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A NOTE ON A ONE-DIMENSIONAL NONLINEAR STOCHASTIC WAVE EQUATION

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Abstract. A class of one-dimensional semilinear stochastic wave equations with additive white noise is solved in the framework of a Colombeau generalized stochastic process space.

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1. Introduction

There is a large class of stochastic processes that appear in applied problems but which can not be defined in a classical way. For example, the white noise process is a good model of fluctuating phenomena which frequently appear in dynamic systems and its concept has proved to be a very useful mathematical idealization. The white noise was first correctly defined in connection with the theory of generalized functions (distributions). In fact, it is a derivative of a Wiener process (i.e. of its version with continuous but nowhere differentiable sample paths), when we consider both processes as generalized stochastic processes. That is the reason why the generalized stochastic processes have been introduced and taken as a standard (see [4], [12]). But they are involving the distribution spaces which are not suitable for multiplication and thus for dealing with nonlinear stochastic partial differential equations. Therefore, some of the authors have avoided distribution spaces in their studies (for example, in paper [10] the weighted L^2-spaces are used). There are also approaches which assume the use of Wick product as is done in [5].

In this paper, we use the theory of Colombeau generalized functions spaces (see [2], [3]) to overcome the multiplication problem. This is also done in the papers [8], [9], [11] and in a similar way in the papers [1] and [6]. More precisely, we use Colombeau-type algebras constructed in [2] and the energy inequality for wave equation (see [7] and references in it).

Basically, we are interested in one-dimensional nonlinear stochastic wave equation that involves additive white noise.

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In other words, we consider the equation

$$(1)\qquad \begin{aligned}&(\partial_t^2-\partial_x^2)U+F(U)+\dot{W}=0,\\ &U|_{t=0}=A,\ \ \partial_t U|_{t=0}=B,\end{aligned}$$

where A and B are certain Colombeau-generalized stochastic processes on $\mathbb{R}$, and $\dot{W}$ is the white noise process on $\mathbb{R}^2$.

The paper is organized as follows. In Section 2, we give some basic notations and definitions from the stochastic analysis and Colombeau generalized functions theory. In Section 3, instead of equations (1), so-called non-regularized, we consider its regularized version, i.e. we substitute function F by a family of smooth functions F_ε, for $\varepsilon \in (0,1)$. We prove the existence and uniqueness of the solution to the regularized equation. Finally, in Section 4, we are interested in questions under what conditions given on initial data and regularized white noise process the solution to the regularized equation is also the solution to the non-regularized one.

2. Preliminaries

At the beginning we recall some basic facts from the stochastic analysis, such as construction of white noise and the Wiener process (Brownian motion).

Let (Ω,Σ,μ) be a probability space. Weakly measurable mapping

$$X:\Omega\ \to \mathcal{D}'(\mathbb{R}^d)$$

is called generalized stochastic process on $\mathbb{R}^d$.

For each fixed function $\varphi\in\mathcal{D}(\mathbb{R}^d)$, the mapping $\Omega\to\mathbb{R}$ defined by

$$\omega\to\langle X(\omega),\varphi\rangle$$

is random variable. The space of generalized stochastic processes will be denoted by $\mathcal{D}'_\Omega(\mathbb{R}^d)$. The characteristic functional of process X is

$$C_X(\varphi)=\int e^{i\langle X(\omega),\varphi\rangle}d\mu(\omega)$$

for $\varphi\in\mathcal{D}(\mathbb{R}^d)$.

Construction of the white noise $\dot{W}$ on $\mathbb{R}^d$ goes as follows. The probability space will be the space of tempered distributions $\Omega=\mathcal{S}'(\mathbb{R}^d)$ and Σ will be the Borel σ-algebra generated by the weak topology.

There is a unique probability measure μ on (Ω,Σ) such that

$$\int e^{i\langle X(\omega),\varphi\rangle}d\mu(\omega)=e^{-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R}^d)}}$$

for $\varphi\in\mathcal{S}(\mathbb{R}^d)$, which is the well-known result given by the Bochner-Minlos theorem.

We define the white noise $\dot{W} : \Omega \to \mathcal{D}'(\mathbb{R}^d)$ as the identity mapping:

$$\langle \dot{W}(\omega), \varphi \rangle = \langle \omega, \varphi \rangle$$

for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Note that (4) determines its characteristic functional. Thus $\dot{W}$ is a generalized Gaussian process with mean zero and variance

$$D(\dot{W}(\varphi)) = E(\dot{W}(\varphi)^2) = \|\varphi\|^2_{L^2(R^d)}$$

where E denotes mathematical expectation. Its covariance is

$$E(\dot{W}(\varphi)\dot{W}(\psi)) = \int_{\mathbb{R}^d} \varphi(y)\psi(y)dy.$$

We now give the relation between the white noise and Wiener process on $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ let us define its signed indicator function

$$m(x, y) = \prod_{j=1}^{d} \operatorname{sign}(x_j)\kappa(x, y),$$

where $\kappa(x, \cdot)$ is the indicator function of the d-dimensional interval from the origin to the point x as extremal corner.

We define the Wiener process on $\mathbb{R}^d$ as follows

$$B(x) := \lim_{\varepsilon \to 0} \langle \dot{W}, m(x, \cdot) * \varphi_\varepsilon \rangle$$

where φ_ε are the molifiers of the form

$$\varphi_\varepsilon(y) = \frac{1}{\varepsilon^d}\varphi\left(\frac{y}{\varepsilon}\right),\ \varphi \in \mathcal{D}(\mathbb{R}^d),\ \int \varphi(y)dy = 1.$$

Note that the limit on the right-hand side exists in $L^2(\Omega)$.

Mapping $(x, \omega) \to B(x, \omega)$ has a version with almost surely continuous paths and it is a *Wiener process* on $\mathbb{R}^d$ (see [8]). It follows from the construction that

$$\dot{W} = \partial_{x_1} \dots \partial_{x_d} B$$

almost surely in $\mathcal{D}(\mathbb{R}^d)$.

Let us now recall the facts from the Colombeau generalized functions theory that we need here. Let O be an open subset of $\mathbb{R}^n$. We consider the following spaces:

$\mathcal{E}(O)$ is the space of all mappings $G : (0, 1) \times O \to \mathbb{C}$ such that

$$G(\varepsilon, \cdot) = G_\varepsilon \in C^\infty(O),\ \varepsilon > 0.$$

$\mathcal{E}_b([0, T) \times \mathbb{R}^n)$ is the space of all $G_\varepsilon \in \mathcal{E}([0, T) \times \mathbb{R}^n)$ with the property that for all $T > 0$ and $\alpha \in \mathbb{N}_0^n$ there exists $N \in \mathbb{N}$ such that

$$\|\partial^\alpha G_\varepsilon\|_{L^\infty([0,T)\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}).$$

$\mathcal{N}_b([0,T)\times\mathbb{R}^n)$ is the space of all $G_\varepsilon \in \mathcal{E}([0,T)\times\mathbb{R}^n)$ with the property that for all $T>0$, $\alpha\in\mathbb{N}_0^n$ and $a\in\mathbb{R}$

$$\|\partial^\alpha G_\varepsilon\|_{L^\infty([0,T)\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon^a).$$

Spaces $\mathcal{E}_b([0,T)\times\mathbb{R}^n)$ and $\mathcal{N}_b([0,T)\times\mathbb{R}^n)$ are multiplicative algebras and $\mathcal{N}_b([0,T)\times\mathbb{R}^n)$ is an ideal of $\mathcal{E}_b([0,T)\times\mathbb{R}^n)$.

Factor algebra

$$\mathcal{G}_b([0,T)\times\mathbb{R}^n) = \mathcal{E}_b([0,T)\times\mathbb{R}^n)/\mathcal{N}_b([0,T)\times\mathbb{R}^n)$$

is called the algebra of Colombeau bounded generalized functions.

One can similarly define spaces $\mathcal{E}_b(\mathbb{R}^n)$, $\mathcal{N}_b(\mathbb{R}^n)$ and $\mathcal{G}_b(\mathbb{R}^n)$. Their elements do not depend on time t.

Let us remark that by $f(\varepsilon)=\mathcal{O}(\varepsilon^b)$ we mean that $|f(\varepsilon)|\le \text{const}\,\varepsilon^b$ holds, and by $f(\varepsilon)=o(\varepsilon^b)$ we mean $\lim\limits_{\varepsilon\to 0} f(\varepsilon)\varepsilon^{-b}=0$.

In [2], the following construction is given.

$\mathcal{E}_{2,2}([0,T)\times\mathbb{R}^n)$ is the multiplicative algebra of all $G_\varepsilon\in\mathcal{E}([0,T)\times\mathbb{R}^n)$ with the property that for all $T>0$ and $\alpha\in\mathbb{N}_0^n$ there exists $N\in\mathbb{N}$ such that

$$\sup_{t\in[0,T)} \|\partial^\alpha G_\varepsilon\|_{L^2([0,T)\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}).$$

We say that $\|\partial^\alpha G_\varepsilon\|_{L^2}$ is moderate or that it has the moderate bound.

$\mathcal{N}_{2,2}([0,T)\times\mathbb{R}^n)$ is the multiplicative algebra of all $G_\varepsilon\in\mathcal{E}([0,T)\times\mathbb{R}^n)$ with the property that for all $\alpha\in\mathbb{N}_0^n$, and $a\in\mathbb{R}$,

$$\sup_{t\in[0,T)} \|\partial^\alpha G_\varepsilon\|_{L^2([0,T)\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{a}).$$

We say that $\|\partial^\alpha G_\varepsilon\|_{L^2}$ is negligible.

Similarly as above, we define

$$\mathcal{G}_{2,2}([0,T)\times\mathbb{R}^n) = \mathcal{E}_{2,2}([0,T)\times\mathbb{R}^n)/\mathcal{N}_{2,2}([0,T)\times\mathbb{R}^n).$$

One can similarly define spaces $\mathcal{E}_{2,2}(\mathbb{R}^n)$, $\mathcal{N}_{2,2}(\mathbb{R}^n)$ and $\mathcal{G}_{2,2}(\mathbb{R}^n)$ with elements independent of the time variable t.

Let Q denote $[0,T)\times O$ or O. The proof that $\mathcal{N}_{2,2}(Q)$ is an ideal of $\mathcal{E}_{2,2}(Q)$ is given in [2]. Sobolev embedding theorems give that $\mathcal{E}_{2,2}(Q)\subset\mathcal{E}_b(Q)$ and $\mathcal{N}_{2,2}(Q)\subset\mathcal{N}_b(Q)$. Thus there exists a canonical mapping $\mathcal{G}_{2,2}(Q)\to\mathcal{G}_b(Q)$. Also, this means that instead of L^2-norm on the strip $[0,T)\times\mathbb{R}$ one can use L^∞-norm on $[0,T)$ and L^2-norm on $\mathbb{R}$ and vice versa.

Definition 1. *$\mathcal{G}_{2,2}$-Colombeau random generalized function on probability space (Ω,Σ,μ) is a mapping $U:\Omega\to\mathcal{G}_{2,2}(Q)$ such that there exists a function $U_\varepsilon:(0,1)\times Q\times\Omega\to\mathbb{R}$ with the following properties:*

1) For fixed $\varepsilon \in (0,1)$, $(x,\omega) \to U_\varepsilon(\varepsilon, x, \omega)$ is jointly measurable in $Q \times \Omega$.

2) $\varepsilon \to U_\varepsilon(\varepsilon, \cdot, \omega)$ belongs to $\mathcal{E}_{2,2}(Q)$ almost surely in $\omega \in \Omega$, and it is a representative of $U(\omega)$.

By $\mathcal{G}^{\Omega}_{2,2}(Q)$ we denote the algebra of $\mathcal{G}_{2,2}$-Colombeau random generalized functions on Ω.

3. Regularized wave equation

We consider the equations (1), given in the introduction of this paper. We suppose that the function F is smooth, polynomially bounded together with all its derivatives, and that $F(0) = 0$.

The white noise process $\dot{W}$ is represented with a smooth function

$$\dot{W}_\varepsilon = (\dot{W} * \phi_\varepsilon)\xi_\varepsilon,$$

where ϕ_ε is a nonnegative model delta net and ξ_ε is a nonnegative net of smooth, compactly supported cut-off functions converging to identity. The cut-off procedure is necessary to obtain L^2-moderate properties of the above function $\dot{W}_\varepsilon$ and finite propagation speed for (1).

Instead of equations (1) we consider this equation given by the representatives:

$$(2) \qquad \begin{aligned} &(\partial_t^2 - \partial_x^2)U_\varepsilon + F(U_\varepsilon) + \dot{W}_\varepsilon = 0, \\ &U_\varepsilon|_{\{t=0\}} = A_\varepsilon, \;\; \partial_t U_\varepsilon|_{\{t=0\}} = B_\varepsilon, \end{aligned}$$

where $A_\varepsilon, B_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}(\mathbb{R})$ and $\dot{W}_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$.

We substitute F by a family of smooth functions F_ε, $\varepsilon \in (0,1)$, which is called the regularization of F. This will be done in the following way.

We choose a smooth function F_ε such that there exists a net a_ε such that for every $\alpha \in \mathbb{N}_0$ there exist ε_0 and $m^\alpha \in \mathbb{N}$ such that

$$F_\varepsilon(y) = F(y) \text{ for } |y| \le a_\varepsilon, \; \varepsilon < \varepsilon_0$$

$$\|D^\alpha F_\varepsilon(y)\|_{L^\infty} = \mathcal{O}(a_\varepsilon^{m_\alpha}).$$

Let us denote $m = \sup_{|\alpha| \le 1} m^\alpha$.

Instead of equations (2), which we call non-regularized, we now consider its regularized equation:

$$(3) \qquad \begin{aligned} &(\partial_t^2 - \partial_x^2)U_\varepsilon + F_\varepsilon(U_\varepsilon) + \dot{W}_\varepsilon = 0, \\ &U_\varepsilon|_{\{t=0\}} = \tilde{A}_\varepsilon, \;\; \partial_t U_\varepsilon|_{\{t=0\}} = \tilde{B}_\varepsilon, \end{aligned}$$

$\tilde{A}_\varepsilon, \tilde{B}_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}(\mathbb{R})$ and $\dot{W}_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$.

Theorem 1. *There exists a net a_ε such that for every $T > 0$ equation (3) has a unique solution almost surely in $\mathcal{E}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$.*

Proof. For each fixed ε, F_ε is globally Lipschitz function. Thus equation (3) has a unique strong solution U_ε.

First, note that the mapping $(x,\omega) \to U_\varepsilon(\varepsilon,x,\omega)$ is jointly measurable in x and ω for every fixed ε.

Let us prove that the solution U_ε belongs to $\mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$. Let $\omega\in\Omega$ be fixed.

We choose net a a_ε such that

$$a_\varepsilon = o\left(\left(\log\varepsilon^{-1}\right)^{\frac{1}{m}}\right). \tag{4}$$

The energy inequality gives

$$\begin{aligned}
&\|(\partial_t U_\varepsilon,\partial_x U_\varepsilon)(t)\|_{L^2}\\
\le&\|(\partial_t U_\varepsilon,\partial_x U_\varepsilon)(0)\|_{L^2}+\int_0^T\|F_\varepsilon(U_\varepsilon)\|_{L^2}ds+\int_0^T\|\dot{W}_\varepsilon\|_{L^2}ds\\
\le&\|(\partial_t U_\varepsilon,\partial_x U_\varepsilon)(0)\|_{L^2}+\int_0^T\|F'_\varepsilon(U_\varepsilon)\|_{L^\infty}\|U_\varepsilon(s)\|_{L^2}ds+\int_0^T\|\dot{W}_\varepsilon\|_{L^2}ds\\
\le&\|(\partial_t U_\varepsilon,\partial_x U_\varepsilon)(0)\|_{L^2}+\int_0^T a_\varepsilon^m\|U_\varepsilon\|_{L^2}ds+\int_0^T\|\dot{W}_\varepsilon\|_{L^2}ds.
\end{aligned}$$

Since the first and second terms are moderate and

$$\|U_\varepsilon\|_{L^2}\le C_T\|\partial_x U_\varepsilon\|_{L^2} \tag{5}$$

one can apply the Gronwall inequality and obtain the moderate bound for $\|\partial_x U_\varepsilon(t,\cdot)\|_{L^2}$. Then, by virtue of the formula (5), $\|U_\varepsilon(t,\cdot)\|_{L^2}$ has also the moderate bound.

To obtain the moderate bounds for L^2-norms of higher order derivatives of U_ε, we differentiate equation (3) with respect to spatial variable x. Then we have

$$(\partial_t^2-\partial_x^2)\partial_x U_\varepsilon + F'_\varepsilon(U_\varepsilon)\partial_x U_\varepsilon + \partial_x\dot{W}_\varepsilon = 0. \tag{6}$$

Using again the energy inequality we obtain

$$\begin{aligned}
&\|(\partial_{tx} U_\varepsilon,\partial_{xx} U_\varepsilon)(t)\|_{L^2}\\
\le&\|(\partial_{tx} U_\varepsilon,\partial_{xx} U_\varepsilon)(0)\|_{L^2}+\int_0^T\|F'_\varepsilon(U_\varepsilon)\partial_x U_\varepsilon\|_{L^2}ds+\int_0^T\|\partial_x\dot{W}_\varepsilon\|_{L^2}ds\\
\le&\|(\partial_{tx} U_\varepsilon,\partial_{xx} U_\varepsilon)(0)\|_{L^2}+\int_0^T\|F'_\varepsilon(U_\varepsilon)\|_{L^\infty}\|\partial_x U_\varepsilon\|_{L^2}ds+\int_0^T\|\partial_x\dot{W}_\varepsilon\|_{L^2}ds\\
\le&\|(\partial_{tx} U_\varepsilon,\partial_{xx} U_\varepsilon)(0)\|_{L^2}+\int_0^T a_\varepsilon^m\|\partial_x U_\varepsilon\|_{L^2}ds+\int_0^T\|\partial_x\dot{W}_\varepsilon\|_{L^2}ds.
\end{aligned}$$

Since we have proved that $\|\partial_x U_\varepsilon(t,\cdot)\|_{L^2}$ is moderate we immediately obtain that $\|\partial_{xx} U_\varepsilon(t,\cdot)\|_{L^2}$ is moderate, too.

We can continue by differentiating equation (6) in order to consider higher order derivatives of U_ε. Then we have

$$(\partial_t^2 - \partial_x^2)\partial_{xx}U_\varepsilon + F''_\varepsilon(U_\varepsilon)(\partial_x U_\varepsilon)^2 + F'_\varepsilon(U_\varepsilon)\partial_{xx}U_\varepsilon + \partial_{xx}\dot{W}_\varepsilon = 0.$$

Similarly as above we obtain

$$\begin{aligned}
&\|\partial_{txx}U_\varepsilon, \partial_{xxx}U_\varepsilon)(t)\|_{L^2} \\
\leq\ & \|(\partial_{txx}U_\varepsilon, \partial_{xxx}U_\varepsilon)(0)\|_{L^2} + \int_0^T \|F''_\varepsilon(U_\varepsilon)(\partial_x U_\varepsilon)^2\|_{L^2} ds \\
+\ & \int_0^T \|F'_\varepsilon(U_\varepsilon)\partial_{xx}U_\varepsilon\|_{L^2} ds + \int_0^T \|\partial_{xx}\dot{W}_\varepsilon\|_{L^2} ds \\
\leq\ & \|(\partial_{txx}U_\varepsilon, \partial_{xxx}U_\varepsilon)(0)\|_{L^2} + \int_0^T \|F''_\varepsilon(U_\varepsilon)\|_{L^\infty} \|(\partial_x U_\varepsilon)^2\|_{L^2} ds \\
+\ & \int_0^T \|F'_\varepsilon(U_\varepsilon)\|_{L^\infty} \|\partial_{xx}U_\varepsilon\|_{L^2} ds + \int_0^T \|\partial_{xx}\dot{W}_\varepsilon\|_{L^2} ds.
\end{aligned}$$

The first and the last terms are obviously moderate. In order to estimate the second term we use that $\|F''_\varepsilon(U_\varepsilon)\|_{L^\infty} \leq a_\varepsilon^m$ and the fact that

$$\|(\partial_x U_\varepsilon)^2\|_{L^2} \leq \|\partial_x U_\varepsilon\|_{L^4}^2 \leq \|\partial_x U_\varepsilon\|_{H^1}^2.$$

In estimating the third term we use $\|F'_\varepsilon(U_\varepsilon)\|_{L^\infty} \leq a_\varepsilon^m$ and the fact that $\|\partial_{xx}U_\varepsilon(t,\cdot)\|_{L^2}$ has the moderate bound, which we have proved in the previous step. Using all those facts we obtain the moderate bound for $\|\partial_{xx}U_\varepsilon(t,\cdot)\|_{L^2}$.

In order to obtain the moderate bounds for L^2-norm of m-th order derivative of U_ε, $\partial_x^m U_\varepsilon$, we only have to give bounds of the term that contains highest order derivative of U_ε because in all other terms derivatives of order at most $m-2$ appear. Their L^∞-norms are bounded by L^2- norms of derivatives of order at most $m-1$ which are moderate from the previous step.

The term that contains derivative of order $m-1$ (highest order derivative) is of the form

$$\int_0^T \|F'_\varepsilon(U_\varepsilon(s))\partial_x^{(m-1)}U_\varepsilon(s)\|_{L^2} ds.$$

Now we have

$$\begin{aligned}
\int_0^T \|F'_\varepsilon(U_\varepsilon(s))\partial_x^{(m-1)}U_\varepsilon(s)\|_{L^2} ds \quad &\leq \int_0^T \|F'_\varepsilon(U_\varepsilon)\|_{L^\infty} \|\partial_x^{(m-1)}U_\varepsilon(s)\|_{L^2} ds \\
&\leq \int_0^T a_\varepsilon^m \|\partial_x^{(m-1)}U_\varepsilon(s)\|_{L^2} ds.
\end{aligned}$$

Since we have from the previous step that $\|\partial_x^{(m-1)}U_\varepsilon(t,\cdot)\|_{L^2}$ has the moderate bound, the moderate bound for L^2-norm of arbitrary order derivative follows. Thus, we proved that $U_\varepsilon \in \mathcal{E}^\Omega_{2,2}([0,T)\times\mathbb{R})$.

It remains to show the uniqueness of the solution U_ε in $\mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$. For that purpose we shall suppose that there are two solutions to equation (3), $U_{1\varepsilon}, U_{2\varepsilon} \in \mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$, and show that $\bar{U}_\varepsilon := U_{1\varepsilon} - U_{2\varepsilon} \in \mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$.

Since both $U_{1\varepsilon}$ and $U_{2\varepsilon}$ are the solutions to equation (3) we have

$$(\partial_t^2 - \partial_x^2)\bar{U}_\varepsilon + (F_\varepsilon(U_{1\varepsilon}) - F_\varepsilon(U_{2\varepsilon})) + N_\varepsilon = 0, \qquad (7)$$
$$\bar{U}_\varepsilon|_{t=0} = N_{1\varepsilon}, \quad \partial_t\bar{U}_\varepsilon|_{t=0} = N_{2\varepsilon},$$

where $N_{1\varepsilon}, N_{2\varepsilon} \in \mathcal{N}^{\Omega}_{2,2}(\mathbb{R})$ and $N_\varepsilon \in \mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$.

Now we have

$$\begin{aligned}
&\|(\partial_t\bar{U}_\varepsilon, \partial_x\bar{U}_\varepsilon)(t)\|_{L^2}\\
\leq\;& \|(N_{2\varepsilon}, \partial_x N_{1\varepsilon})\|_{L^2} + \int_0^T \|N_\varepsilon\|_{L^2}ds + \int_0^T \|F_\varepsilon(U_{1\varepsilon}) - F_\varepsilon(U_{2\varepsilon})\|_{L^2}ds\\
\leq\;& \|(N_{2\varepsilon}, \partial_x N_{1\varepsilon})\|_{L^2} + \int_0^T \|N_\varepsilon\|_{L^2} + \int_0^T \|F'_\varepsilon(\tilde{U}_\varepsilon)\|_{L^\infty}\|\bar{U}_\varepsilon\|_{L^2}ds\\
\leq\;& \|(N_{2\varepsilon}, \partial_x N_{1\varepsilon})\|_{L^2} + \|N_\varepsilon\|_{L^2} + C\int_0^T a_\varepsilon^m\|\partial_x\bar{U}_\varepsilon\|_{L^2}ds,
\end{aligned}$$

for some $\tilde{U}_\varepsilon \in (\min(U_{1\varepsilon}, U_{2\varepsilon}), \max(U_{1\varepsilon}, U_{2\varepsilon}))$.

Since the first and the second terms are negligible and a_ε^m satisfies (4), we apply Gronwall's type inequality and obtain that $\|\partial_x\bar{U}_\varepsilon(t,\cdot)\|_{L^2}$ is negligible. Since (5) holds, $\|\bar{U}_\varepsilon(t,\cdot)\|_{L^2}$ is negligible, too.

To show that L^2-norms of higher order derivatives of $\bar{U}_\varepsilon$ are also negligible we start by differentiating equation (7). Then we obtain

$$(\partial_t^2 - \partial_x^2)\partial_x\bar{U}_\varepsilon + F'_\varepsilon(U_{1\varepsilon})\partial_x U_{1\varepsilon} - F'_\varepsilon(U_{2\varepsilon})\partial_x U_{2\varepsilon} + \partial_x N_\varepsilon = 0.$$

Now we have

$$\begin{aligned}
&(\partial_t^2 - \partial_x^2)\partial_x\bar{U}_\varepsilon + F'_\varepsilon(U_{1\varepsilon})\partial_x U_{1\varepsilon}\dot{W}_\varepsilon - F'_\varepsilon(U_{2\varepsilon})\partial_x U_{2\varepsilon}\dot{W}_\varepsilon\\
&+(F_\varepsilon(U_{1\varepsilon}) - F_\varepsilon(U_{2\varepsilon}))\partial_x\dot{W}_\varepsilon + \partial_x N_\varepsilon = 0.
\end{aligned}$$

The energy inequality gives

$$\begin{aligned}
&\|(\partial_{tx}\bar{U}_\varepsilon, \partial_{xx}\bar{U}_\varepsilon)(t)\|_{L^2} \leq \|(\partial_x N_{2\varepsilon}, \partial_{xx}N_{1\varepsilon})\|_{L^2} + \int_0^T \|\partial_x N_\varepsilon\|_{L^2}ds\\
&+\int_0^T \|F'_\varepsilon(U_{1\varepsilon})\partial_x U_{1\varepsilon}\|_{L^2}ds + \int_0^T \|F'_\varepsilon(U_{2\varepsilon})\partial_x U_{2\varepsilon}\|_{L^2}\\
&+\int_0^T \|F'_\varepsilon(\tilde{U}_\varepsilon)\|_{L^\infty}\|\bar{U}_\varepsilon\|_{L^2}ds\\
&\leq \|(\partial_x N_{2\varepsilon}, \partial_{xx}N_{1\varepsilon})\|_{L^2} + \int_0^T \|\partial_x N_\varepsilon\|_{L^2}ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \|F_\varepsilon'(U_{1\varepsilon})\partial_x U_{1\varepsilon} - F_\varepsilon'(U_{1\varepsilon})\partial_x U_{2\varepsilon}\|_{L^2} ds \\
&+\int_0^T \|F_\varepsilon'(U_{1\varepsilon})\partial_x U_{2\varepsilon} - F_\varepsilon'(U_{2\varepsilon})\partial_x U_{2\varepsilon}\|_{L^2} ds \\
&+\int_0^T \|F_\varepsilon'(\tilde{U}_\varepsilon)\|_{L^\infty} \|\bar{U}_\varepsilon\|_{L^2} ds \\
&\le \|(\partial_x N_{2\varepsilon}, \partial_{xx} N_{1\varepsilon})\|_{L^2} + \int_0^T \|\partial_x N_\varepsilon\|_{L^2} ds \\
&+\int_0^T a_\varepsilon^m \|\partial_x \bar{U}_\varepsilon\|_{L^2} ds + \int_0^T a_\varepsilon^m \|\bar{U}_\varepsilon\|_{L^\infty} \|\partial_x U_{2\varepsilon}\|_{L^2} ds + \int_0^T a_\varepsilon^m \|\bar{U}_\varepsilon\|_{L^2} ds \\
&\le \|(\partial_x N_{2\varepsilon}, \partial_{xx} N_{1\varepsilon})\|_{L^2} + \int_0^T \|\partial_x N_\varepsilon\|_{L^2} ds \\
&+\int_0^T a_\varepsilon^m \|\partial_x \bar{U}_\varepsilon\|_{L^2} ds + C\int_0^T a_\varepsilon^m \|\partial_x \bar{U}_\varepsilon\|_{L^2} \|\partial_x U_{2\varepsilon}\|_{L^2} ds + \int_0^T a_\varepsilon^m \|\bar{U}_\varepsilon\|_{L^2} ds,
\end{aligned}
$$

for some $\tilde{U}_\varepsilon \in (\min(U_{1\varepsilon}, U_{2\varepsilon}), \max(U_{1\varepsilon}, U_{2\varepsilon}))$.

Since $\|\bar{U}_\varepsilon(t,\cdot)\|_{L^2}$ and $\|\partial_x \bar{U}_\varepsilon(t,\cdot)\|_{L^2}$ are negligible we immediately obtain that $\|\partial_{xx} \bar{U}_\varepsilon(t,\cdot)\|_{L^2}$ is negligible, too. We have used that $H^1 \subset L^\infty$ for $n = 1$. Similarly, one can show that the L^2-norm of an arbitrary order derivative of $\bar{U}_\varepsilon$ is negligible.

Note that in both existence and uniqueness proof, the derivatives of U_ε with respect to the time variable t can be estimated directly by using the equation we solve and by differentiating it. Thus the proof is completed.

4. The non-regularized equation

Theorem 2. *Let the primitive function of F_ε that equals to zero in zero be nonnegative and*

$$\|(B_\varepsilon, \partial_x A_\varepsilon)\|_{L^2} + T\|\dot{W}_\varepsilon\|_{L^2} = o(a_\varepsilon) \ \text{as } \varepsilon \to 0, \tag{8}$$

where $T > 0$ and a_ε is the corresponding net used in the regularization of the function F (which depends on T).

Then the solution to the regularized equation

$$
\begin{aligned}
&(\partial_t^2 - \partial_x^2) U_\varepsilon + F_\varepsilon(U_\varepsilon) + \dot{W}_\varepsilon = 0, \\
&U_\varepsilon|_{t=0} = \tilde{A}_\varepsilon, \ \ \partial_t U_\varepsilon|_{t=0} = \tilde{B}_\varepsilon,
\end{aligned}
$$

is also the solution to the non-regularized one

$$
\begin{aligned}
&(\partial_t^2 - \partial_x^2) U_\varepsilon + F(U_\varepsilon) + \dot{W}_\varepsilon = 0, \\
&U_\varepsilon|_{t=0} = A_\varepsilon, \ \ \partial_t U_\varepsilon|_{t=0} = B_\varepsilon.
\end{aligned}
$$

Proof. It is well known that for any $t \in [0, T)$ the following inequality holds

$$\|U_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C\|\partial_x U_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}. \tag{9}$$

Using the energy inequality

$$\|\partial_x U_\varepsilon\|_{L^2} \le \|(B_\varepsilon, \partial_x A_\varepsilon)\|_{L^2} + \int_0^T \|\dot{W}_\varepsilon\|_{L^2} ds$$

and (8) we obtain

$$\|\partial_x U_\varepsilon(t)\|_{L^2} \le a_\varepsilon,\ \forall t \in [0, T).$$

In other words, it holds that

$$\|U_\varepsilon(t)\|_{L^\infty} \le C_T a_\varepsilon,\ \forall t \in [0, T)$$

from where it follows that

$$F_\varepsilon(U_\varepsilon) = F(U_\varepsilon)$$

which completes the proof.

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