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σ-NULL-ADDITIVE SET FUNCTIONS

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Abstract. There is introduced the notion of σ-null-additive set function as a generalization of the classical measure. There are proved the relations to disjoint and chain variations. The general Lebesgue decomposition theorem is obtained.

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1. Introduction

We introduce in this paper the notion of σ-null-additive set functions as a generalization of the notion of the classical measure. This class of set functions is a subclass of general class of non-additive set functions the class of null-additive set functions. We prove some general properties of σ–null-additive set functions. We investigate the relation with the disjoint variation and chain variation and prove a general Lebesgue decomposition theorem.

2. Basic definitions

We have now the definition of the main notion in this article.

Definition 1. *A set function m, $m : \mathcal{D} \to [-\infty, \infty]$, is called null-additive, if we have*

$$m(A \cup B) = m(A)$$

whenever $A, B \in \mathcal{D}$, $A \cap B = \emptyset$, *and* $m(B) = 0$.

It is obvious that for null-additive set function m we have $m(\emptyset) = 0$ whenever there exists $B \in \mathcal{D}$ such that $m(B) = 0$. We shall always suppose that $m(\emptyset) = 0$, if otherwise is not explicitly stated.

In the next section we present a number of important examples of σ–null-additive set functions.

We give two simple examples of set functions.

Example 1. Let $m(A) \neq 0$ whenever $A \in \Sigma, A \neq \emptyset$. Then m is null-additive.

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Example 2. Let $X = \{x, y\}$ and define m in the following way:

$$m(X) = 1 \ \ and \ \ m(A) = 0 \ \ for \ \ A \neq X.$$

Then m is not null-additive.

We have some obvious properties of null-additive set functions.

Theorem 1. *Let m be a positive monotone set function defined on a ring $\mathcal{R}$. Then the following statements are equivalent:*

(1) m is null-additive.

(2) If $E \in \mathcal{R}, F \in \mathcal{R}, m(F) = 0,$ then $m(E \cup F) = m(E)$.

(3) If $E \in \mathcal{R}, F \in \mathcal{R}, F \subset E$ and $m(F) = 0,$ then $m(E \setminus F) = m(E)$.

(4) If $E \in \mathcal{R}, F \in \mathcal{R}, m(F) = 0,$ then $m(E \setminus F) = m(F)$.

(5) If $E \in \mathcal{R}, F \in \mathcal{R}, m(F) = 0,$ then $m(E \Delta F) = m(F)$.

Now we have an example of null-additive non-monotone set function.

Example 3. Let $m : \Sigma \to [0, 1]$ be defined by

$$m(A) = \begin{cases} 2\mu(A), & \text{if } \mu(A) \leq \frac{1}{2}, \\ -2\mu(A) + 2 & \text{if } \mu(A) \geq \frac{1}{2}, \end{cases}$$

where $\mu, \mu : \Sigma \to [0, 1]$, is a σ-additive measure. Then m is null-additive and not monotone, but it is continuous from above and from below.

We can introduce a more general class of set functions.

Definition 2. *A set function m, $m : \mathcal{D} \to [-\infty, \infty]$, is called supernull-additive, if we have*

$$m(A \cup B) \geq m(A)$$

whenever $A, B \in \mathcal{D}, \ A \cap B = \emptyset, \ and \ m(B) = 0$.

The class of supernull-additive set functions includes all null-additive set functions, all monotone set functions (defined on a ring) and all superadditive set functions.

Theorem 2. *Let m be a null-additive, positive, monotone and continuous from above set function defined on Σ. If $A \in \Sigma$, then*

$$m(A \cup B_n) \to m(A)$$

for any decreasing sequence $\{B_n\}$ from Σ for which $m(B_n) \to 0$ and there exists at least one n_0 such that $m(A \cup B_{n_0}) < \infty$ as $m(A) < \infty$.

Theorem 3. *Let m be a null-additive, positive, monotone and continuous set function on Σ. If $A \in \Sigma$, then we have*

$$m(A \setminus B_n) \to m(A)$$

for any decreasing sequence $\{B_n\}$ from $\mathcal{R}$ for which $\lim_{n\to\infty} m(B_n) = 0$.

3. σ-null-additive set function

We introduce the following generalization of the $\sigma-$additiveness.

Definition 3. *Let $\mathcal{R}$ be a $\sigma-$ring, a set function $m : \mathcal{R} \to [-\infty, \infty]$ with $m(\emptyset) = 0$ is $\sigma-$null-additive if for every sequence $\{B_i\}$ of pairwise disjoint sets from $\mathcal{R}$ such that $A \cap B_i = \emptyset$ and $m(B_i) = 0$ we have*

$$m(A \cup \bigcup_{i=1}^{\infty} B_i) = m(A).$$

Proposition 1. *Let $\mathcal{R}$ be a $\sigma-$ring and let m be a function $m : \mathcal{R} \to [-\infty, \infty]$. The function m is $\sigma-$null-additive if and only if m is null-additive and $m(B_i) = 0 \ \ (i \in \mathbf{N})$ implies $m(\bigcup_{i=1}^{\infty} B_i) = 0$ for a sequence $\{B_i\}$ of pairwise disjoint sets from $\mathcal{R}$.*

Proof. We have for $\sigma-$null-additive set function m

$$m(\bigcup_{i=1}^{\infty} B_i) = m(B_1 \cup \bigcup_{i=2}^{\infty} B_i) = m(B_1) = 0, \quad \text{where } B_1 \cap \bigcup_{i=2}^{\infty} B_i = \emptyset.$$

If $m(B) = 0$ for $B \in \mathcal{R}$, then taking $B_1 = B$ and $B_i = \emptyset$ for $i \geq 2$ we have by Definition 3

$$m(A \cup B) = m(A \cup \bigcup_{i=1}^{\infty} B_i) = m(A)$$

for any $A \in \mathcal{R}$, i.e., m is null-additive. The inverse statement is obvious. □

Proposition 2. *Let $\mathcal{R}$ be a $\sigma-$ring. If m is null - additive and continuous from below, then it is $\sigma-$null-additive.*

Proof. Let $\{B_i\}$ be a sequence of pairwise disjoint sets from $\mathcal{R}$ such that $A \cap B_i = \emptyset$ and $m(B_i) = 0 \quad (i \in \mathbf{N})$. Then, since

$$A \cup \bigcup_{i=1}^{n} B_i \nearrow A \cup \bigcup_{i=1}^{\infty} B_i,$$

we have by continuity of m

$$\lim_{n \to \infty} m(A \cup \bigcup_{i=1}^{n} B_i) = m(A \cup \bigcup_{i=1}^{\infty} B_i).$$

But the left part of the preceding equality by the null-additivity of m is the limit of a stationary sequence with all members equal to $m(A)$. Hence the limit is also equal to $m(A)$ and so we have

$$m(A) = m(A \cup \bigcup_{i=1}^{\infty} B_i),$$

i.e., m is $\sigma-$null-additive. □

Example 1. (Decomposable measures)

We introduce an operation which generalizes the usual addition on the interval $[0,1]$.

Definition 4. *A triangular conorm S ($t-$conorm briefly) is a function $S : [0,1]^2 \to [0,1]$ such that*

(s_1) $S(x,y) \le S(x,z)$ *for* $y \le z$ *(monotonicity)*

(s_2) $S(x,y) = S(y,x)$ *(commutativity)*

(s_3) $S(x, S(y,z)) = S(S(x,y), z)$ *(associativity)*

(s_4) $S(x,0) = S(0,x) = x$ *(boundary condition).*

Example 4. The following are the most important t-conorms

$$S_{\mathbf{M}}(x,y) = \max(x,y), \qquad S_{\mathbf{P}}(x,y) = x + y - xy,$$

the bounded sum is given by

$$S_{\mathbf{L}}(x,y) = \min(1, x+y),$$

and a non-continuous $t-$conorm

$$S_{\mathbf{W}}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

There are many other important $t-$conorms.

Definition 5. *A set function $m, m : \mathcal{D} \to [0,1]$, is σ-S-decomposable if it satisfies $m(\emptyset) = 0$ and*

$$m(\cup_{i=1}^{\infty} A_i) = S_{i=1}^{\infty} m(A_i)$$

for every sequence $\{A_i\}$ from $\mathcal{D}$ of pairwise disjoint sets such that $\cup_{i=1}^{\infty} A_i \in \mathcal{D}$.

We remark that $\sigma - S-$decomposable measure is always $\sigma-$null-additive .

Example 2. ($k-$triangular set functions)

Definition 6. *A set function $m : \Sigma \to [0,\infty)$ is said to be k-triangular for $k \ge 1$ if $m(\emptyset) = 0$ and*

$$m(A) - km(B) \le m(A \cup B) \le m(A) + km(B),$$

whenever $A, B \in \Sigma, A \cap B = \emptyset$.

We remark that even for the classical signed measure μ, the set function $|\mu(\ .\ )|$ is not monotone, but it is 1-triangular.

Proposition 3. *Let $\mathcal{R}$ be a $\sigma-$ring. The monotone and $\sigma - k-$subadditive set function $m : \mathcal{R} \to [0, \infty]$ with $m(\emptyset) = 0$ is $\sigma-$null-additive.*

Proof. Follows by Proposition 1 and the inequality

$$m(B_1) \leq m(\bigcup_{i=1}^{\infty} B_i) \leq m(B_1) + k \sum_{i=2}^{\infty} m(B_i),$$

where $\{B_i\}$ is a sequence of pairwise disjoint sets from $\mathcal{R}$, such that $m(B_i) = 0 \quad (i \in \mathbf{N})$. □

4. The relation of the disjoint variation and chain variation

Definition 7. *For an arbitrary but fixed subset A of X and a set function m we define the disjoint variation $\overline{m}$ by*

$$\overline{m}(A) = \sup_{I} \sum_{i \in I} \mid m(D_i) \mid,$$

where the supremum is taken over all finite families $\{D_i\}_{i \in I}$ of pairwise disjoint sets of $\mathcal{D}$ such that $D_i \subset A \ \ (i \in I)$.

Remark. If $A \in \mathcal{A}$, then we can take in the previous definition the supremum for all finite families $\{D_i\}_{i \in I}$ of disjoint sets such that $\bigcup_{i \in I} D_i = A$.

The relation of the disjoint variation with the notion of null-additivity is given in the next theorem.

Theorem 4. *If a set function m, $m(\emptyset) = 0$, is null-additive, then its disjoint variation $\overline{m}$ is also null-additive.*

Proof. Suppose that m is null-additive. Let $B \in \mathcal{D}$ be such that $\overline{m}(B) = 0$. Then by $\mid m(B) \mid \leq \overline{m}(B)$ it follows $m(B) = 0$. For an arbitrary $A \in \mathcal{D}$ such that $A \cup B \in \mathcal{D}$ and $A \cap B = \emptyset$ we have

$$\begin{aligned}
\overline{m}(A \cup B) &= \sup\{\sum_{i=1}^{n} \mid m(D_i) \mid : \ \{D_i\} \ \text{ disjoint}\} \\
&= \sup\{\sum_{i=1}^{n} \mid m((D_i \cap A) \cup (D_i \cap B)) \mid : \ \{D_i\} \ \text{ disjoint}\} \\
&= \sup\{\sum_{i=1}^{n} \mid m(D_i \cap A) \mid : \ \{D_i\} \ \text{ disjoint } \} = \overline{m}(A),
\end{aligned}$$

where we have used that $\overline{m}(D_i \cap B) = 0, \ \ i = 1, ..., n$, holds by the monotonicity of $\overline{m}$, which implies $m(D_i \cap B) = 0$ and hence by null-additivity of m

$$m((D_i \cap A) \cup (D_i \cap B)) = m(D_i \cap A).$$ □

Definition 8. *A real-valued set function m with $m(\emptyset) = 0$ is of bounded chain variation if $\mid m \mid (X) \; < \; +\infty$.*

We denote the family of set functions which vanish on an empty set and with bounded chain variation on (X, Σ) by BV. Since for $m, v \in BV$ and $a \in \mathbf{R}$ we have

1^0 $\mid m \mid (X) = 0$ if and only if $m = 0$;

2^0 $\mid am \mid (X) = \mid a \mid \cdot \mid m \mid (X)$;

3^0 $\mid m + v \mid (X) \;\leq\; \mid m \mid (X) + \mid v \mid (X)$,

the functional

$$\|m\| = \mid m \mid (X)$$

is a norm on the vector space BV.

The second type of variation of set function is given in the next definition.

Definition 9. *The chain variation of a real-valued set function m with $m(\emptyset) = 0$ on the set $A \in \mathcal{D}$ is given by*

$$\mid m \mid (A) = \sup\{\sum_{i=1}^{n} \mid m(A_i) - m(A_{i-1}) \mid :$$

$$\emptyset = A_0 \subset A_1 \subset ... \subset A_n = A, \quad A_i \in \mathcal{D},\; i = 1, ..., n\}$$

We remark that the supremum in the previous definition is taken over all chains between $\emptyset$ and A.

There are null-additive set functions which do not belong to BV.

Example 5. Let $X = [-1, 1]$ and Σ be a family of all Borel subsets of $[-1, 1]$. Taking a measure

$$\mu(A) = \int_A x \, dx \quad (A \in \Sigma),$$

we have that

$$m(A) = \sqrt{\mid \mu(A) \mid} \quad (A \in \Sigma)$$

is a null-additive set functions. But $m \notin BV$. Namely, take a special chain

$$\emptyset = A_0 \subset A_1 \subset ... \subset A_{2n},$$

with $A_{2i-1} = [-\frac{i-1}{n}, \frac{i}{n}]$ and $A_{2i} = [-\frac{i}{n}, \frac{i}{n}]$ for $i = 1, 2, ..., n$. Then

$$\|m\| \geq \sum_{i=1}^{2n} \mid m(A_i) - m(A_{i-1}) \mid = 2\sqrt{n} \to \infty$$

as $n \to \infty$. Hence $m \notin BV$.

There are also set functions which belong to BV, but which are not null-additive.

5. Signed fuzzy measures

Σ always denotes a σ-algebra of subsets of the given set X.

Definition 10. *A signed fuzzy measure (revised monotone continuous from above and below set function)* $m, m : \Sigma \to [-\infty, \infty]$, *is an extended real-valued set function* m *defined on* σ*-algebra* Σ *and with the properties:*

($\mathbf{SFM_1}$) $m(\emptyset) = 0;$

($\mathbf{SFM_2}$) *if* $E, F \in \Sigma,\ \ E \cap F = \emptyset$, *then*

(**a**) $m(E) \geq 0,\ \ m(F) \geq 0,\ \ \max(m(E), m(F)) > 0$ *implies*

$$m(E \cup F) \geq \max(m(E), m(F));$$

(**b**) $m(E) \leq 0,\ \ m(F) \leq 0,\ \ \min(m(E), m(F)) < 0$ *implies*

$$m(E \cup F) \leq \min(m(E), m(F));$$

(**c**) $m(E) > 0,\ \ m(F) < 0$ *implies* $m(E) \geq m(E \cup F) \geq m(F)$.

($\mathbf{SFM_3}$) $(E_1 \subset E_2 \subset ... \quad, E_n \in \Sigma \quad \Rightarrow \quad m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n);$

($\mathbf{SFM_4}$) $(E_1 \supset E_2 \supset ... \quad, E_n \in \Sigma$, *there exists* $n_0 \in \mathbf{N}$, $|m(E_{n_0})| < \infty) \quad \Rightarrow \quad m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$.

The property ($\mathbf{SFM_2}$) of m is called the revised monotonicity.

Example 6. Any non-negative signed fuzzy measure is a usual fuzzy measure. Namely, the condition ($\mathbf{SFM_2}$), (**a**) implies the monotonicity of m. But also each fuzzy measure is a signed fuzzy measure.

Example 7. The classical signed measure is a signed fuzzy measure.

Definition 11. *A set* $A \subset X$ *is called a positive set (resp. negative set) for signed fuzzy measure* m *on* (Σ, X) *if for every subset* E *of* A *which belongs to* Σ *we have* $m(E) \geq 0$ *(resp.* $m(E) \leq 0$*).*

We have the following Hahn decomposition type theorem.

Theorem 5. *Let* m *be a null-additive revised monotone set function on* (Σ, X), *which is continuous from above and from below (i.e., a null-additive signed fuzzy measure). If* m *takes at most one of the values* $-\infty$ *or* $+\infty$, *and if*

$$E \in \Sigma, \quad |m(E)| < +\infty \ \ \Rightarrow \ \ |m(F)| < +\infty \ \ (F \subset E, F \in \Sigma),$$

then there exist two disjoint sets A *and* B *from* Σ *such that* $A \cup B = X$ *and* A *is a positive set and* B *is a negative set.*

In this section we shall suppose, without loss of generality, that the signed fuzzy measure m satisfies the condition

$$-\infty < m(E) \leq +\infty \quad (E \in \Sigma).$$

Now we can prove the following version of the Jordan decomposition theorem

Theorem 6. *Let m be a null-additive signed fuzzy measure. Then there exist uniquely determined null-additive fuzzy measures m^+ and m^- such that*

$$m^+ \geq m \geq -m^-.$$

Moreover, if m has a representation in the form

$$m = v_1 - v_2,$$

where v_1 and v_2 are null-additive fuzzy measures, then $v_1 \geq m^+$ and $v_2 \geq m^-$.

6. Lebesgue Decomposition Theorem

Let m and v be two non-negative monotone set functions defined on a ring $\mathcal{R}$.

Definition 12. *Let m and v be two finite monotone set functions. If $E \in \mathcal{R}, v(E) = 0$ implies $m(E) = 0$, then we say that m is absolutely continuous with respect to v.*

Definition 13. *Let m and v be two finite monotone set functions. If for every $\epsilon > 0$ there is a $\delta > 0$ such that $E \in \mathcal{R}, v(E) < \delta$ implies $m(E) < \epsilon$, then we say that m is absolutely $\epsilon-$continuous with respect to v.*

Theorem 7. *Let m and v be two finite monotone set functions defined on a $\sigma-$ring $\mathcal{R}$ such that they are continuous from above and continuous from below. If v is autocontinuous from above, then m is absolutely continuous with respect to v if and only if m is absolutely $\epsilon-$continuous with respect to v.*

Definition 14. *Let m and v be two finite monotone set functions defined on $\mathcal{R}$. The monotone set function m is called singular with respect to v, denoted by $m \perp v$, if there exists a set A from $\mathcal{R}$ such that*

$$m(E \setminus A) = v(E) = 0 \quad (E \in \mathcal{R}).$$

Remark 1. We have that if for null-additive monotone set functions m and v, which are continuous from above and continuous from below, $m \perp v$ holds, then we have $v \perp m$, too.

Now we have the following theorem of Lebesgue decomposition type for continuous null-additive monotone set functions.

Theorem 8. *Let m and v be two finite null-additive monotone continuous set functions on $\sigma-$ring $\mathcal{R}$. Then there exist two null-additive monotone set functions m_c and m_s such that $m_c(E) = m(E \setminus A)$ and $m_s(E) = m(E \cap A)$ for a set $A \in \mathcal{R}$ and m_c is absolutely continuous with respect to v and m_s is singular with respect to v.*

We will use now the ideal approach. Let $\mathcal{R}$ be a ring and m a function from $\mathcal{R}$ into $[0, \infty]$. We do not suppose the monotonicity of m. We write

$$\mathcal{N}(m) = \{A \in \mathcal{R} : \ m(A \cap Y) = 0, \ \forall Y \in \mathcal{R}\}.$$

Theorem 9. *Let $\mathcal{R}$ be a ring. If $m : \mathcal{R} \to [0, \infty]$ is null-additive set function then the set $\mathcal{N}(m)$ is an ideal in $\mathcal{R}$.*

Proof. By the definition we have for $B \in \mathcal{R}$ and $A \in \mathcal{N}(m)$ such that $B \subset A$ that $m(B) = 0$, i.e., $B \in \mathcal{N}(m)$.

For $A_1, A_2 \in \mathcal{N}(m)$ and for arbitrary but fixed subset B of $A_1 \cup A_2$ which belongs to $\mathcal{R}$ we have $B \setminus A_1$ and $B \cap A_1$, and so

$$m(B \setminus A_1) = m(B \cap A_1) = 0.$$

Hence, since $B \setminus A_1$ and $B \cap A_1$ are disjoint sets and m is null-additive,

$$m(B) = m((B \setminus A_1) \cup (B \cap A_1)) = 0,$$

i.e. $A_1 \cup A_2 \in \mathcal{N}(m)$. □

Corollary 1. *Let $\mathcal{R}$ be a $\sigma-$ring. If $m : \mathcal{R} \to [0, \infty]$ is $\sigma-$null-additive then the set $\mathcal{N}(m)$ is a $\sigma-$ideal of $\mathcal{R}$.*

Definition 15. *Let $\mathcal{R}$ be a ring and $\mathcal{M}$ a subset of $\mathcal{R}$. $\mathcal{M}$ is said to satisfy the countable chain condition (CCC) if every disjoint subset (consisting of disjoint sets from $\mathcal{M}$) of $\mathcal{M} \setminus \{\emptyset\}$ is at most countable. Let m be a function from $\mathcal{R}$ into $[0, \infty]$, m is said to satisfies the (CCC) if $\mathcal{R} \setminus \mathcal{N}(m)$ satisfy the (CCC).*

Definition 16. *Let $\mathcal{R}$ be a ring and let m and v be two functions from $\mathcal{R}$ into $[0, \infty]$. We say that m is $v-$continuous if $\mathcal{N}(v) \subset \mathcal{N}(m)$. We say that m is $v-$singular if there exists an element $A \in \mathcal{N}(\mu)$ such that $T \setminus A \in \mathcal{N}(m)$ for every T in $\mathcal{R}$.*

Theorem 10. *Let $\mathcal{R}$ be a $\sigma-$ring, m a null-additive set function, and v a finite σ-null-additive set function from $\mathcal{R}$ into $[0, +\infty]$. If $\mathcal{N}(m) \setminus \mathcal{N}(v)$ satisfies (CCC) then there exists $A \in \mathcal{N}(v)$ such that null-additive set functions*

$$m_1 : Y \in \mathcal{R} \to m(Y \setminus A), \qquad m_2 : Y \in \mathcal{R} \to m(Y \cap A)$$

are, respectively, $v-$continuous and $v-$singular.

Corollary 2. *Let $\mathcal{R}$ be a $\sigma-$ring , $\oplus$ a pseudo-addition. Let m be a $\oplus-$decomposable measure on $\mathcal{R}$ and v a $\sigma-$null-additive and exhaustive function on $\mathcal{R}$. Then m can be uniquely represented as the $\oplus-$sum of two $\oplus-$decomposable functions m_1 and m_2 such that m_1 is $v-$continuous and m_2 is $v-$singular. Moreover m_2 is m_1-singular.*

Remark 2. The main point in Corollary 2 is the possibility of supposition of $\sigma - \oplus-$decomposability of v (which implies $\sigma-$null-additivity) without order continuity. In this way it can be considered also the possibility measure, i.e. the set function $\pi : \mathcal{R} \to [0,1]$ which satisfies

$$\pi(A \cup B) = \pi(A) \vee \pi(B)$$

for $A, B \in \mathcal{R}$ such that $A \cap B = \emptyset$, which in general does not be order continuous although it is $\sigma - \vee-$decomposable.

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