Novi Sad J. Math.
Vol. 32, No. 1, 2002, 59-71

ON A THEOREM OF ZABCZYK FOR SEMIGROUPS OF OPERATORS IN LOCALLY CONVEX SPACES

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Abstract. The purpose of this paper is to extend a stability theorem of Zabczyk to the case of semigroups of operators in locally convex topological vector spaces. Obtained results generalize the similar theorems proved by Datko, Pazy, Rolewicz and Littman for the case of C_0-semigroups of operators in Banach spaces.

AMS Mathematics Subject Classification (2000): 34D05,34D20

Key words and phrases: $\Phi - semigroup$, exponential stability

1. Introduction

Let X be a locally convex space whose topology is generated by the family of seminorms $\{|\cdot|_\gamma : \gamma \in \Gamma\}$. The space of all continuous linear operators from X into itself will be denoted by $B(X)$. For all $A \in B(X)$ and for all $\beta, \gamma \in \Gamma$ we shall denote

$$\|A\|_{\beta,\gamma} = \sup\{|Ax|_\gamma : |x|_\beta \leq 1\}.$$

It is obvious that $A \in B(X)$ if and only if for every $\gamma \in \Gamma$ there exists $\beta = \beta(\gamma) \in \Gamma$ such that $\|A\|_{\beta,\gamma} < \infty$.

Recall that a family $\mathbf{S} = (S(t))_{t\geq 0}$ of continuous linear operators from X into itself is a C_0-*semigroup* on X, if

s_1) $S(0) = I$ (the identity operator on X);

s_2) $S(t+s) = S(t)S(s)$, for all $t, s \geq 0$;

s_3) $\lim\limits_{t\to 0} |S(t)x - x|_\gamma = 0$, for all $x \in X$ and all $\gamma \in \Gamma$.

For details about C_0-semigroups in locally convex spaces see for instance [2] and [5].

In what follows we denote by Φ the set of all functions $\varphi : \mathbb{R}_+ \times \Gamma \to \Gamma$ with the properties

φ_1) $\varphi(0, \gamma) = \gamma$, for all $\gamma \in \Gamma$;

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φ_2) $\varphi(t+s,\gamma)=\varphi(t,\varphi(s,\gamma))$, for all
$t,s\geq 0$ and all $\gamma\in\Gamma$.
($\mathbb{R}_+=[0,\infty)$, $\mathbb{R}_+^*=(0,\infty)$.)

In this paper we consider a particular class of C_0-semigroups defined by

Definition 1.1 *A C_0-semigroup $\mathbf{S}=(S(t))_{t\geq 0}$ is called Φ-semigroup, if there exists $\varphi\in\Phi$ such that*

$$\|S(t)\|_{\varphi(t,\gamma),\gamma}<\infty \quad \textit{for all} \quad (t,\gamma)\in\mathbb{R}_+\times\Gamma.$$

Hence if $\mathbf{S}$ is an Φ-semigroup, then there exists $\varphi\in\Phi$ with

$$|S(t)x|_\gamma\leq\|S(t)\|_{\varphi(t,\gamma),\gamma}|x|_{\varphi(t,\gamma)}.$$

Definition 1.2 *An Φ-semigroup $\mathbf{S}=(S(t))_{t\geq 0}$ is said to be*

(i) exponentially bounded (and denote e.b.) if there exists $\varphi\in\Phi$ and $M,\omega:\Gamma\to\mathbb{R}_+^*$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma}\leq M(\gamma)e^{t\omega(\gamma)} \quad \textit{for all} \quad (t,\gamma)\in\mathbb{R}_+\times\Gamma;$$

(ii) uniformly exponentially bounded (and denote u.e.b.) if there exist the functions M and ω from (i) satisfying the conditions:

$$M_0(\gamma):=\sup_{t\geq 0}M(\varphi(t,\gamma))<\infty \;\; \textit{and} \;\; \omega_0(\gamma):=\sup_{t\geq 0}\omega(\varphi(t,\gamma)<\infty \;\; \textit{for all} \;\; \gamma\in\Gamma.$$

It is obvious that if $\mathbf{S}$ is u.e.b. then it is e.b.

Remark 1.1. If X is a Banach space then every C_0-semigroup $\mathbf{S}$ is an Φ-semigroup with u.e.b. (see [6]).

Definition 1.3 *An Φ-semigroup $\mathbf{S}=(S(t))_{t\geq 0}$ is said to be*

(i) stable (denoted s.) if there are $\varphi\in\Phi$ and $M:\Gamma\to\mathbb{R}_+^*$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma}\leq M(\gamma) \quad \textit{for all} \quad (t,\gamma)\in\mathbb{R}_+\times\Gamma;$$

(ii) uniformly stable (denoted u.s.) if it is stable and the function M from (i) satisfies the condition

$$M_0(\gamma):=\sup_{t\geq 0}M(\varphi(t,\gamma))<\infty \quad \textit{for all} \quad \gamma\in\Gamma;$$

(iii) exponentially stable (denoted e.s.) if there are $\varphi\in\Phi$ and $N,\nu:\Gamma\to\mathbb{R}_+^*$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma}\leq N(\gamma)e^{-t\nu(\gamma)} \quad \textit{for all} \quad (t,\gamma)\in\mathbb{R}_+\times\Gamma;$$

(iv) *uniformly exponentially stable (denoted u.e.s.) if it is e.s. and the functions N and ν from (iii) satisfy*

$$N_0(\gamma) := \sup_{t\geq 0} N(\varphi(t,\gamma)) < \infty \quad \textit{and} \quad \nu_0(\gamma) := \inf_{t\geq 0} \nu(\varphi(t,\gamma)) > 0 \quad \textit{for all} \quad \gamma \in \Gamma.$$

Remark 1.2. It is obvious that

$$\begin{array}{ccccc} \text{u.e.s.} & \Rightarrow & \text{u.s.} & \Rightarrow & \text{u.e.b.} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{e.s.} & \Rightarrow & \text{s.} & \Rightarrow & \text{e.b.} \end{array}$$

In Banach spaces we have that

$$\text{u.e.s.} \quad \Leftrightarrow \quad \text{e.s.} \quad \Rightarrow \quad \text{u.s.} \quad \Leftrightarrow \quad \text{s.}$$

In stability theory in Banach spaces a well-known result due to Zabczyk ([8]) is

Theorem 1.4 *Let $\mathbf{S} = (S(t))_{t\geq 0}$ be a C_0-semigroup on the Banach space X with the norm $\|\cdot\|$. $\mathbf{S}$ is u.e.s. if and only if there exists a strictly non-decreasing continuous convex function $R : \mathbb{R}_+ \to \mathbb{R}_+$ with $R(0) = 0$,such that for all $x \in X$ there exists $\alpha(x) > 0$ with*

$$\int_0^\infty R(\alpha(x)\|S(t)x\|)dt < \infty.$$

We observe that from Theorem 1.1. results the following corollary.

Corollary 1.5 *A C_0-semigroup $\mathbf{S}$ on the Banach space X is u.e.s. if and only if there exists a strictly increasing continuous convex function $R : \mathbb{R}_+ \to \mathbb{R}_+$ with $R(0) = 0$ such that for all $x \in X$ there is $\alpha(x) > 0$ with*

$$\sum_{n=0}^\infty R(\alpha(x)\|S(n)x\|) < \infty.$$

Firstly, we observe that in contrast to the case of Banach spaces Theorem 1.1. and Corollary 1.1. are not valid in locally convex spaces.

Example 1.1. Let X be the space of all complex continuous functions on $\mathbb{R}_+$ and $\Gamma = \mathbb{R}_+^*$.
The family $\{|\cdot|_\gamma : \gamma \in \Gamma\}$ given by

$$|x|_\gamma = |x(\gamma)| \quad \text{for all} \quad \gamma \in \Gamma$$

determines the structure of a locally convex space on X. It is easy to see that

$$\varphi : \mathbb{R}_+ \times \Gamma \to \Gamma, \quad \varphi(t,\gamma) = \gamma e^{2t}$$

belongs to Φ and $\mathbf{S} = (S(t))_{t\geq 0}$ defined by

$$S(t)x(s) = e^{-t}x(se^{2t}) \quad \text{for all} \quad (t,s) \in \mathbb{R}_+ \times \mathbb{R}_+^* \quad \text{and all} \quad x \in X,$$

is an Φ-semigroup with

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} = e^{-t} \quad \text{for all} \quad (t,\gamma) \in \mathbb{R}_+ \times \Gamma.$$

Hence $\mathbf{S}$ is u.e.s. We observe that for $x(s) = s$ we have that

$$\int_0^\infty R(\alpha(x)|S(t)x|_\gamma)dt = \int_0^\infty R(\alpha(x)\gamma e^t)dt = \infty$$

and

$$\sum_{n=1}^\infty R(\alpha(x)|S(n)x|_\gamma) = \sum_{n=1}^\infty R(\alpha(x)\gamma e^n) = \infty \quad \text{for all} \quad \gamma \in \Gamma, \alpha(x) > 0,$$

and for all strictly increasing, continuous, convex functions $R : \mathbb{R}_+ \to \mathbb{R}_+$.

Example 1.2. Let $\Gamma = \mathbb{R}$ and let X be the space of all complex continuous functions x with the property that there is $M_x > 0$ such that

$$|x(t)| \leq M_x|t| \quad \text{for all} \quad t \in \mathbb{R}.$$

The family $\{|\cdot|_\gamma : \gamma \in \Gamma\}$ defined by

$$|x|_\gamma = |x(\gamma)| \quad \text{for all} \quad \gamma \in \Gamma$$

determines the structure of a locally convex space on X. The function

$$\varphi : \mathbb{R}_+ \times \Gamma \to \Gamma, \quad \varphi(t,\gamma) = \gamma e^{+2t}$$

belongs to Φ and $\mathbf{S} = (S(t))_{t\geq 0}$ defined by

$$S(t)x(s) = e^t x(se^{-2t}) \quad \text{for all} \quad (t,s) \in \mathbb{R}_+ \times \mathbb{R} \quad \text{and all} \quad x \in X,$$

is an Φ-semigroup on X.

Because

$$\|S(t)\|_{\varphi(\gamma),\gamma} = e^t \quad \text{for all} \quad (t,\gamma) \in \mathbb{R}_+ \times \Gamma,$$

it follows that $\mathbf{S}$ is not u.e.s., even if

$$\int_0^\infty |S(t)x|_\gamma dt = \int_0^\infty |e^t x(\gamma e^{-2t})|dt \leq M_x|\gamma| \int_0^\infty e^{-t}dt = |\gamma|M_x < \infty$$

and

$$\sum_{n=1}^\infty |S(n)x|_\gamma \leq |\gamma|M_x \sum_{n=1}^\infty e^{-n} \leq |\gamma|M_x < \infty \quad \text{for all} \quad (t,\gamma,x) \in \mathbb{R}_+ \times \Gamma \times X.$$

To obtain a new characterization of the u.e.s. C_0-semigroups in Banach spaces we prove

Lemma 1.6 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$. Then*

$$F : \mathbb{R}_+ \to \mathbb{R}_+, F(t) = \int_0^t f(s)ds,$$

is a continuous, convex and strictly increasing bijection.

Proof. We observe that $F(0) = 0$ and F is a non-decreasing function. If there exists $t_1 < t_2$ such that $F(t_1) = F(t_2)$ then

$$0 = F(t_2) - F(t_1) = \int_{t_1}^{t_2} f(s)ds$$

which is a contradiction because $f(t) > 0$ for all $t > 0$. Hence F is strictly increasing.

Since $\lim\limits_{t\to\infty} f(t) > 0$, it follows that

$$\lim_{t\to\infty} F(s)ds = \int_0^\infty f(s)ds = \infty.$$

which shows that F is a continuous bijection.

If $t_1, t_2 \in \mathbb{R}_+$ then

$$F(\frac{t_1 + t_2}{2}) - F(t_1) = \int_{t_1}^{\frac{t_1+t_2}{2}} f(s)ds = \frac{1}{2} \int_{t_1}^{t_2} f(\frac{t_1 + t}{2})dt \leq$$

$$\leq \frac{1}{2} \int_{t_1}^{t_2} f(t)dt = \frac{F(t_2) - F(t_1)}{2},$$

and hence

$$F(\frac{t_1 + t_2}{2}) \leq \frac{F(t_1) + F(t_2)}{2},$$

which shows that F is convex. □

Theorem 1.7 *A C_0-semigroup* **S** *on the Banach space X is u.e.s. if and only if there exists a non-decreasing continuous function $R : \mathbb{R}_+ \to \mathbb{R}_+$ with the properties:*

(i) $R(0) = 0$;

(ii) $R(t) > 0$ *for all* $t > 0$;

(iii) $\sum\limits_{n=0}^{\infty} R(\|S(n)\|) < \infty.$

Proof. Necessity. It is a simple verification for $R(t) = t$.
Sufficiecy. We will prove that

$$\sup_{n\geq 0} \|S(n)\| < \infty.$$

If the inequality does not hold then there is a strictly increasing sequence $(k_n)_{n\geq 0}$ of natural numbers such that

$$\lim_{n\to\infty} \|S(k_n)\| = \infty.$$

By (iii) $\lim\limits_{n\to\infty} R(\|S(n)\|) = 0$ wich implies that $\lim\limits_{n\to\infty} R(\|S(k_n)\|) = 0$. It follows that

$$R(1) \leq \lim_{n\to\infty} R(\|S(k_n)\|) = 0,$$

which is a contradiction.

Let $M = \sup\limits_{n\geq 0} \|S(n)\|$ and $R_1 : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$R_1(t) = \int_0^t R(s)ds.$$

By Lemma 1.1 R_1 is a strictly increasing, continuous, convex bijection with $R_1(0) = 0$.

For $x = 0, \alpha(x) = 1$ we have

$$\sum_{n=0}^{\infty} R_1(\alpha(x)\|S(n)x\|) = 0.$$

For $x \neq 0, \alpha(x) = \frac{1}{\|x\|}$ we have

$$\sum_{n=0}^{\infty} R_1(\alpha(x)\|S(n)x\|) \leq \sum_{n=0}^{\infty} R_1(\alpha(x)\|S(n)\|\|x\|) \leq$$

$$\leq \sum_{n=0}^{\infty} R_1(\|S(n)\|) \leq \sum_{n=0}^{\infty} \|S(n)\|R(\|S(n)\|) \leq M \sum_{n=0}^{\infty} R(\|S(n)\| < \infty.$$

By Corollary 1.1. it follows that $\mathbf{S}$ is u.e.s. □

In this paper we generalize this theorem for the case of u.e.b. Φ-semigroups in locally convex spaces. Thus we shall extend results of Zabczyk [8] in two directions. First, we shall consider the case of Φ-semigroups in locally convex spaces and second, we shall not assume the convexity of R. The obtained results can be regarded as generalizations of the well-known result of Datko [1], Pazy [6], Rolewicz [7] and Littman [3].

2. Preliminaries

We start with the following

Lemma 2.1 *If* $\mathbf{S} = (S(t))_{t\geq 0}$ *is an* Φ*-semigroup then*

(i) $\|S(t+s)\|_{\varphi(t+s,\gamma),\gamma} \leq \|S(t)\|_{\varphi(t,\gamma),\gamma}\|S(s)\|_{\varphi(s,\varphi(t,\gamma)),\varphi(t,\gamma)}$ *for all* $(t,s,\gamma) \in \mathbb{R}_+^2 \times \Gamma$;

(ii) $\|S(nt)\|_{\varphi(nt,\gamma),\gamma} \leq \prod_{k=1}^{n} \|S(t)\|_{\varphi(kt,\gamma),\varphi((k-1)t,\gamma)}$ *for all* $(t,n,\gamma) \in \mathbb{R}_+ \times \mathbb{N}^* \times \Gamma$ $(\mathbb{N} = \{0,1,2,\ldots\},$; $\mathbb{N}^* = \{0,1,2,\ldots\}$.

Proof. (i) We observe that

$$|S(t+s)x|_\gamma = |S(t)S(s)x|_\gamma \leq \|S(t)\|_{\varphi(t,\gamma),\gamma}|S(s)x|_{\varphi(t,\gamma)} \leq$$
$$\leq \|S(t)\|_{\varphi(t,\gamma),\gamma}\|S(s)\|_{\varphi(s,\varphi(t,\gamma)),\varphi(t,\gamma)}|x|_{\varphi(s,\varphi(t,\gamma))} =$$
$$= \|S(t)\|_{\varphi(t,\gamma),\gamma}\|S(s)\|_{\varphi(t+s,\gamma),\varphi(t,\gamma)}|x|_{\varphi(t+s,\gamma)},$$

and hence

$$\|S(t+s)\|_{\varphi(t+s,\gamma),\gamma} \leq \|S(t)\|_{\varphi(t,\gamma),\gamma}\|S(s)\|_{\varphi(t+s,\gamma),\varphi(t,\gamma)}$$

for all $(t,s,\gamma) \in \mathbb{R}_+^2 \times \Gamma$.

(ii) It follows from (i) by induction. □

Lemma 2.2 *If* $\mathbf{S}$ *is an* Φ*-semigroup with u.e.b. then*

$$\|S(t+1)\|_{\varphi(t+1,\gamma),\gamma} \leq M_0(\gamma)e^{\omega_0(\gamma)}\|S(s)\|_{\varphi(s,\gamma),\gamma}$$

for all $(t,s,\gamma) \in \mathbb{R}_+^2 \times \Gamma$ *with* $s \in [t,t+1]$.

Proof. Indeed, if $t \geq 0, \gamma \in \Gamma$ and $s \in [t,t+1]$ then by Lemma 2.1. and Definition 1.2., we have that

$$\|S(t+1)\|_{\varphi(t+1,\gamma),\gamma} \leq \|S(s)\|_{\varphi(s,\gamma),\gamma}\|S(t+1-s)\|_{\varphi(t+1,\gamma),\varphi(s,\gamma)} \leq$$
$$\leq M(\varphi(s,\gamma))e^{(t+1-s)\omega(\varphi(s,\gamma))}\|S(s)\|_{\varphi(s,\gamma),\gamma} \leq$$
$$\leq M_0(\gamma)e^{\omega_0(\gamma)}\|S(s)\|_{\varphi(s,\gamma),\gamma}.$$

Lemma 2.3 *Let* $\mathbf{S}$ *be an* Φ*-semigroup with u.e.b. If there exists* $P : \mathbb{N}^* \times \Gamma \to \mathbb{R}_+$ *such that* *(i)* $\|S(n)\|_{\varphi(n,\gamma),\gamma} \leq P(n,\gamma)$ *for all* $(n,\gamma) \in \mathbb{N}^* \times \Gamma$*;*

(ii) $\lim_{n\to\infty} P(n,\gamma) = 0$ *for all* $\gamma \in \Gamma$*;*

(iii) $(iii) P(n,\varphi(s,\gamma)) \leq P(n,\gamma)$ *for all* $(n,s,\gamma) \in \mathbb{N}^* \times \mathbb{R}_+ \times \Gamma$

then $\mathbf{S}$ *is u.e.s.*

Proof. Let $A : \Gamma \to \mathcal{P}\ (\mathbb{N}^*)$ be the function defined by

$$A(\gamma) = \{n \in \mathbb{N}^* : P(n,\gamma) < e^{-1}\}.$$

From (ii) it results that $A(\gamma)$ is non-empty for all $\gamma \in \Gamma$.

If we denote by $n(\gamma) = \min A(\gamma)$ then from $A(\gamma) \subset A(\varphi(t,\gamma))$ and (iii) it results that

$$n(\varphi(t,\gamma)) \leq n(\gamma) \quad \text{for all} \quad (t,\gamma) \in \mathbb{R}_+ \times \Gamma.$$

For all $(t,\gamma) \in \mathbb{R}_+ \times \Gamma$ there is a $p \in \mathbb{N}$ such that $pn(\gamma) \leq t < (p+1)n(\gamma)$.

If $p = 0$ then

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M(\gamma)e^{t\omega(\gamma)} \leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)}.$$

If $p \geq 1$ then

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq \|S(pn(\gamma))\|_{\varphi(pn(\gamma),\gamma),\gamma}\|S(t-pn(\gamma))\|_{\varphi(t,\gamma),\varphi(pn(\gamma),\gamma)} \leq$$

$$\leq M(\varphi(pn(\gamma),\gamma))e^{(t-pn(\gamma))\omega(\varphi(pn(\gamma),\gamma))}\prod_{k=1}^{p}\|S(n(\gamma))\|_{\varphi(kn(\gamma),\gamma),\varphi((k-1)n(\gamma),\gamma)} \leq$$

$$\leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)}\prod_{k=1}^{p}P(n(\gamma),\varphi((k-1)n(\gamma),\gamma)) \leq$$

$$\leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)}\prod_{k=1}^{p}P(n(\gamma),\gamma) \leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)-p} \leq$$

$$\leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)+1}e^{-t\nu(\gamma)} = N(\gamma)e^{-t\nu(\gamma)},$$

where

$$N(\gamma) = M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)+1} \quad \text{and} \quad \nu(\gamma) = \frac{1}{n(\gamma)}.$$

Because

$$N_0(\gamma) = \sup_{t \geq 0} N(\varphi(t, \gamma)) \leq M_0(\gamma) e^{n(\gamma)\omega_0(\gamma)+1} < \infty$$

and

$$\nu_0(\gamma) = \inf_{t \geq 0} \nu(\varphi(t, \gamma)) = \inf_{t \geq 0} \frac{1}{n(\varphi(t, \gamma))} \geq \frac{1}{n(\gamma)} > 0,$$

finally we obtain that **S** is u.e.s.

3. The main results

In what follows for every $\varphi : \mathbb{R}_+ \times \Gamma \to \Gamma$ we shall denote by $\mathcal{R}_\varphi$ the set of all functions $R : \mathbb{R}_+ \times \Gamma \to \mathbb{R}_+$ with the following properties

r_1) $R(0, \gamma) = 0$ for all $\gamma \in \Gamma$;

r_2) $R(t, \gamma) > 0$ for all $t > 0$ and all $\gamma \in \Gamma$;

r_3) $\lim\limits_{t \to \infty} R(t, \gamma) = \infty$ for every $\gamma \in \Gamma$;

r_4) $R(t, \gamma) \leq R(t, \varphi(s, \gamma))$ for all $(t, s, \gamma) \in \mathbb{R}_+^2 \times \Gamma$;

r_5) $R(s, \gamma) \leq R(t, \gamma)$ for all $(s, t, \gamma) \in \mathbb{R}_+^2 \times \Gamma$ with $s \leq t$.

Lemma 3.1 *Let $\varphi \in \Phi$ and $R \in \mathcal{R}_\varphi$. Then for every $(r, \gamma) \in \mathbb{R}_+ \times \Gamma$ the set*

$$B_r(\gamma) = \{t \geq 0 : R(t, \gamma) \leq r\}$$

is bounded and the function

$$\delta : \mathbb{R}_+ \times \Gamma \to \mathbb{R}_+^*, \quad \delta(r, \gamma) = 1 + \sup B_r(\gamma)$$

satisfies the inequality

$$\delta(r, \varphi(t, \gamma)) \leq \delta(r, \gamma) \quad \textit{for all} \quad (r, t, \gamma) \in \mathbb{R}_+^2 \times \Gamma.$$

Proof. From $\lim\limits_{t \to \infty} R(t, \gamma) = \infty$ it follows that $B_r(\gamma)$ is a bounded set for all $(r, \gamma) \in \mathbb{R}_+ \times \Gamma$.

On the other hand, r_4) implies

$$B_r(\varphi(t, \gamma)) \subset B_r(\gamma) \quad \text{for all} \quad (r, t, \gamma) \in \mathbb{R}_+^2 \times \Gamma$$

which proves the lemma. □

The main result of this paper is

Theorem 3.2 *If the Φ-semigroup* **S** *is u.e.b., then it is u.e.s. if and only if there exists $\varphi \in \Phi$, $R \in \mathcal{R}_\varphi$ and $K : \Gamma \to \mathbb{R}_+^*$ such that*

(i)

$$K_0(\gamma) := \sup_{t\geq 0} K(\varphi(t,\gamma)) < \infty$$

and

(ii)

$$\sum_{n=1}^{\infty} R(\|S(n)\|_{\varphi(n,\gamma),\gamma},\gamma)dt \leq K(\gamma)$$

for all $\gamma \in \Gamma$.

*Proof.*Necessity. It results from Definition 1.3. for

$$R(t,\gamma) = t \quad \text{and} \quad K(\gamma) = \frac{N(\gamma)}{\nu(\gamma)},$$

where N and ν are given by Definition 1.3.
Sufficiency. Because

$$R(\|S(n)\|_{\varphi(n,\gamma),\gamma},\gamma) \leq \sum_{m=1}^{\infty} R(\|S(m)\|_{\varphi(m,\gamma),\gamma},\gamma) \leq K(\gamma) \leq K_0(\gamma),$$

for all $\gamma \in \Gamma$, by Lemma 3.1. it results that there exists $M_1 : \Gamma \to \mathbb{R}_+^*$, $M_1(\gamma) = \delta(K_0(\gamma),\gamma)$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M_1(\gamma) \quad \text{and} \quad M_1(\varphi(t,\gamma)) \leq M_1(t,\gamma),$$

for all $(t,\gamma) \in \mathbb{R}_+ \times \Gamma$ (i.e. $\mathbf{S}$ is u.s.).

Let $F : \mathbb{R}_+ \times \Gamma \to \mathbb{R}_+$ be the function defined by

$$F(t,\gamma) = \int_0^t R(s,\gamma)ds.$$

By Lemma 1.1. the function $t \to F(t,\gamma)$ is an increasing continuous bijection for every $\gamma \in \Gamma$. If we denote by $f_\gamma = F(\cdot,\gamma)^{-1}$ then from $R \in \mathcal{R}_\varphi$ it follows that $F \in \mathcal{R}_\varphi$,

$$f_{\varphi(t,\gamma)}(s) \leq f_\gamma(s) \quad \text{for all} \quad (t,s,\gamma) \in \mathbb{R}_+^2 \times \Gamma.$$

and

$$\sum_{n=1}^{\infty} F(\|S(n)\|_{\varphi(n,\gamma),\gamma},\gamma) \leq \sum_{n=1}^{\infty} \|S(n)\|_{\varphi(n,\gamma),\gamma} R(\|S(n)\|_{\varphi(n,\gamma),\gamma},\gamma) \leq$$

$$\leq M_1(\gamma)\sum_{n=1}^{\infty} R(\|S(n)\|_{\varphi(n,\gamma),\gamma},\gamma) \leq M_1(\gamma)K(\gamma) \leq M_1(\gamma)K_0(\gamma) = M_2(\gamma)$$

for all $\gamma \in \Gamma$.

If we denote by

$$g(n,\gamma) = \frac{\|S(n)\|_{\varphi(n,\gamma),\gamma}}{M_1(\gamma)}$$

then for $n \geq 1$ and $\gamma \in \Gamma$ we have

$$nF(g(n,\gamma),\gamma) = \sum_{m=1}^{n} F(g(n,\gamma) \leq \sum_{m=1}^{n} F\Big(g(m,\gamma)\|S(n-m)\|_{\varphi(n,\gamma),\varphi(m,\gamma)},\gamma\Big) \leq$$

$$\leq \sum_{m=1}^{n} F(g(m,\gamma)M_1(\varphi(m,\gamma),\gamma),\gamma) \leq \sum_{m=1}^{n} F(\|S(m)\|_{\varphi(m,\gamma),\gamma},\gamma) \leq M_2(\gamma)$$

and hence

$$\|S(n)\|_{\varphi(n,\gamma),\gamma} \leq P(n,\gamma)$$

for all $t \geq 0$ and $\gamma \in \Gamma$, where

$$P(n,\gamma) = M_1(\gamma) f_\gamma\Big(\frac{M_2(\gamma)}{n}\Big).$$

It is easy to see that

$$\lim_{n\to\infty} P(n,\gamma) = 0 \quad \text{for every} \quad \gamma \in \Gamma$$

and

$$P(n,\varphi(s,\gamma)) \leq P(n,\gamma), \quad \text{for all} \quad (n,s,\gamma) \in \mathbb{N}^* \times \mathbb{R}_+ \times \Gamma.$$

An application of Lemma 2.3. proves that **S** is u.e.s. □

The continuous variant of Theorem 3.1. is given by

Corollary 3.3 *If the Φ-semigroup* **S** *is u.e.b., then it is u.e.s. if and only if there exists $\varphi \in \Phi$, $R \in \mathcal{R}_\varphi$ and $K : \Gamma \to \mathbb{R}_+^*$ such that*

$$(i) K_0(\gamma) = \sup_{t\geq 0} K(\varphi(t,\gamma)) < \infty$$

and

$$(ii) \int_0^\infty R(\|S(t)\|_{\varphi(t,\gamma),\gamma},\gamma)dt \leq K(\gamma)$$

for all $\gamma \in \Gamma$.

Proof. Necessity. It is a simple verification for

$$R(t,\gamma) = t \quad \text{and} \quad K(\gamma) = \frac{N(\gamma)}{\nu(\gamma)},$$

where N and ν are given by Definition 1.3.
Sufficiency. Let $M_3 : \Gamma \to \mathbb{R}_+^*$ be the function defined by

$$M_3(\gamma) = \sup_{t\geq 0} M(\varphi(t,\gamma))e^{\omega(\varphi(t,\gamma))},$$

where M and ω are given by Definition 1.2.

By Lemma 2.2 we have that

$$\|S(t+1)\|_{\varphi(t+1,\gamma),\gamma} \leq M_3(\gamma)\|S(s)\|_{\varphi(s,\gamma)},$$

for all $(t,s,\gamma) \in \mathbb{R}_+^2 \times \Gamma$ with $s \in [t,t+1]$. If we denote by

$$R_1 : \mathbb{R}_+ \times \Gamma \to \mathbb{R}_+, \quad R_1(t,\gamma) = R(\frac{t}{M_3(\gamma)},\gamma)$$

then $R_1 \in \mathcal{R}_\varphi$ and

$$\sum_{n=1}^{\infty} R_1(\|S(n)\|_{\varphi(n,\gamma),\gamma},\gamma) \leq \sum_{n=1}^{\infty}\int_{n-1}^{n} R(\|S(s)\|_{\varphi(s,\gamma),\gamma},\gamma)ds =$$

$$\int_0^{\infty} R(\|S(s)\|_{\varphi(s,\gamma),\gamma},\gamma)ds \leq K(\gamma), \quad \text{for all} \quad \gamma \in \Gamma.$$

From Theorem 3.1. it results that **S** is u.e.s. □

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Received by the editors July 3, 2000