tag{4}$$

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we may take the u_1 coordinate to be

$$\begin{array}{rc} u_1 = & u(x_1, x_2, ...x_n) - t \\ u_2 = & x_2 \\ \vdots & \vdots \\ u_n = & x_n \end{array} \tag{5}$$

then

$$w_u = D\binom{x}{u} du_2 ... du_n \tag{6}$$

where

$$D\binom{x}{u} = \left[D\binom{u}{x}^{-1} \right]^{-1} = \frac{1}{\frac{\partial u}{\partial x_1}}. \tag{7}$$

Therefore, from(6) and(7) we have

$$w_u = w_u(u, du) = \frac{du_2 ... du_n}{\frac{\partial u}{\partial x_1}} \tag{8}$$

From (5) and taking into account(4), it follws that there exists such a function $\alpha = \alpha(u_2, ...u_n) \in C^\infty(R^n)$ such that

$$x_1 = \alpha(u_2, ...u_n) \tag{9}$$

Therefore, from (2) and considering the formulae (5),(6) and(7) we have,

$$\psi(t) = \int_{u_1 = o} \varphi_1(u_1, u_2, ...u_n) w_u(u, du) \tag{10}$$

where

$$\varphi_1(u_1, u_2, ...u_n) = \varphi(x_1, x_2, ...x_n) \tag{11}$$

and $w_u(u,du)$ is defined by(8).

On the other hand from [1], p. 230, formula 6,we have,

$$\left\langle \delta^{(k)}(G(x_1, x_2, ...x_n), \varphi(x_1, x_2, ...x_n) \right\rangle = (-1)^k \int_{G(x)=0} w_k(\varphi) \tag{12}$$

$k = 0, 1, 2, ...$, where $x = (x_1, x_2, ...x_n), G(x_1, x_2, ...x_n)$ is such an infinite differentiable function that

$$grad\ G = \left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, ... \frac{\partial G}{\partial x_n} \right) \neq 0, \tag{13}$$

$$w_k(\varphi) = \frac{\partial^k}{\partial u_1^k} \left\{ D\binom{x}{u} \varphi_1(u_1, u_2, ...u_n) \right\} du_2 ... du_n, \tag{14}$$

$$w_o = \varphi.w, \tag{15}$$

$$\begin{aligned} u_1 &= G(x_1, x_2, ...x_n) \\ u_2 &= x_2 \\ &\vdots \\ u_n &= x_n, \end{aligned} \tag{16}$$

φ_1 is defined by equation(11), $w = w_u$ is the differential form defined by (6) and

$$D\binom{x}{u} = \left[D\binom{u}{x}^{-1}\right]^{-1} = \frac{1}{\frac{\partial G}{\partial x_1}} \tag{17}$$

with

$$\frac{\partial G}{\partial x_1} > 0. \tag{18}$$

Otherwise, from [1], p. 211, formula 8, $\delta^{(k)}(G(x_1, x_2, ...x_n)$ can be written as,

$$\begin{aligned} \left\langle \delta^{(k)}(G(x), \varphi \right\rangle = (-1)^k \int_{G=0} f^{(k)}_{u_1}(0, u_2, ...u_n) du_2...du_n = \\ (-1)^k \int_{G=0} \left[\frac{\partial^k}{\partial u_1^k} f^{(k)}_{u_1}(0, u_2, ...u_n)\right]_{u_1=0} du_2...du_n \end{aligned} \tag{19}$$

where

$$f(u_1, u_2, ...u_n) = \varphi_1(u_1, u_2, ...u_n) D\binom{x}{u} \tag{20}$$

φ_1 is defined by equation (11) and $D\binom{x}{u}$ by (17).

From(2) and(4), taking into account(15), we have

$$\psi(0) = \int_{u(x)=0} \varphi w = \langle \delta(u(x), \varphi(x) \rangle \tag{21}$$

In this paper we obtain an expansion of Taylor style of the distribution $\delta^{(k)}(u(x_1, ...x_n) - t)$ where $u(x_1, ...x_n) \in C^\infty(R^n)$ whithout critical point, t is a real number and $\delta^{(k)}(u(x_1, ...x_n) - t)$ is defined by (19).

2. The expansion of $\delta^{(k)}(u(x_1, ...x_n) - t)$

We begin by showing a lemma of expansion of $\psi(t)$ defined by (2).

Lemma 1 *Let $\psi(t)$ be the function defined by (2). Then the following expansion for $\psi(t)$ in powers of t is valid:*

$$\psi(t) = \sum_{\nu=o}^{\infty} a_\nu(\varphi) t^\nu \tag{22}$$

for every $\varphi \in C_o^\infty(R^n)$, *where,*

$$a_\nu(\varphi) = \frac{1}{\nu!} \int_{G=0} w_\nu(\varphi), \tag{23}$$

$$G = G(x_1, ...x_n) = u(x_1, ...x_n), \tag{24}$$

$w_\nu(\varphi)$ *is defined by (14).*

Proof. From(10) and (4) we have

$$\psi(t) = \int_{u_1=o} \varphi_1(u(x) - t, u_2, ...u_n) w_u(u, du) \tag{25}$$

where

$$u_1 = u(x) - t \tag{26}$$

On the other hand, $\varphi_1(u(x) - t, u_2, ...u_n)$ has the following Taylor series expansion in the neighborhood of $t = 0$

$$\varphi_1(u(x) - t, u_2, ...u_n) = \sum_{\nu \geq o} \frac{(-1)^\nu}{\nu!} \left\{ \left[\frac{d^\nu}{dt^\nu} \varphi_1(u(x) - t, u_2, ...u_n) \right]_{t=o} \right\} t^\nu. \tag{27}$$

Considering that $\varphi_1 \in C_o^\infty(R^n)$ and the convergence uniform of the series (27), from (25) and (27) we have

$$\psi(t) = \sum_{\nu \geq o} \frac{(-1)^\nu}{\nu!} \left\{ \int_{u_1=o} \left[\frac{d^\nu}{dt^\nu} \varphi_1(u(x) - t, u_2, ...u_n) \right]_{t=o} w_u(u, du) \right\} t^\nu. \tag{28}$$

On the other hand, taking into account (26), we obtain

$$\left[\frac{d^\nu}{dt^\nu} \varphi_1(u(x) - t, u_2, ...u_n) \right]_{t=o} = \left[\frac{\partial^\nu}{\partial u_1^\nu} \varphi_1(u_1, u_2, ...u_n) \right]_{u_1=u(x)} (-1)^\nu, \tag{29}$$

therefore, from (29) we have

$$\begin{aligned} &\textstyle\int_{u_1=o} \left[\frac{d^\nu}{dt^\nu} \varphi_1(u(x) - t, u_2, ...u_n) \right]_{t=o} w_u(u, du) = \\ &\textstyle\int_{u_1=o} \left[(-1)^\nu \frac{\partial^\nu}{\partial u_1^\nu} \varphi_1(u_1, u_2, ...u_n) \right]_{u_1=u(x)} w_u(u, du). \end{aligned} \tag{30}$$

Considering that the form w_u does not depend on the choice of $u_1, u_2, ...u_n$ coordinate system (see[1], p. 222), then using (26), if $t = o$ is $u_1 = u(x_1, x_2, ...x_n)$, thus, from (30) we obtain

$$\begin{aligned} &\textstyle\int_{u_1=o} \left[\frac{\partial^\nu}{\partial u_1^\nu} \varphi_1(u_1, u_2, ...u_n) \right]_{u_1=u(x)} w_u(u, du) = \\ &\textstyle\int_{u(x)=o} \left[\frac{\partial^\nu}{\partial u_1^\nu} \varphi_1(u_1, u_2, ...u_n) \right]_{u_1=o} w_u(u, du). \end{aligned} \tag{31}$$

Substituting (30) in(28) and taking into account(31) we arrive at

$$(32)\quad \psi(t)=\sum_{\nu\geq o}\frac{(-1)^{\nu}}{\nu!}\left\{\int_{u(x)=o}\left[\frac{\partial^{\nu}}{\partial u_1^{\nu}}\varphi_1(u_1,u_2,...u_n)\right]_{u_1=o}w_u(u,du)\right\}t^{\nu}.$$

On the other hand, considering the invariance from $w(u,du)$ on the hypersuface S given by the equation $G(x_1,...x_n)=0$ ([1], p. 222),the following property is valid:

$$(33)\quad \begin{aligned}&\left[\tfrac{\partial}{\partial u_1}\varphi_1(u_1,u_2,...u_n)\right]_{u_1=u(x)=o}.w_u(u,du)=\\&\left[\tfrac{\partial}{\partial u_1}\left\{\varphi_1(u_1,u_2,...u_n).w_u(u,du)\right\}\right]_{u_1=u(x)=o}\end{aligned}$$

In fact, from (8) we have

$$(34)\quad \begin{aligned}&\tfrac{\partial}{\partial u_1}w_u(u,du)=\tfrac{\partial}{\partial u_1}\left\{\left(\tfrac{\partial u}{\partial x_1}\right)^{-1}\right\}du_2...du_n=\\&(-1)\left\{\tfrac{\partial}{\partial u_1}\left(\tfrac{\partial u}{\partial x_1}\right)\right\}.\left(\tfrac{\partial u}{\partial x_1}\right)^{-2}du_2...du_n\end{aligned}$$

On the other hand, considering (9) we obtain

$$(35)\quad \begin{aligned}&\tfrac{\partial}{\partial u_1}\left(\tfrac{\partial u}{\partial x_1}\right)=\tfrac{\partial}{\partial u_1}\left(\tfrac{\partial}{\partial\alpha}u(\alpha,u_2,...u_n)\right)=\\&\tfrac{\partial}{\partial\alpha}\left(\tfrac{\partial u}{\partial\alpha}\right).\tfrac{\partial\alpha}{\partial u_1}+\tfrac{\partial}{\partial u_2}\left(\tfrac{\partial u}{\partial\alpha}\right).\tfrac{\partial u_2}{\partial u_1}+...+\tfrac{\partial}{\partial u_n}\left(\tfrac{\partial u}{\partial\alpha}\right).\tfrac{\partial u_n}{\partial u_1}=\quad\tfrac{\partial^2 u}{\partial\alpha^2}.\tfrac{\partial\alpha}{\partial u_1}\end{aligned}$$

From (9),

$$(36)\quad \frac{\partial\alpha}{\partial u_1}=0,$$

thus, from (35) and (36) we have

$$(37)\quad \frac{\partial}{\partial u_1}\left(\frac{\partial}{\partial x_1}u(\alpha,u_2,...u_n)\right)=0.$$

From (34) and considering (37) we arrive at

$$(38)\quad \frac{\partial}{\partial u_1}w_u(u,du)=0.$$

Therefore, from(38) we conclude that the property (33) is valid.

Now, using the property (33), we have,

$$(39)\quad \begin{aligned}&\left[\tfrac{\partial^{\nu}}{\partial u_1^{\nu}}\left\{\varphi_1(u_1,u_2,...u_n)\right\}\right]_{u_1=u(x)=o}.w_u(u,du)=\\&\left[\tfrac{\partial^{\nu}}{\partial u_1^{\nu}}\left\{\varphi_1(u_1,u_2,...u_n).w_u(u,du)\right\}\right]_{u_1=u(x)=o}\end{aligned}$$

$\nu = 0, 1, 2, ...$

Substituting (39) in (32) we obtain

$$\psi(t) = \sum_{\nu \geq o} \frac{1}{\nu!} \int_{u(x)=o} \left\{ \frac{\partial^\nu}{\partial u_1^\nu} \left[\varphi_1(u_1, u_2, ...u_n) w_u(u, du)\right] \right\}_{u_1=o} t^\nu. \tag{40}$$

From (40) and considering (8), (11),(19) and (20), we have

$$\psi(t) = \sum_{\nu \geq o} \frac{(-1)^\nu}{\nu!} \left\langle \delta^{(\nu)}(u(x)), \varphi(x) \right\rangle .t^\nu. \tag{41}$$

Otherwise, from (41) and (12) we have

$$\psi(t) = \sum_{\nu \geq o} \frac{1}{\nu!} \left(\int_{u(x)=o} w_\nu(\varphi) \right) t^\nu. \tag{42}$$

where $w_\nu(\varphi)$ is defined by(14).

Finally, from (42) we conclude the Lemma 1, formula (22). □

We observe from (41) that we have obtained a series expansion of $\delta(u(x)-t)$.

In fact, from (2) and (21) we obtain

$$\begin{aligned} \psi(t) = \int_{u(x)=t} \varphi w = \int_{u(x)-t=o} \varphi(x) w(x.dx) = \\ \langle \delta(u(x)-t), \varphi(x) \rangle \end{aligned} \tag{43}$$

and from(41) we have

$$\psi(t) = \left\langle \sum_{\nu \geq o} \frac{(-1)^\nu}{\nu!} \delta^{(\nu)}(u(x)) t^\nu, \varphi(x) \right\rangle. \tag{44}$$

Therefore, from (43) and (44) we obtain the following formula:

$$\delta(u(x)-t) = \sum_{\nu \geq o} \frac{(-1)^\nu}{\nu!} \delta^{(\nu)}(u(x)) t^\nu. \tag{45}$$

On the other hand, a series expansion of $\delta^{(k)}(u(x)-t)$ can be considered as a generalization from the formula (45) which we will study in the following theorem:

Theorem 2 *Let $u(x_1, ...x_n) \in C^\infty(R^n)$ be such that (n-1)-dimensional manifold $u(x_1, ...x_n) - t = 0$ has no critical point, then the following formula is valid,*

$$\delta^{(k)}(u(x)-t) = \sum_{q \geq o} \frac{(-1)^q}{q!} \delta^{(k+q)}(u(x)) t^q. \tag{46}$$

where t is a real number and $\delta^{(\nu)}(G(x_1, ...x_n))$ is defined by (12) or(19).

Proof. From (40) we have,

$$\psi(t) = \sum_{\nu \geq o} L_\nu \frac{t^\nu}{\nu!} \tag{47}$$

where,

$$L_\nu = \int_{u(x)=o} \frac{\partial^\nu}{\partial u_1^\nu} \{\varphi_1(u_1, u_2, ...u_n) w_u(u, du)\}. \tag{48}$$

Considering that the series (27) exibits uniform convergence, from (28) and (48) we have,

$$\begin{gathered} \tfrac{d^k \psi(t)}{dt^k} = \textstyle\sum_{\nu \geq k} L_\nu \tfrac{t^{\nu-k}}{(\nu-k)!} = \sum_{q \geq o} L_{q+k} \tfrac{t^q}{q!} = \\ \textstyle\sum_{q \geq o} \left[\int_{u(x)=o} \tfrac{\partial^{q+k}}{\partial u_1^{q+k}} \{\varphi_1(u_1, u_2, ...u_n).w_u(u, du)\}\right] \tfrac{t^q}{q!} = \\ \textstyle\sum_{q \geq o} \left\{\int_{u(x)=o} \left[\tfrac{\partial^{q+k}}{\partial u_1^{q+k}} \left\{\varphi_1(u_1, u_2, ...u_n).D\left(\tfrac{x}{u}\right)\right\}\right] du_2...du_n\right\} \tfrac{t^q}{q!}. \end{gathered} \tag{49}$$

From (49) and taking into account (14) and (15) we obtain

$$\begin{aligned} \tfrac{d^k \psi(t)}{dt^k} = & \textstyle\sum_{q \geq o} \left(\int_{u(x)=o} w_{q+k}(\varphi)\right) \tfrac{t^q}{q!} = \\ & \sum_{q \geq o} (-1)^{q+k} \left\langle \delta^{(k+q)}(u(x)), \varphi \right\rangle \tfrac{t^q}{q!}. \end{aligned} \tag{50}$$

Now, from (12) and (14) we have

$$\left\langle (-1)^k \delta^{(k)}(u(x) - t), \varphi \right\rangle = \int_{u_1=o} w_k(\varphi), \tag{51}$$

where,

$$w_k(\varphi) = \frac{\partial^k}{\partial u_1^k} \left\{\varphi_1(u_1, u_2, ...u_n).D\left(\frac{x}{u}\right)\right\} du_2...du_n. \tag{52}$$

From (51), (52) and considering (29) we obtain

$$\begin{gathered} \left\langle (-1)^k \delta^{(k)}(u(x) - t), \varphi \right\rangle = \\ \textstyle\int_{u_1=o} \tfrac{\partial^k}{\partial u_1^k} \left\{\varphi_1(u_1, u_2, ...u_n).D\left(\tfrac{x}{u}\right)\right\} du_2...du_n = \\ \textstyle\int_{u_1=o} \tfrac{\partial^k}{\partial u_1^k} \varphi_1(u(x) - t), u_2, ...u_n).w_u(u, du) = \\ \textstyle\int_{u_1=o} (-1)^k \tfrac{d^k}{dt^k} \{\varphi_1(u(x) - t), u_2, ...u_n).w_u(u, du)\} = \\ \tfrac{d^k}{dt^k} \left\{\textstyle\int_{u_1=o} (-1)^k \{\varphi_1(u(x) - t), u_2, ...u_n).w_u(u, du)\}\right\} = \\ \tfrac{d^k}{dt^k} \left\{\textstyle\int_{u_1=t} \varphi(x) w(x, dx)\right\} = \tfrac{d^k \psi(t)}{dt^k}. \end{gathered} \tag{53}$$

By (50) and (53) we conclude the proof of the theorem. □

In particular, putting

$$u(x) - t = P(x) + m^2 \tag{54}$$

in (46), where m is a positive real number,

$$P = P(x) = x_1^2 + x_2^2 + ...x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2, \tag{55}$$

$p + q = n$ (dimension of the space), we obtain the following

$$\delta^{(k)}(m^2 + P) = \sum_{q \geq o} \frac{(m^2)^q}{q!} \delta^{(k+q)}(P) \tag{56}$$

The formula (56) appears in [3] (formula (77)) under conditions n being odd, in [4], (formulae (38) and (39)) for two cases: a) p and q being even and b) p and $q$being odd. Finally, the formula (56) appears in [5] independently of p, q and n, where $p + q = n$ is the dimension of the space.

References

[1] Gelfand, I. M., Shilov, G. E., Generalized Functions, Vol. I, Academic Press, New York 1964.

[2] Leray, J., Hyperbolic Differential Equations, The Institute for Advanced Study, Princeton, New jersey, 1957.

[3] Manuel Aguirre T., The expansion and Fourier's transform of $\delta^{(k-1)}(m^2 + P)$, Integral Transforma and Special Functions, Vol. 3, No. 2, pp. 113-134, 1995.

[4] Manuel Aguirre T., The expansion of $\delta^{(k-1)}(m^2 + P)$, Integral Transform and Special Functions, Vol. 8, No. 1-2, pp. 139-148, 1999.

[5] Manuel Aguirre T., The expansion in series (of Taylor types) of $(k-1)$ derivative of Dirac's delta in $m^2 + P$, to appear.

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