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ON UPPER AND LOWER WEAKLY α-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper, the authors defined a multifunction $F : X \to Y$ to be upper (resp. lower) weakly α-continuous if for each $x \in X$ and each open set V of Y such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$), there exists an α-open set U of X containing x such that $U \subset F^{+}(\mathrm{Cl}(V))$ (resp. $U \subset F^{-}(\mathrm{Cl}(V))$). They give some characterizations and several properties concerning upper (lower) weakly α-continuous multifunctions.

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1. Introduction

In 1965, Njåstad [13] introduced a weak form of open sets called α- sets. Mashhour et al. [11] defined a function to be α-continuous if the inverse image of each set is an α-set. Noiri [16] called α-continuous functions strongly semi-continuous and in [17] he further investigated α-continuous functions. In [18], Noiri introduced a class of functions called weakly α-continuous functions. Some properties of weakly α-continuous functions are studied in [25], [31] and [32].

In 1986, Neubrunn [12] introduced and investigated the notion of upper (lower) α-continuous multifunctions. These multifunctions are further investigated by the present authors [26]. In [27], the present authors introduced a class of multifunctions called weakly α-continuous multifunctions. Some properties of weakly α-continuous multifunctions are investigated in [4] and [27].

The purpose of the present paper is to obtain some characterizations of upper (lower) weakly α-continuous multifunctions and several properties of such multifunctions.

2. Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be α-open (or α-set) [13] (resp. semi-open [8], preopen [10]) if $A \subset$

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$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A)), A \subset \mathrm{Int}(\mathrm{Cl}(A)))$. The family of all α-open (resp. semi-open, preopen) sets of X containing a point $x \in X$ is denoted by $\alpha(X,x), PO(X,x))$. The family of all α-open (resp. semi-open, preopen) sets in X is denoted by $\alpha(X)$ (resp.SO(X),PO(X)). For these three families, it is shown in [17, Lemma 3.1] that $SO(X) \cap PO(X) = \alpha(X)$. Since $\alpha(X)$ is a topology for X [13, Proposition 2], by $\alpha\mathrm{Cl}(A)$ (resp. $\alpha\mathrm{Int}(A)$) we denote the closure (resp. interior) of A with respect to $\alpha(X)$. The complement of a semi-open (resp. preopen, α-open) set is said to be semi-closed (resp. preclosed, α-closed). The intersection of all semi-closed sets of X containing A is called the semi- closure [5] of A and is denoted by $s\mathrm{Cl}(A)$. The union of all semi-open (resp. preopen) sets of X contained in A is called the semi-interior (resp. preinterior) of A and is denoted by $s\mathrm{Int}(A)$ (resp. $p\mathrm{Int}(A)$). A subset A of a space X is said to be regular-open (resp. regular closed) if $A = \mathrm{Int}(\mathrm{Cl}(A)$(resp. $A = \mathrm{Cl}(\mathrm{Int}(A)))$. The family of regular open (resp. regular closed) sets of X is denoted by $RO(X)$ (resp. $RC(X)$). The θ-closure [35] of A, denoted by $\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \emptyset$ for every open neighborhood U of x. It is shown in [35] that $\mathrm{Cl}_\theta(A)$ is closed in X and $\mathrm{Cl}(U) = Cl_\theta(U)$ for each open set U of X.

Lemma 1. *The following properties hold for a subset A of a topological space X:*

(1) If A is open in X, then sCl(A)=Int(Cl(A)).

(2) A is α-open in X if and only if $U \subset A \subset sCl(U)$ for some open set U of X.

(3) $\alpha Cl(A) = A \cup Cl(A)))$.

Proof. This follows from [17, Lemma 4.12] and [1, Theorem 2.2]. □

Throughout this paper, spaces (X,τ) and (X,σ) (or simply X and Y) always mean topological spaces and $F : X \to Y(resp. f : X \to Y)$ represents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set G of Y of $F^+(G)$ and $F^-(G)$ [3], respectively, that is

$$F^+(G) = \{x \in X : F(x) \subset G\} \quad \text{and} \quad F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}. \quad \Box$$

Definition 1. *A multifunction $F : X \to Y$ is said to be*

(1) *upper weakly continuous [22, 34] if for each $x \in X$ and each open set V of Y containing F(x), there exists an open set U of X containing x such that $F(U) \subset Cl(V)$,*

(2) *upper weakly quasi continuous [19] if for each $x \in X$ and each open set U containing x and each open set V containing F(x), there exists a nonempty open set G of X such that $G \subset U$ and $F(G) \subset Cl(V)$,*

(3) upper almost weakly continuous if for each $x \in X$ and each open set V containing $F(x)$, $x \in Int(Cl(F^+(Cl(V))))$.

Definition 2. *A multifunction $F : X \to Y$ is said to be*

(1) upper α-continuous [26] at a point x in X if for each open set V of Y containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset V$,

(2) lower α-continuous [26] at $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,

(3) upper(lower) α-continuous [12] if it is upper (lower) α-continuous at every point of X.

Definition 3. *A multifunction $F : X \to Y$ is said to be*

(1) upper almost α-continuous [27] at a point $x \in X$ if for each $U \in SO(X, x)$ and each open set V containing $F(x)$, there exists a nonempty open set $G \subset U$ such that $F(G) \subset sCl(V)$,

(2) lower almost α-continuous [27] at a point $x \in X$ if for each $U \in SO(X, x)$ and each open set V such that $F(x) \cap V = \emptyset$, there exists a nonempty open set $G \subset U$ such that $F(g) \cap sCl(V) \neq \emptyset$ for every $g \in G$,

(3) upper (lower) α-continuous if F has this property at every point of X.

Definition 4. *A multifunction $F : X \to Y$ is said to be*

(1) upper weakly α-continuous (briefly u.w.α.c.) at a point $x \in X$ if for each $U \in SO(X, x)$ and each open set V containing $F(x)$, there exists a nonempty open set $G \subset U$ such that $F(G) \subset Cl(V)$,

(2) lower weakly α-continuous (briefly l.w.α.c.) at a point $x \in X$ if for each $U \in SO(X, x)$ and each open set V such that $F(x) \cap V = \emptyset$, there exists a nonempty open set $G \subset U$ such that $F(g) \cap Cl(V) \neq \emptyset$ for every $g \in G$,

(3) upper (lower) weakly α-continuous if F has this property at every point of X.

For the properties of multifunctions defined above we have the following diagram:

upper weakly quasicontinuous
$\uparrow$
upper α-continuous $\to$upper almost α-continuous $\to$upper weakly α-continuous
$\downarrow$
upper almost weakly continuous

3. Characterizations

In [4,Theorem 7], Cao and Dontchev have stated several characterizations of upper weakly α-continuous multifunctions without the proof. In this section, we obtain many characterizations of upper weakly α-continuous (lower weakly α-continuous) multifunctions.

Theorem 1. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) F is u.w.α.c. at a point $x \in X$;

(2) for any open set V of Y containing $F(x)$, there exists $S \in \alpha(X, x)$ such that $F(S) \subset Cl(V)$;

(3) $x \in \alpha Int(F^+(Cl(V)))$ for every open set V containing $F(x)$;

(4) $x \in Int(Cl(Int(F^+(Cl(V)))))$ for every open set V containg $F(x)$.

Proof. (1) $\to$ (2): Let V be any open set of Y containing $F(x)$. For each $U \in SO(X, x)$, there exists a nonempty open set G_U such that $G_U \subset U$ and $F(G_U) \subset \mathrm{Cl}(V)$. Let $W = \cup\{G_U : U \in SO(X, x)\}$. Put $S = W \cup \{x\}$, then W is open in X, $x \in s\mathrm{Cl}(W)$ and $F(W) \subset \mathrm{Cl}(V)$. Therefore, we have $S \in \alpha(X, x)$ by Lemma 1 and $F(S) \subset \mathrm{Cl}(V)$.

(2) $\to$ (3): Let V be any open set of Y containing $F(x)$. Then there exists $S \in \alpha(X, x)$ such that $F(S) \subset \mathrm{Cl}(V)$. Thus we obtain $x \in S \subset F^+(\mathrm{Cl}(V))$ and hence $x \in \alpha\mathrm{Int}(F^+(\mathrm{Cl}(V)))$.

(3) $\to$ (4): Let V be any open set of Y containing $F(x)$. Now put $\alpha\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. Then $U \in \alpha(X)$ and $x \in U \subset F^+(\mathrm{Cl}(V))$. Thus we have $x \in U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(Cl(V)))))$.

(4) $\to$ (1): Let $U \in SO(X, x)$ and V be any open set of Y containing $F(x)$. Then we have $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{Cl}(V))))) = s\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{Cl}(V))))$. It follows from [15, Lemma 3] and [14, Lemma 1] that $\emptyset \neq U \cap \mathrm{Int}(F^+(\mathrm{Cl}(V))) \in SO(X)$. Put $G = \mathrm{Int}(U \cap \mathrm{Int}(F^+(\mathrm{Cl}(V))))$. Then G is a nonempty open set of Y [14, Lemma 4], $G \subset U$ and $F(G) \subset \mathrm{Cl}(V)$. □

Theorem 2. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) F is l.w.α.c. at a point x of X;

(2) for any open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap Cl(V) \neq \emptyset$ for every $u \in U$;

(3) $x \in \alpha Int(F^-(Cl(V)))$ for every open set V of Y such that $F(x) \cap V \neq \emptyset$;

(4) $x \in Int(Cl(Int(F^-(Cl(V)))))$ for every open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. The proof is similar to that of Theorem 1. □

The following theorem is stated by Cao and Dontchev [4] without the proof. We shall give the proof since it is important.

Theorem 3. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) F is u.w.α.c.;

(2) for each $x \in X$ and each open set V of Y containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset Cl(V)$;

(3) $F^+(V) \subset Int(Cl(Int(F^+(Cl(V)))))$ for every open set V of Y;

(4) $Cl(Int(Cl(F^-(Int(K))))) \subset F^-(K)$ for every closed set K of Y;

(5) $\alpha Cl(F^-(Int(K))) \subset F^-(K)$ for every closed set K of Y;

(6) $\alpha Cl(F^-(Int(Cl(B)))) \subset F^-(Cl(B))$ for every subset B of Y;

(7) $F^+(Int(B)) \subset \alpha Int(F^+(Cl(Int(B))))$ for every subset B of Y;

(8) $F^+(V) \subset \alpha Int(F^+(Cl(V)))$ for enery open set V of Y;

(9) $\alpha Cl(F^-(Int(K))) \subset F^-(K)$ for every regular closed set K of Y;

(10) $\alpha Cl(F^-(V)) \subset F^-(Cl(V))$ for every open set V of Y;

(11) $\alpha Cl(F^-(Cl_\theta(B))) \subset F^-(Cl_\theta(B))$ for every subset B of Y.

Proof. (1) → (2): The proof follows immediately from Theorem 1.

(2) → (3): Let V be any open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$ and there exists $U \in \alpha(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$. Therefore, we have $x \in U \subset F^+(\mathrm{Cl}(V))$. Since $U \in \alpha(X, x)$, we have $x \in U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{Cl}(V)))))$.

(3) → (4): Let K be any closed set of Y. Then $Y - K$ is an open set in Y. By (3), we have $F^+(Y-K) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{Cl}(Y-K)))))$. By the straighforward calculations, we obtain

$$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{Int}(K))))) \subset F^-(K).$$

(4) → (5): Let K be any closed set of Y. Then, we have $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{Int}(K))))) \subset f^-(K)$ and hence $\alpha\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ by Lemma 1.

(5) → (6): Let B be an arbitrary subset of Y, then $\mathrm{Cl}(B)$ is closed in Y. Therefore, by (5) we have $\alpha\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^-(\mathrm{Cl}(B))$.

(6) → (7): Let B be any subset of Y. Then, we obtain

$$\begin{gathered} X - F^+(\mathrm{Int}(B)) = F^-(\mathrm{Cl}(Y-B)) \supset \alpha\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(Y-B)))) = \\ \alpha\mathrm{Cl}(F^-(Y - \mathrm{Cl}(\mathrm{Int}(B))))) = \\ \alpha\mathrm{Cl}(X - F^+(\mathrm{Cl}(\mathrm{Int}(B)))) = X - \alpha\mathrm{Int}(F^+(\mathrm{Cl}(\mathrm{Int}(B)))). \end{gathered}$$

Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset \alpha\mathrm{Int}(F^+(\mathrm{Cl}(\mathrm{Int}(B))))$.

(7) → (8): The proof is obvious.

(8) → (1): Let x be any point of X and V be any open set of Y containing $F(x)$. Then, it follows from [1, Theorem 23] that $x \in F^+(V) \subset \alpha\mathrm{Int}(F^+(\mathrm{Cl}(V))))) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(\mathrm{Cl}(V)))))$ and hence F is u.w.α.c. at x by Theorem 1.

(5) → (9): The proof is obvious.

(9) → (10): Let V be any open set of Y. Then $\mathrm{Cl}(V)$ is regular closed in Y and hence we have $\alpha\mathrm{Cl}(F^-(V)) \subset \alpha\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(V))$.

(10) → (8): Let V be any open set of Y. Then we have

$$X - \alpha\mathrm{Int}(F^+(\mathrm{Cl}(V))))) = \alpha\mathrm{Cl}(X - F^+(Cl(V))))) = \alpha\mathrm{Cl}(F^-(Y - \mathrm{Cl}(V))) \\ \subset F^-(\mathrm{Cl}(Y - \mathrm{Cl}(V)))) = X - F^+(\mathrm{Int}(\mathrm{Cl}(V))).$$

Therefore, we obtain $F^+(V) \subset F^+(\mathrm{Int}(\mathrm{Cl}(V))) \subset \alpha\mathrm{Int}(F^+(\mathrm{Cl}(V)))$.

(10) → (11): Let B any subset of Y. Put $V = \mathrm{Int}(Cl_\theta(B))$ in (10). Then, since $Cl_\theta(B)$ is closed in Y, we have $\alpha\mathrm{Cl}(F^-(\mathrm{Int}(Cl_\theta(B)))) \subset F^-(Cl_\theta(B))$.

(11) → (9): Let K be any regular closed set of Y. In general, we have $\mathrm{Cl}(V) = Cl_\theta(V)$ for every open set V of Y. Therefore, we have

$$\alpha\mathrm{Cl}(F^-(\mathrm{Int}(K))) = \alpha\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(K))))) = \alpha\mathrm{Cl}(F^-(\mathrm{Int}(Cl_\theta(\mathrm{Int}(K))))) \\ \subset F^-(Cl_\theta(\mathrm{Int}(K))) = F^-(\mathrm{Cl}(\mathrm{Int}(K))) = F^-(K). \quad \square$$

Theorem 4. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) F is l.w.α.c.;

(2) for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $U \subset F^-(Cl(V))$;

(3) $F^-(V) \subset Int(Cl(Int(F^-(Cl(V)))))$ for every open set V of Y;

(4) $Cl(Int(Cl(F^+(Int(K))))) \subset F^+(Int(K))$ for every closed set K of Y;

(5) $\alpha Cl(F^+(Int(K))) \subset F^+(K)$ for every closed set K of Y;

(6) $\alpha Cl(F^+(Int(Cl(B)))) \subset F^+(Cl(B))$ for every closed set B of Y;

(7) $F^-(Int(B)) \subset \alpha Int(F^-(Cl(Int(B))))$ for every subset B of Y;

(8) $F^-(V) \subset \alpha Int(F^-(Cl(V)))$ for every open set V of Y;

(9) $\alpha Cl(F^+(Int(K))) \subset F^+(K)$ for every regular set K of Y;

(10) $\alpha Cl(F^+(V)) \subset F^+(Cl(V))$ for every open set V of Y;

(11) $\alpha Cl(F^+(Int(Cl_\theta(B))))) \subset F^+(Cl_\theta(B))$ for every subset B of Y;

Proof. The proof is similar to that of Theorem 3. □

Lemma 2. *If $F : X \to Y$ is l.w.α.c., then for each $x \in X$ and each subset B of Y with $F(x) \cap Int_\theta(B) \neq \emptyset$ there exists $U \in \alpha(X, x)$ such that $U \subset F^-(B)$.*

Proof. Since $F(x) \cap Int_\theta(B) \neq \emptyset$, there exists a nonempty open set V of Y such that $V \subset \mathrm{Cl}(V) \subset B$ and $F(x) \cap V \neq \emptyset$. Since F is l.w.α.c., there exists $U \in \alpha(X, x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$ and hence $U \subset F^-(B)$.
□

Theorem 5. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) F is l.w.α.c.;

(2) $\alpha Cl(F^+(B)) \subset F^+(Cl_\theta(B))$ *for every subset B of Y;*

(3) $F(\alpha Cl(A)) \subset Cl_\theta(F(A))$ *for every subset A of X.*

Proof. (1) → (2): Let B be any subset of Y. Suppose that $x \in F^-(Y - Cl_\theta(B)) = F^-(Int_\theta(Y - B))$. By Lemma 2, there exists $U \in \alpha(X, x)$ such that $U \subset F^-(Y - B) = X - F^+(B)$. Thus $U \cap F^+(B) = \emptyset$ and hence $x \in X - \alpha\mathrm{Cl}(F^+(B))$.

(2) → (1): Let V be any open set of Y. Since $\mathrm{Cl}(V) = Cl_\theta(V)$ for every open set V of Y, we have $\alpha\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ and by Theorem 4 F is l.w.α.c.

(2) → (3): Let A be any subset of X. By (2), we have

$$\alpha\mathrm{Cl}(A) \subset \alpha\mathrm{Cl}(F^+(F(A))) \subset F^+(Cl_\theta(F(A))).$$

Thus we obtain $F(\alpha\mathrm{Cl}(A)) \subset Cl_\theta(F(A))$.

(3) → (2): Let B be any subset of Y. By (3), we obtain

$$F(\alpha\mathrm{Cl}(F^+(B))) \subset Cl_\theta(F(F^+(B))) \subset Cl_\theta(B).$$

Thus we obtain $\alpha\mathrm{Cl}(F^+(B)) \subset F^+(Cl_\theta(B))$. □

A function $f : X \to Y$ is said to be weakly α-continuous [18] if for each $x \in X$ and each open set V containing $f(x)$, there exists $U \in \alpha(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$.

Corollary 1. *(Noiri [18], Sen and Bhattacharyya [32]). The following are equivalent for a function* $f : X \to Y$*:*

(1) f is weakly α-continuous;

(2) $f^{-1}(V) \subset \alpha Int(f^{-1}(Cl(V)))$ *for every open set V of Y;*

(3) $\alpha Cl(f^{-1}(Int(K))) \subset f^{-1}(K)$ *for every regular closed set K of Y;*

(4) $\alpha Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ *for every open set V of Y;*

(5) $\alpha Cl(f^{-1}(Int(Cl_\theta(B)))) \subset f^{-1}(Cl_\theta(B))$ *for every open set B of Y;*

(6) $Cl(Int(Cl(f^{-1}(V))))) \subset f^{-1}(Cl(V))$ *for every open set V of Y;*

(7) $f^{-1}(V) \subset Int(Cl(Int(f^{-1}(Cl(V)))))$ *for every open set V of Y;*

(8) $f(Cl(Int(Cl(A)))) \subset Cl_\theta(f(A)))$ *for every subset A of X;*

(9) $Cl(Int(Cl(f^{-1}(B)))) \subset f^{-1}(Cl_\theta(B))$ *for every subset B of Y.*

For a multifunction $F : X \to Y$, by $ClF : X \to Y$ [2] (resp. $\alpha ClF : X \to Y$ [26]) we denote a multifunction defined as follows: $(ClF)(x) = \mathrm{Cl}(F(x))$ (resp. $(\alpha ClF)(x) = \alpha\mathrm{Cl}(F(x))$) for each $x \in X$.

Definition 5. *A subset A of a topogical space X is said to be*

(1) α-paracompact [36] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,

(2) α-regular [6] (resp. α-almost-regular [7]) if for each $a \in A$ and each open (resp. regular open) set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

Lemma 3. *(Kovačević [6]). If A is an α-regular α- paracompact set of a topological space X and U is an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Lemma 4. *(Popa and Noiri [28]). If $F : X \to Y$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then for each open set V of Y $G^+(V) = F^+(V)$, where G denotes αClF or ClF.*

Theorem 6. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$. Then the following are equivalent:*

(1) F is u.w.α.c.;

(2) αClF is u.w.α.c.;

(3) ClF is u.w.α.c.

Proof. Similarly to Lemma 4, we put $G = \alpha ClF$ or ClF. First, suppose that F is u.w.α.c.
Let $x \in X$ and V be any open set of Y containing $G(x)$. By Lemma 4, $x \in G^+(V) = F^+(V)$ and there exists $U \in \alpha(X, x)$ such that $F(u) \subset \mathrm{Cl}(V)$ for each $u \in U$. Therefore, we have $(\alpha ClF)(u) \subset (ClF)(u) \subset \mathrm{Cl}(V)$; hence $G(u) \subset \mathrm{Cl}(V)$ for each $u \in U$. This shows that G is u.w.α.c.

Conversely, suppose that G is u.w.α.c. Let $x \in X$ and V be any open set of Y containing $F(x)$. By Lemma 4, $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. There exists $U \in \alpha(X, x)$ such that $G(U) \subset \mathrm{Cl}(V)$; hence $F(U) \subset \mathrm{Cl}(V)$. This shows that F is u.w. α.c. □

Lemma 5. *(Popa and Noiri [28]). If $F : X \to Y$ is a multifunction, then for each open set V of Y $G^-(V) = F^-(V)$, where G denotes αClF or ClF.*

Theorem 7. *For a multifunction $F : X \to Y$, the following are equivalent:*

(1) F is l.w.α.c.;

(2) αClF is l.w.α.c.;

(3) ClF is l.w.α.c.

Proof. By utilizing Lemma 5, this can be proved in a similar way as Theorem 6. □

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows:

$$G_F(x) = \{x\} \times F(x) \quad \text{for every} \quad x \in .X$$

Lemma 6. *(Noiri and Popa [20]). For a multifunction $F : X \to Y$, the following hold:*

$$(a) \quad G_F^+(A \times B) = A \cap F^+(B) \text{ and } (b) \quad G_F^-(A \times B) = A \cap F^-(B)$$

for any subsets $A \subset X$ and $B \subset Y$.

Theorem 8. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is u.w.α.c. if and only if $G_F : X \to Y$ is u.w.α.c.*

Proof. Necessity. Suppose that $F : X \to Y$ is u.w.α.c. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family $\{V(y) : y \in F(x)\}$ is open cover of $F(x)$ and $F(x)$ is compact. Therefore, there exists a finite number of points,say, $y_1, y_2, , ...y_n$ in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \leq i \leq n\}$. Set

$$U = \cap\{U(y_i) : 1 \leq i \leq n\} \text{ and } V = \cap\{V(y_i) : 1 \leq i \leq n\}.$$

Then U and V are open in X and Y, respectively, and $\{x\} \times F(x) \subset U \times V \subset W$. Since F is u.w.α.c., there exists $U_0 \in \alpha(X, x)$ such that $F(U_0) \subset \mathrm{Cl}(V)$. By Lemma 6, we have

$$U \cap U_0 \subset U \cap F^+(\mathrm{Cl}(V)) = G_F^+(U \times \mathrm{Cl}(V)) \subset G_F^+(\mathrm{Cl}(W)).$$

Therefore, we obtain $U \cap U_0 \in \alpha(X, x)$ and $G_F(U \cap U_0) \subset \mathrm{Cl}(W)$. This shows that G_F is u.w.α.c.

Sufficiency. Suppose that $G_F : X \to X \times Y$ is u.w.α.c. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \alpha(X, x)$ such that $G_F(U) \subset \mathrm{Cl}(X \times V = X \times \mathrm{Cl}(V)$. By Lemma 6, we have $U \subset G_F^+(X \times \mathrm{Cl}(V)) = F^+(\mathrm{Cl}(V))$ and $F(U) \subset \mathrm{Cl}(V)$. This shows that F is u.w.α.c. □

Theorem 9. *A multifunction $F : X \to Y$ is l.w.α.c. if and only if $G_F : X \to X \times Y$ is l.w.α.c.*

Proof. Necessity. Suppose that F is l.w.α.c. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets $U \subset X$ and $V \subset Y$. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \alpha(X, x)$ such that $G \subset F^-(\mathrm{Cl}(V))$. By Lemma 6, we have

$$U \cap G \subset U \cap F^-(\mathrm{Cl}(V)) = G_F^-(U \times \mathrm{Cl}(V)) \subset G_F^-(\mathrm{Cl}(W))$$

Moreover, we have $U \cap G \in \alpha(X, x)$ and hence G_F is l.w.α.c.

Sufficiency. Suppose that G_F is l.w.α.c. Let $x \in X$ and V be any open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and

$$G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset.$$

Since G_F is l.w.α.c., there exists $U \in \alpha(X, x)$ such that $U \subset G_F^-(\mathrm{Cl}(X \times V)) = G_F^-(X \times \mathrm{Cl}(V))$. By Lemma 6, we obtain $U \subset F^-(\mathrm{Cl}(V))$. This shows that F is l.w.α.c. □

Corollary 2. *(Noiri [18]). A function $F : X \to Y$ is weakly α-continuous if and only if the graph function $g : X \to X \times Y$, defined as follows: $g(x) = (x, f(x))$ for each $x \in X$, is weakly α-continuous.*

Lemma 7. *(Mashhour et al. [11], Reilly and Vamanamurthy [30]). Let U and X_0 be subsets of a topological space X. The following properties hold:*

(1) if $U \in \alpha(X)$ and $X_0 \in SO(X) \cup PO(X)$, then $U \cap X_0 \in \alpha(X_0)$.

(2) If $U \subset X_0 \subset X$, $U \in \alpha(X_0)$ and $X_0 \in \alpha(X)$, ,then $U \in \alpha(X)$.

Theorem 10. *If a multifunction $F : X \to Y$ is u.w.α.c. (resp. l.w.α.c.) and $X_0 \in SO(X) \cup PO(X)$, then the restriction $F/X_0 : X_0 \to Y$ is u.w.α.c. (resp. l.w.α.c.).*

Proof. We prove only the first case, the proof of the second being analogous. Let $x \in X_0$ and V be any open sets of Y such that $(F/X_0)(x) \subset V$. Since $(F/X_0)(x) = F(x)$ and F is u.w.α.c., there exists $U \in \alpha(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$. Let $U_0 = U \cap X_0$, then $U_0 \in \alpha(X_0, x)$ by Lemma 7 and $(F/X_0)(U_0) = F(U_0) \subset \mathrm{Cl}(V)$. This shows that F/X_0 is u.w.α.c. □

Corollary 3. *(Noiri [18]). If $f : X \to Y$ is weakly α- continuous and $X_0 \in SO(X) \cup PO(X)$, then the restriction $f/X_0 : X_0 \to Y$ is weakly α-continuous.*

Theorem 11. *A multifunction $F : X \to Y$ is u.w.α.c. (resp. l.w.α.c.) if for each $x \in X$ there exists $X_0 \in \alpha(X, x)$ such that the restriction $F/X_0 : X_0 \to Y$ is u.w.α.c. (resp. l.w.α.c.).*

Proof. We prove only the first case, the proof of the second being analogous. Let $x \in X$ and V be any open sets of Y such that $F(x) \subset V$. There exists $X_0 \in \alpha(X, x)$ such that $F/X_0 \to Y$ is u.w.α.c. Therefore, there exists $U_0 \in \alpha(X_0, x)$ such that $(F/X_0)(U_0) \subset \mathrm{Cl}(V)$. By Lemma 7, $U_0 \in \alpha(X, x)$ and $F(u) = (F/X_0)(u)$ for each $u \in U_0$. This shows that F is u.w.α.c. □

Corollary 4. *Let $\{U_\alpha : \alpha \in \nabla\}$ be a cover of X by α-open sets of X. Then, a multifunction $F : X \to Y$ is u.w.α.c. (resp. l.w.α.c.) if and only if the restriction $F/U_\alpha : U_\alpha \to Y$ is u.w.α.c. (resp. l.w.α.c.) for each $\alpha \in \nabla$.*

Proof. This is an immediate consequence of Theorems 10 and 11. □

Corollary 5. *(Sen and Bhattacharyya [32]). Let $f : X \to Y$ be a function and $X = X_1 \cup X_2$, where X_1 and X_2 are α-open in X. If the restrictions $f/X_1 : X_1 \to Y$ are weakly α- continuous for each i=1,2, then f is weakly α-continuous.*

4. Weak α-continuity, almost α-continuity and α- continuity

Theorem 12. *If $F : X \to Y$ is a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is a normal space, then the following are equivalent:*

(1) F is upper α-continuous;

(2) F is upper almost α-continuous;

(3) F is u.w.α.c.

Proof. We prove only the implication (3) $\to$ (1). Suppose that F is u.w.α.c. Let $x \in X$ and V be any open sets of Y such that $F(x) \subset V$. Since $F(x)$ is closed in Y, by the normality of Y there exists an open set W of Y such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since F is u.w.α.c., there exists $U \in \alpha(X, x)$ such that $F(U) \subset \mathrm{Cl}(W)$; hence $F(U) \subset V$. This shows that F is upper α-continuous. □

Definition 6. *A multifunction $F : X \to Y$ is said to be α-preopen if for every $U \in \alpha(X), F(U) \subset Int(Cl(F(U)))$.*

Theorem 13. *If a multifunction $F : X \to Y$ is u.w.α.c. and α-preopen, then F is upper almost α-continuous.*

Proof. For any $x \in X$ and any open set V of Y containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$. Since F is α- preopen, we have $F(U) \subset \mathrm{Int}(\mathrm{Cl}(F(U))) \subset \mathrm{Int}(\mathrm{Cl}(V)) = s\mathrm{Cl}(V)$. It follows from [27, Theorem 3] that F is upper almost α-continuous. □

Theorem 14. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is open in Y for each $x \in X$. Then the following are equivalent:*

(1) F is lower α-continuous;

(2) F is lower almost α-continuous;

(3) F is l.w.α.c.

Proof. We shall only show that (3) implies (1). Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. There exists $U \in \alpha(X, x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$. Since $F(u)$ is open in Y, $F(u) \cap V \neq \emptyset$ for every $u \in U$ and hence F is lower α-continuous. □

Definition 7. *A topological space X is said to be almost regular [33] if for each $x \in X$ and each regular closed set F of X not containing x, there exists disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.*

Theorem 15. *If a multifunction $F : X \to Y$ is u.w.α.c. and $F(x)$ is an α-almost regular and α-paracompact subset of Y for each $x \in X$, then F is upper almost α-continuous.*

Proof. Let V be any regular open set of Y containing $F(x)$. Since $F(x)$ is α-almost regular and α-paracompact, by [24, Lemma 2] there exists an open set H of Y such that $F(x) \subset H \subset \mathrm{Cl}(H) \subset V$. Since F is u.w.α.c. and $F(x) \subset H$, there exists $U \in \alpha(X,x)$ such that $F(U) \subset Cl(H) \subset V$. Therefore, it follows from [27, Theorem 3] that F is upper almost α-continuous. □

Corollary 6. *If a multifunction $F : X \to Y$ is u.w.α.c., Y is almost regular and $F(x)$ is α-paracompact for each $x \in X$, then F is upper almost α-continuous.*

Theorem 16. *If a multifunction $F : X \to Y$ is l.w.α.c. and $F(x)$ is an α-almost regular subset of Y for each $x \in X$, then F is lower almost α-continuous.*

Proof. Let V be a regular open set of Y such that $F(x) \cap V \neq \emptyset$. Since $F(x)$ is α-almost regular, by [24, Lemma 5] there exists an open set H of Y such that $F(x) \cap H \neq \emptyset$ and $\mathrm{Cl}(H) \subset V$. Since F is l.w.α.c. and $F(x) \cap H \neq \emptyset$, there exists $U \in \alpha(X,x)$ such that $F(u) \cap \mathrm{Cl}(H) \neq \emptyset$; hence $F(u) \cap V \neq \emptyset$ for every $u \in U$. It follows from [27, Theorem 5] that F is lower almost α-continuous. □

Corollary 7. *If a multifunction $F : X \to Y$ is l.w.α.c. and Y is almost regular, then F is lower almost α-continuous.*

Definition 8. *A topological space X is said to be*

(1) α-compact [9] if every cover of X by α-open sets of X has a finite subcover,

(2) quasi H-closed [29] if for every open cover $\{U_\alpha : \alpha \in \nabla\}$ of X, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{Cl(U_\alpha) : \alpha \in \nabla_0\}$.

Theorem 17. *Let $F : X \to Y$ be a surjective multifunction, X α-compact and Y a T_4-space. If F is u.w.α.c. and $F(x)$ is compact for each $x \in X$, then F is upper almost α-continuous.*

Proof. It follows from [27, Theorem 19] that Y is quasi H-closed. Every quasi H-closed T_4-space is almost regular [21, p. 139]. Therefore, it follows from Corollary 6 that F is upper almost α-continuous. □

Definition 9. *A multifunction $F : X \to Y$ is said to be weak* $\alpha-$continuous if for each open set V of Y, $F^-(Fr(V))$ is α- closed in X, where $Fr(V)$ denotes the frontier of V.*

Theorem 18. *A multifunction $F : X \to Y$ is upper α- continuous if and only if it is u.w.α.c. and weak* α- continuous.*

Proof. Necessity. The proof follows from definition of upper α-continuous, u.w.α.c. and *weak** α-continuous and [26, Theorem 3.3].

Sufficiency. Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. By Theorem 3, there exists $G \in \alpha(X,x)$ such that $F(G) \subset \mathrm{Cl}(V)$. Now put $U =$

$G \cap (X - F^-(Fr(V)))$. Since $F^-(Fr(V))$ is α-closed in X, by [16, Lemma 3.2] $U \in \alpha(X)$. Moreover we have $F(x) \cap Fr(V) = \emptyset$ and hence $x \in X - F^-(Fr(V))$. Therefore, we obtain $x \in U$ and $F(U) \subset V$ since $F(U) \subset F(G) \subset \mathrm{Cl}(V)$ and $F(U) \subset Y - Fr(V)$. Thus, F is upper α-continuous. □

A function $f : X \to Y$ is said to be *weak** $\alpha -$ *continuous* [32] (resp. $\alpha -$ *continuous* [11]) if for each open set V of Y, $f^{-1}(Fr(V))$ is α-closed (resp. $f^{-1}(V)$ is α-open) in X.

Corollary 8. Corollary 8 *(Sen and Bhattacharyya [32]). A function $f : X \to Y$ is α-continuous if and only if it is weakly α-continuous and weak* α-continuous.*

5. Weakly α-continuous multifunctions into Urysohn spaces

A topological space X is said to be *Urysohn* if for each pair of distinct points x and y of X, there exist open sets U and V such that $x \in U, y \in V$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(V) = \emptyset$.

Lemma 8. *(Smithson [34]). If A and B are disjoint compact subsets of a Urysohn space X, then there exists open sets U and V of X such that $A \subset U, B \subset V$ and $Cl(U) \cap Cl(V) = \emptyset$.*

Theorem 19. *If $F, G : (X, \tau) \to (Y, \sigma)$ are u.w.α.c. multifunctions into a Urysohn space Y and for each $x \in X$ F(x) and G(x) are compact in (Y, σ) , then $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is α-closed in (X, τ).*

Proof. By [27, Teorem 7], multifunctions $F, G : (X, \tau^\alpha) \to (Y, \sigma)$ are upper weakly continuous and A is closed in (X, τ^α) [34, Theorem 17]. Therefore, A is α-closed in (X, τ). □

Corollary 9. *(Sen and Bhattacharyya [32]). If $f, g : X \to Y$ are weakly α-continuous functions and Y is a Urysohn space, then $\{x \in X : f(x) = g(x)\}$ is α-closed in X.*

Theorem 20. *Let $F, G : X \to Y$ be multifunctions into an Urysohn space Y and $F(x)$, $G(x)$ compact in Y for each $x \in X$. If F is u.w.α.c. and G is upper almost weakly continuous, then $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is preclosed in X.*

Proof. Let $x \in X - A$. Then we have $F(x) \cap G(x) = \emptyset$. By Lemma 8 there exist open sets V and W such that $F(x) \subset V, G(x) \subset W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since F is u.w.αc., there exists $U_1 \in \alpha(X, x)$ such that $F(U_1) \subset \mathrm{Cl}(V)$. Since G is upper almost weakly continuous, by [20, Theorem 3.1] there exists $U_2 \in PO(X, x)$ such that $G(U_2) \subset \mathrm{Cl}(W)$. Now, put $U = U_1 \cap U_2$, then we have

$U \in PO(X,x)$ [25, Lemma 4.1] and $U \cap A = \emptyset$. Therefore, A is preclosed in X. □

A function $f : X \to Y$ is said to be *almost weakly continuous* [25] if for each set V of Y, $f^{-1}(V) \subset \text{Int}(\text{Cl}(f^{-1}(\text{Cl}(V))))$.

Corollary 10. *(Popa and Noiri [25]). Let $f, g : X \to Y$ be functions into a Urysohn space Y. If f is weakly α-continuous and g is almost weakly continuous, then $\{x \in X : f(x) = g(x)\}$ is preclosed in X.*

Theorem 21. *Let $F : X_1 \to Y$ and $G : X_2 \to Y$ be multifunctions into a Urysohn space Y and $F(x)$, $G(x)$ compact in Y for each $x \in X_1$ and each $i = 1, 2$. If F is u.w.α.c. and G is upper almost weakly continuous, then $A = \{(x_1, x_2) : F(x_1) \cap G(x_2) \neq \emptyset\}$ is preclosed set of the product space $X_1 \times X_2$.*

Proof. We shall show that $X_1 \times X_2 - A$ is preopen in $X_1 \times X_2$. Let $(x_1, x_2) \in X_1 \times X_2 - A$. Then we have $F(x_1) \cap G(x_2) = \emptyset$. By Lemma 8, there exist open sets V and W such that $F(x) \subset V, G(x) \subset W$ and $\text{Cl}(V) \cap \text{Cl}(W) = \emptyset$. Since F is u.w.α.c., by Theorem 3 we have $x_1 \in F^+(V) \subset \alpha\text{Int}(F^+(\text{Cl}(V)))$. Since G is upper almost weakly continuous, by [20, Theorem 3.1] we have $x_2 \in G^+(W) \subset p\text{Int}(G^+(\text{Cl}(W)))$. Now, put $U = \alpha\text{Int}(F^+(\text{Cl}(V))) \times p\text{Int}(G^+(\text{Cl}(W)))$, then we have $U \in PO(X_1 \times X_2)$ [23, Lemma 2] and $(x_1, x_2) \in U \subset X_1 \times X_2 - A$. Therefore, A is preclosed in $X_1 \times X_2$. □

Theorem 22. *Let $F, G : X \to Y$ be multifunctions into a Urysohn space Y and $F(x)$, $G(x)$ compact in Y for each $x \in X$. If F is u.w.α.c. and G is upper weakly quasicontinuous, then $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is semi-closed in X.*

Proof. The proof is similar to that of Theorem 20. □

Theorem 23. *Let $F : X_1 \to Y$ and $G : X_2 \to Y$ be multifunctions into a Urysohn space Y and $F(x)$, $G(x)$ compact in Y for each $x \in X_1$ and each $i = 1, 2$. If F is u.w.α.c. and G is upper weakly quasicontinuous, then $\{(x_1, x_2) : F(x_1) \cap G(x_2) \neq \emptyset\}$ is a semi-closed set of the product space $X_1 \times X_2$.*

Proof. The proof is similar to that of Theorem 21. □

Definition 10. *For a multifunction $F : X \to Y$, the graph $G(F) = \{(x, F(x)) : x \in X\}$ is said to be strongly α-closed if for each $(x, y) \in (X \times Y) - G(F)$, there exists $U \in \alpha(X, x)$ and $V \in \alpha(Y, y)$ such that $[U \times \alpha Cl(V)] \cap G(F) = \emptyset$.*

Lemma 9. *A multifunction $F : X \to Y$ has a strongly α-closed graph if and only if for each $(x, y) \in (X \times Y) - G(F)$, there exist $U \in \alpha(X, x)$ and $V \in \alpha(Y, y)$ such that $F(U) \cap Cl(V) = \emptyset$.*

Proof. For any $V \in \alpha(Y)$, we have $\mathrm{Cl}(V) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(V)))) = \mathrm{Cl}(\mathrm{Int}(V))$ and hence by Lemma 1 $\alpha\mathrm{Cl}(V) = V \cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V))) = V \cup \mathrm{Cl}(\mathrm{Int}(V)) = \mathrm{Cl}(V)$. Therefore, the proof is obvious. □

Theorem 24. *If $F : X \to Y$ is u.w.α.c. multifunction such that $F(x)$ is compact for each $x \in X$ and Y is a Urysohn space, then $G(F)$ is strongly α-closed.*

Proof. Let $(x, y) \in (X \times Y) - G(F)$, then $y \in Y - F(x)$. By Lemma 8, there exist open sets V and W of Y such that $y \in V, F(x) \subset W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since F is u.w.α.c., there exists $U \in \alpha(X, x)$ such that $F(U) \subset \mathrm{Cl}(W)$. Therefore, we have $F(U) \cap \mathrm{Cl}(V) = \emptyset$ and by Lemma 9 $G(F)$ is strongly α-closed. □

Corollary 11. *(Sen and Bhattacharyya [32]). If $f : X \to Y$ is a weakly α-continuous function and Y is a Urysohn space, then $G(f)$ is strongly α-closed.*

Theorem 25. *Let $F_1, F_2 : (X, \tau) \to (Y, \tau)$ be u.w.α. c. multifunctions into a Urysohn space (Y, σ) and $F_i(x)$ compact in Y for each $x \in X_1$ and each i=1,2. If $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$, then a multifunction $F : (X, \tau) \to (Y, \sigma)$, defined as follows $F(x) = F_1(x) \cap F_2(x)$ for each $x \in X$,is u.w.α .c.*

Proof. By [27, Theorem 7] $F_1, F_2 : (X, \tau^\alpha) \to (Y, \sigma)$ are upper weakly continuous and by [34, Theorem 18] $F : (X, \tau^\alpha) \to (Y, \sigma)$ is upper weakly continuous. Therefore, $F : (X, \tau) \to (Y, \sigma)$ is u.w.α.c. [27, Theorem 7]. □

Lemma 10. *If A is α-open and α-closed in a space X, then A is closed in X.*

Proof. Let A be an α-open and α-closed set of X. Then we have $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset A$. Therefore, we have $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))) = \mathrm{Cl}(\mathrm{Int}(A))$ and hence $\mathrm{Cl}(A) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset A$. This shows that A is closed in X. Therefore, we have $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \subset \mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(A)$ and hence A is open. Consequently, A is clopen in X. □

Lemma 11. *If a multifunction $F : X \to Y$ is u.w.α.c., and l.w.α.c., then $F^+(V)$ is clopen in X for every clopen set V of Y.*

Proof. Let V be any clopen set of Y. It follows from Theorem 3 that

$$F^+(V) \subset \alpha\mathrm{Int}(F^+(\mathrm{Cl}(V))) = \alpha\mathrm{Int}(F^+(V)).$$

This shows that $F^+(V)$ is α-open in X. Furthermore, since V is open, it follows from Theorem 4 that $\alpha\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V)) = F^+(V)$. Thus, $F^+(V)$ is α-closed. Therefore, it follows from Lemma 10 $F^+(V)$ is clopen in X. □

Theorem 26. *Let $F : X \to Y$ be an u.w.α.c. and l.w.α.c. surjective multifunction. If X is connected and $F(x)$ is connected for each $x \in X$, then Y is connected.*

Proof. Suppose that Y is not connected. There exist nonempty open sets U and V of Y such that $U \cup V = Y$ and $U \cap V = \emptyset$. Since $F(x)$ is connected for each $x \in X$, we have either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V = X)$,

(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$,

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

By Lemma 11, $F^+(U)$ and $F^+(V)$ are clopen. Consequently, X is not connected. □

Corollary 12. *(Noiri [18]). If $f : X \to Y$ is a weakly α- continuous surjection and X is connected, then Y is connected.*

Definition 11. *An u.w.α.c. multifunction $F : X \to A$ of a space X onto a subset A of X is called a retraction [34] if F(a)=a for all $a \in A$.*

Theorem 27. *If $F : (X, \tau) \to A$ is an u.w.α.c. retraction, (X, τ) is Hausdorff and F(x) is compact for each $x \in X$, then A is α-closed in (X, τ).*

Proof. By [27, Theorem 7], $F : (X, \tau^\alpha) \to A$ is upper weakly continuous. by [34, theorem 10], A is closed in (X, τ^α) and hence A is α-closed in (X, τ). □

Corollary 13. *(Sen and Bhattacharyya [32]). Let $A \subset X$ and $f : (X, \tau) \to A$ be a surjective weakly α-continuous retraction. If X is Hausdorff, then A is α-closed in X.*

Definition 12. *The $\alpha - frontier$ of a subset A of a space X, denoted by $\alpha Fr(A)$, is defined by $\alpha Fr(A) = \alpha Cl(A) \cap \alpha Cl(X - A) = \alpha Cl(A) - \alpha Int(A)$.*

Theorem 28. *The set of all points x of X at which a multifunction $F : X \to Y$ is not u.w.α.c. (resp. l.w.α.c.) is identical with the union of the α-frontier of the upper (resp. lower) inverse images of the closures of open sets containing (resp. meeting) F(x).*

Proof. Let x be a point of X at which F is not u.w.α.c. Then, there exists an open set V containing $F(x)$ such that $U \cap (X - F^+(\mathrm{Cl}(V))) \neq \emptyset$ for every $U \in \alpha(X, x)$. Then, we have $x \in \alpha Cl(X - F^+(\mathrm{Cl}(V)))$. Since $x \in F^+(V)$, we have $x \in \alpha Cl(F^+(Cl(V)))$ and hence $x \in \alpha Fr(F^+(\mathrm{Cl}(V)))$. If F is u.w.α.c. at x, then there exists $U \in \alpha(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$; hence $U \subset F^+(\mathrm{Cl}(V))$. Therefore, we obtain $x \in U \subset \alpha\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. This contradicts that $x \in \alpha Fr(F^+(\mathrm{Cl}(V)))$. Thus F is not u.w.α.c. at x. The case of l.w.α.c. is similarly shown. □

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