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PARTIAL MULTIPLIERS ON PARTIALLY ORDERED SETS[1]

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Abstract. For a partially ordered set $\mathcal{A}$, we denote by $\mathcal{A}_o$ the family of all elements $D \in \mathcal{A}$ such that $A \wedge D = \inf\{A, D\}$ exists for all $A \in \mathcal{A}$. We investigate all those functions F that map various subsets $\mathcal{D}_{\mathcal{F}}$ of $\mathcal{A}_o$ into $\mathcal{A}$ so that $F(D) \wedge E = F(E) \wedge D$ for all $D, E \in \mathcal{D}_{\mathcal{F}}$.

The results obtained naturally extend and complement some former statements of G. Szász, J. Szendrei, M. Kolibiar, W. H. Cornish and J. Schmid on multipliers of lattices. Moreover, they are also closely related to the works of R. E. Johnson, Y. Utumi, G. D. Findlay and J. Lambek on generalized rings of quotients.

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0. Introduction

According to Larsen [15, p. 13], a function F from a nonvoid subset $\mathcal{D}_F$ of a commutative semigroup $\mathcal{A}$ into $\mathcal{A}$ is called a partial multiplier on $\mathcal{A}$ if $F(D) \cdot E = F(E) \cdot D$ for all $D, E \in \mathcal{D}_F$. Note that if in particular $\mathcal{D}_F$ is a subsemigroup of $\mathcal{A}$ and $F(D \cdot E) = F(D) \cdot E$ for all $D, E \in \mathcal{D}_F$, then F is a partial multiplier on $\mathcal{A}$.

A partial multiplier F on $\mathcal{A}$ is called total if $\mathcal{D}_F = \mathcal{A}$. Clearly, the identity function $\Delta_{\mathcal{A}}$ of $\mathcal{A}$ is a total multiplier on $\mathcal{A}$. Moreover, if $A \in \mathcal{A}$, then the function F_A, defined by $F_A(D) = A \cdot D$ for all $D \in \mathcal{A}$, is also a total multiplier on $\mathcal{A}$. A total multiplier F on $\mathcal{A}$ is called inner if $F = F_A$ for some $A \in \mathcal{A}$. Note that each total multiplier on $\mathcal{A}$ is inner if and only if $\mathcal{A}$ has an identity element.

The investigation of total multipliers is completely justified by Larsen [15]. To motivate the appropriateness of considering partial multipliers on $\mathcal{A}$, we make here a slight modification of the classical definition of fractions in accordance with the ideas of Lambek [14, p. 36].

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For this, we call an element D of $\mathcal{A}$ effective if it is cancellable in the sense that $A \cdot D = B \cdot D$ implies $A = B$ for all $A, B \in \mathcal{A}$. Note that if the semigroup $\mathcal{A}$ has a zero element, then the effective members of $\mathcal{A}$ are not proper divisors of zero in $\mathcal{A}$. Moreover, if $\mathcal{A}$ is, in particular, the multiplicative semigroup of a ring, then the converse statement is also true.

Now, if the family $\mathcal{S}$ of all effective elements of $\mathcal{A}$ is nonvoid, and moreover $P \in \mathcal{A}$ and $Q \in \mathcal{S}$, then in contrast to the classical definition of quotients we define

$$F = P/Q = \big\{ (D, A) \in \mathcal{A} \times \mathcal{A} : \quad A \cdot Q = P \cdot D \big\}$$

Namely, it can be easily seen that the relation F is a partial multiplier on $\mathcal{A}$. Moreover, F is maximal in the sense that it cannot be extended to a larger domain without violating the multiplier property. Therefore, the domain $\mathcal{D}_F$ of F is actually an ideal of $\mathcal{A}$ and $F(D \cdot E) = F(D) \cdot E$ for all $D \in \mathcal{D}_F$ and $E \in \mathcal{A}$.

In this respect, it is also worth noticing that $\Delta_{\mathcal{A}} = Q/Q$ and $F_A = (A{\cdot}Q)/Q$ for all $Q \in \mathcal{S}$ and $A \in \mathcal{A}$. Moreover, if $F = P/Q$, $G = R/S$ and $H = \big(P \cdot R\big) / \big(Q \cdot S\big)$ for some $P, R \in \mathcal{A}$ and $Q, S \in \mathcal{S}$, then $H(D) = F\big(G(D)\big)$ for all $D \in G^{-1}(\mathcal{D}_{\mathcal{F}})$. Therefore, $F \circ G$ is also a partial multiplier on $\mathcal{A}$ and H is a maximal extension of $F \circ G$.

By using the Hausdorff maximality principle, it can be easily shown that each partial multiplier on $\mathcal{A}$ has a maximal extension. However, in general, this maximal extension need not be unique. Therefore, to obtain a natural generalization of the classical quotient semigroup of $\mathcal{A}$, we have to consider only those partial multipliers F on $\mathcal{A}$ whose domains $\mathcal{D}_F$ are effective subsets of $\mathcal{A}$ in the sense that for each $A, B \in \mathcal{A}$, with $A \neq B$, there exists $D \in \mathcal{D}_F$ such that $A \cdot D \neq B \cdot D$.

Namely, if F is an effective partial multiplier on $\mathcal{A}$ in the sense that the domain $\mathcal{D}_F$ of F is an effective subset of $\mathcal{A}$, then it can be easily seen that

$$F^- = \big\{ (D, A) \in \mathcal{A} \times \mathcal{A} : \quad \forall\ Q \in \mathcal{D}_F : \quad A \cdot Q = F(Q) \cdot D \big\}$$

is the unique maximal extension of F. Note that if in particular the domain $\mathcal{D}_F$ of F contains an effective member Q of $\mathcal{A}$, then under the above definitions we have $F^- = F(Q)/Q = \{ (Q, F(Q)) \}^-$.

Moreover, it can also be easily shown that if F and G are maximal effective partial multipliers on $\mathcal{A}$, then $F \circ G$ is also an effective partial multiplier on $\mathcal{A}$. Therefore, in accordance with the usual multiplication of fractions, it is natural to define $F \bullet G = (F \circ G)^-$.

Namely, thus the family $\mathfrak{M}(\mathcal{A})$ of all maximal effective partial multipliers F on an effective and commutative semigroup $\mathcal{A}$ becomes a commutative semigroup with identity such that the mapping $A \mapsto F_A$ gives an isomorphism of $\mathcal{A}$ into $\mathfrak{M}(\mathcal{A})$ such that $F \bullet F_D = F_{F(D)}$ for all $F \in \mathfrak{M}(\mathcal{A})$ and $D \in \mathcal{D}_F$.

One of the most important particular instances of commutative semigroups is the class of semilattices [2, p. 9]. Total multipliers on semilattices and lattices have already been studied by Szász [23], [24], [25], Szász and Szendrei [26], Kolibiar [13], Figá-Talamanca and Franklin [7], Cornish [6], Nieminen [17], [18], and Noor and Cornish [19]. On the other hand, some partial multipliers on Boolean rings, semilattices and distributive lattices seem to have been investigated only by Brainerd and Lambek [3], Berthiaume [1], and Schmid [21], [22].

The main purpose of this paper is to extend some of the results of the above mentioned authors to certain partial multipliers of a larger class of partially ordered sets. For this, we shall assume that $\mathcal{A}$ is a partially-ordered set such that the family $\mathcal{A}_o$ of all elements $D \in \mathcal{A}$ such that $A \wedge D = \inf\{A, D\}$ exists for all $A \in \mathcal{A}$ is nonvoid. Note that this is the case if in particular $\mathcal{A}$ has a least (greatest) element or $\mathcal{A}$ is a semilattice.

We shall investigate all those functions F that map various subsets $\mathcal{D}_F$ of $\mathcal{A}_o$ into $\mathcal{A}$ so that $F(D) \wedge E = F(E) \wedge D$ for all $D, E \in \mathcal{D}_F$. To obtain some less trivial results about such partial multipliers F, we shall, in addition, assume that $F(D) \leq D$ for all $D \in \mathcal{D}_F$. The latter condition is however, automatically satisfied whenever the domain $\mathcal{D}_F$ of F is supposed to be effective in $\mathcal{A}$.

1. Partially ordered sets

According to Birkhoff [2, p. 1], a nonvoid set $\mathcal{A}$ together with a reflexive, transitive and antisymmetric relation $\leq$ is called a poset (partially ordered set). The use of the script capital letter is mainly motivated by the fact that each poset is isomorphic to a family of sets partially ordered by set inclusion.

As usual, a poset $\mathcal{A}$ is called (1) totally ordered if for each $A, B \in \mathcal{A}$ either $A \leq B$ or $B \leq A$, (2) directed if for each $A, B \in \mathcal{A}$ there exists $C \in \mathcal{A}$ such that $A \leq C$ and $B \leq C$. Moreover, a subset $\mathcal{D}$ of $\mathcal{A}$ is called (1) descending if $A \in \mathcal{A}$, $D \in \mathcal{D}$ and $A \leq D$ imply $A \in \mathcal{D}$, and (2) cofinal if for each $A \in \mathcal{A}$ there exists $D \in \mathcal{D}$ such that $A \leq D$.

The infimum (greatest lower bound) and the supremum (least upper bound) of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ will be understood in the usual sense. However, instead of $\inf \mathcal{D}$ and $\sup \mathcal{D}$, we shall use the lattice theoretic notations $\bigwedge \mathcal{D}$ and $\bigvee \mathcal{D}$, respectively. Thus, for instance $E = \bigwedge \mathcal{D}$ if and only if $E \in \mathcal{A}$ such that for each $A \in \mathcal{A}$ we have $A \leq E$ if and only if $A \leq D$ for all $D \in \mathcal{D}$.

However, in the sequel, we shall only need some very particular cases of the above definitions whenever we write $A \wedge B = \inf\{A, B\}$ and $A \vee B = \sup\{A, B\}$. Concerning the operation $\wedge$, we shall frequently use the next simple theorems which, in their present forms, are usually not included in the standard books on lattices.

Theorem 1.1 *If $\mathcal{A}$ is a poset and $A, B, C, D \in \mathcal{A}$, then*

(1) $A \leq B$ *if and only if* $A = A \wedge B$;

(2) $A \leq B$ *and* $C \leq D$ *imply* $A \wedge C \leq B \wedge D$ *whenever* $A \wedge C$ *and* $B \wedge D$ *exist.*

Corollary 1.2 *If* $\mathcal{A}$ *is a poset and* $A, B, C \in \mathcal{A}$, *then*

(1) $A = A \wedge A$; *(2)* $A = A \wedge (A \vee B)$ *whenever* $A \vee B$ *exists;*

(3) $A \leq B$ *implies* $A \wedge C \leq B \wedge C$ *whenever* $A \wedge C$ *and* $B \wedge C$ *exist.*

Theorem 1.3 *If* $\mathcal{A}$ *is a poset and* $A, B, C \in \mathcal{A}$, *then*

(1) $A \wedge B = B \wedge A$ *whenever either* $B \wedge A$ *or* $A \wedge B$ *exist;*

(2) $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ *whenever* $A \wedge B$ *and* $B \wedge C$ *and moreover either* $(A \wedge B) \wedge C$ *or* $A \wedge (B \wedge C)$ *exist.*

Remark 1.4 A slightly weaker form of the assertion (2) can be found in Birkhoff [2, Lemma 1, p. 8]. Moreover, a somewhat weaker form of the dual of this assertion can be found in Grätzer [10, Exercise 31, p. 8].

In the sequel, we shall also need the following obvious theorem.

Theorem 1.5 *If* $\mathcal{A}$ *is a poset and* $\mathcal{D} \subset \mathcal{A}$, *then the following assertions are equivalent:*

(1) $\mathcal{D}$ *is descending;*

(2) $A \in \mathcal{A}$ *and* $D \in \mathcal{D}$ *imply* $A \wedge D \in \mathcal{D}$ *whenever* $A \wedge D$ *exists.*

Remark 1.6 From the above theorems, by using the dual $\mathcal{A}(\geq)$ of the poset $\mathcal{A}(\leq)$, one can easily get some analogous theorems for the operation $\vee$ and the ascending subsets of $\mathcal{A}(\leq)$. However, in the sequel, we shall mainly need the operation $\wedge$. Therefore, we shall assume here some rather particular terminology.

A nonvoid subset $\mathcal{B}$ of poset $\mathcal{A}$ is called a semilattice in $\mathcal{A}$ if $D \wedge E$ exists in $\mathcal{A}$ and belongs to $\mathcal{B}$ for all $D, E \in \mathcal{B}$. Moreover, a nonvoid subset $\mathcal{D}$ of a semilattice $\mathcal{B}$ in a poset $\mathcal{A}$ is called an ideal of $\mathcal{B}$ if $D \wedge E$ is in $\mathcal{D}$ for all $D \in \mathcal{D}$ and $E \in \mathcal{B}$. Note that, by Theorem 1.5, $\mathcal{D}$ is an ideal of $\mathcal{B}$ if and only if $\mathcal{D}$ is descending subset of $\mathcal{B}$.

If $\mathcal{D}$ and $\mathcal{E}$ are subsets of a poset $\mathcal{A}$ such that $D \wedge E$ exits for all $D \in \mathcal{D}$ and $E \in \mathcal{E}$, then we write $\mathcal{D} \wedge \mathcal{E} = \{ D \wedge E : \ D \in \mathcal{D}, \ E \in \mathcal{E} \}$. Note that if $\mathcal{B}$ is a semilattice in a poset $\mathcal{A}$, then $\mathcal{B} = \mathcal{B} \wedge \mathcal{B}$. Moreover, if $\mathcal{D}$ and $\mathcal{E}$ are ideals of $\mathcal{B}$, then $\mathcal{D} = \mathcal{D} \wedge \mathcal{B}$ and $\mathcal{D} \cap \mathcal{E} = \mathcal{D} \wedge \mathcal{E}$. Therefore, the ideal $\mathcal{D} \cap \mathcal{E}$ inherits some useful properties of $\mathcal{D}$ and $\mathcal{E}$.

2. Effective sets

Definition 2.1 *If $\mathcal{A}$ is a poset, then the set*

$$\mathcal{A}_o = \big\{ D \in \mathcal{A} : \quad \forall\ A \in \mathcal{A} : \quad \exists\ A \wedge D \big\}$$

is called the centre of $\mathcal{A}$.

Concerning this notion, which is very similar to that of Lambek [14, p. 17], but quite different from that of Birkhoff [2, p. 67], we can easily establish the next useful propositions.

Proposition 2.2 *If $\mathcal{A}$ is a poset with a least (resp. greatest) element O (resp. X), then $O \in \mathcal{A}_o$ (resp. $X \in \mathcal{A}_o$).*

Proposition 2.3 *If $\mathcal{A}$ is a poset such that $\mathcal{A}_o \neq \emptyset$, then $\mathcal{A}_o$ is a semilattice in $\mathcal{A}$.*

Proof. If $D, E \in \mathcal{A}_o$ and $A \in \mathcal{A}$, then by Definition 2.1 $A \wedge D$, $D \wedge E$ and $(A \wedge D) \wedge E$ exist. Therefore, by Theorem 1.3 (2), $A \wedge (D \wedge E) = (A \wedge D) \wedge E$ also exists. Thus, by Definition 2.1, $D \wedge E \in \mathcal{A}_o$. □

Proposition 2.4 *If $\mathcal{A}$ is a poset, then $\mathcal{A}_o = \mathcal{A}$ if and only if $\mathcal{A}$ is a semilattice.*

In the sequel, we shall also need the following.

Definition 2.5 *A subset $\mathcal{D}$ of a poset $\mathcal{A}$ is called effective (resp. supereffective) if $\mathcal{D} \subset \mathcal{A}_o$ and for each $A, B \in \mathcal{A}$, with $A \neq B$ (resp. $A \not\leq B$) there exists $D \in \mathcal{D}$ such that $A \wedge D \neq B \wedge D$ (resp. $A \wedge D \not\leq B \wedge D$).*

Moreover, a poset $\mathcal{A}$ is called effective (supereffective) if its centre $\mathcal{A}_o$ is effective (supereffective).

Remark 2.6 The above definition is correct in the sense that if $\mathcal{D}$ is a supereffective subset of a poset $\mathcal{A}$, then $\mathcal{D}$ is, in particular, effective.

Namely, if $A, B \in \mathcal{A}$ such that $A \neq B$, then because of the antisymmetry of $\leq$ we have either $A \not\leq B$ or $B \not\leq A$. Therefore, by the supereffectiveness of $\mathcal{D}$, there exists $D \in \mathcal{D}$ such that either $A \wedge D \not\leq B \wedge D$ or $B \wedge D \not\leq A \wedge D$. Hence, by the reflexivity of $\leq$ it follows that $A \wedge D \neq B \wedge D$.

The following example shows that the converse implication need not be true.

Example 2.7 *Define $A = \{0, 1, 2\}$, $B = \{1, 2, 3\}$, $C = \{2, 3\}$, $D = \{2\}$, $E = \{1\}$ and $F = \emptyset$. Then the family $\mathcal{A} = \big\{ A, B, C, D, E, F \big\}$, with the ordinary set inclusion, is an effective poset such that $\mathcal{A}$ is not supereffective.*

To check this, give the meet table of $\mathcal{A}$, and note that $\mathcal{A}_o = \{C, D, E, F\}$. Moreover, note that $\{C, E\}$, and thus $\mathcal{A}_o$ is effective. But, $\mathcal{A}_o$ is not supereffective. Namely, $A \not\subset B$, but $A \wedge C \subset B \wedge C$, $A \wedge D = B \wedge D$, $A \wedge E = B \wedge E$ and $A \wedge F = B \wedge F$.

Fortunately, in the most important particular cases, the effective subsets are also supereffective.

Theorem 2.8 *If $\mathcal{D}$ is an effective subset of a semilattice $\mathcal{A}$, then $\mathcal{D}$ is supereffective.*

Proof. If $A, B \in \mathcal{A}$ such that $A \not\leq B$, then by Theorem 1.1 (1) $A \neq A \wedge B$. Therefore, by the effectiveness of $\mathcal{D}$, there exists $D \in \mathcal{D}$ such that $A \wedge D \neq (A \wedge B) \wedge D$. Moreover, by Corollary 1.2 (1) and Theorem 1.3, $(A \wedge B) \wedge D = (A \wedge D) \wedge (B \wedge D)$. Therefore, $A \wedge D \neq (A \wedge D) \wedge (B \wedge D)$. Thus, by Theorem 1.1 (1), $A \wedge D \not\leq B \wedge D$. □

To provide some examples of effective sets, we can only prove here the following theorem.

Theorem 2.9 *If $\mathcal{D}$ is a cofinal subset of a poset $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{A}_o$, then $\mathcal{D}$ is effective.*

Proof. If $A, B \in \mathcal{A}$ such that $A \wedge D = B \wedge D$ for all $D \in \mathcal{D}$, then by choosing $D, E \in \mathcal{D}$ such that $A \leq D$ and $B \leq E$, we can see that $A = A \wedge D = B \wedge D = (B \wedge E) \wedge D = (B \wedge D) \wedge E = (A \wedge D) \wedge E = A \wedge E = B \wedge E = B$. □

Now, as an immediate consequence of Proposition 2.4 and Theorems 2.9 and 2.8, we can also state

Corollary 2.10 *If $\mathcal{A}$ is a semilattice, then $\mathcal{A}$ is supereffective.*

Moreover, as a partial converse to Theorem 2.9, we can also prove the following

Theorem 2.11 *If $\mathcal{A}$ is a totally ordered set without a greatest element and $\mathcal{D}$ is an effective subset of $\mathcal{A}$, then $\mathcal{D}$ is cofinal.*

Proof. Assume on the contrary that $\mathcal{D}$ is not cofinal. Then, since $\mathcal{A}$ is totally ordered, there exists $A \in \mathcal{A}$ such that $D < A$ for all $D \in \mathcal{D}$. Moreover, since $\mathcal{A}$ does not have a greatest element, there exists $B \in \mathcal{A}$ such that $A < B$. Furthermore, since $\mathcal{D}$ is effective, there exists $D \in \mathcal{D}$ such that $A \wedge D \neq B \wedge D$. This is already a contradiction since $A \wedge D = D$ and $B \wedge D = D$. □

On the other hand, in addition to Theorem 2.9, it is also worth proving the following

Theorem 2.12 *If $\mathcal{D}$ is a cofinal subset of a poset $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{A}_o$ and $\mathcal{D}$ is directed, then $\mathcal{D}$ is supereffective.*

Proof. If $A, B \in \mathcal{A}$ such that $A \wedge D \leq B \wedge D$ for all $D \in \mathcal{D}$, then by choosing $D \in \mathcal{D}$ such that $A \leq D$ and $B \leq D$, we can see that $A = A \wedge D \leq B \wedge D = B$. □

Now, as an immediate consequence of Theorem 2.12, we can also state

Corollary 2.13 *If $\mathcal{A}$ is a poset with a greatest element X and $\mathcal{D} \subset \mathcal{A}_o$ such that $X \in \mathcal{D}$, then $\mathcal{D}$ is supereffective.*

Hence, by Proposition 2.2, it is clear that in particular we also have

Corollary 2.14 *If $\mathcal{A}$ is a poset with a greatest element, then $\mathcal{A}$ is supereffective.*

Moreover, in the sequel, we shall also need the following

Theorem 2.15 *If $\mathcal{D}$ and $\mathcal{E}$ are effective (supereffective) subsets of a poset $\mathcal{A}$, then the set $\mathcal{D} \wedge \mathcal{E}$ is also effective (supereffective).*

Hint. If $A, B \in \mathcal{A}$ such that $A \neq B$, then since $\mathcal{D}$ is effective there exists $D \in \mathcal{D}$ such that $A \wedge D \neq B \wedge D$. Moreover, since $\mathcal{E}$ is effective, there exists $E \in \mathcal{E}$ such that $(A \wedge D) \wedge E \neq (B \wedge D) \wedge E$. Hence, by Theorem 1.3 (2), $A \wedge (D \wedge E) \neq B \wedge (D \wedge E)$. □

From Theorem 2.15, it is clear that in particular we also have

Corollary 2.16 *If $\mathcal{D}$ and $\mathcal{E}$ are effective (supereffective) subsets of the poset $\mathcal{A}$ such that $\mathcal{D}$ and $\mathcal{E}$ are ideals of $\mathcal{A}_o$, then the ideal $\mathcal{D} \cap \mathcal{E}$ of $\mathcal{A}_o$ is also an effective (supereffective) subset of $\mathcal{A}$.*

Proof. Namely, in this case, we have $\mathcal{D} \cap \mathcal{E} = \mathcal{D} \wedge \mathcal{E}$. □

3. Partial multipliers

Definition 3.1 *If $\mathcal{A}$ is a poset such that $\mathcal{A}_o \neq \emptyset$, then a function F from a nonvoid subset $\mathcal{D}_F$ of $\mathcal{A}_o$ into $\mathcal{A}$ is called a partial multiplier on $\mathcal{A}$ if*

$$F(D) \wedge E = F(E) \wedge D$$

for all $D, E \in \mathcal{D}_F$. The family of all partial multipliers F on $\mathcal{A}$ is denoted by $\mathcal{M}(\mathcal{A})$.

Remark 3.2 A multiplier $F \in \mathcal{M}(\mathcal{A})$ is called total if $\mathcal{D}_F = \mathcal{A}_o$. Clearly, the identity function $\Delta_{\mathcal{A}_o} = \{ (A, A) : \ A \in \mathcal{A} \}$ of $\mathcal{A}_o$ is a total member of $\mathcal{M}(\mathcal{A})$.

Moreover, to provide some less trivial examples of total multipliers, we can at once state

Proposition 3.3 *If $\mathcal{A}$ is a poset, with $\mathcal{A}_o \neq \emptyset$, and $A \in \mathcal{A}$, then the function F_A, defined by*

$$F_A(D) = A \wedge D$$

for all $D \in \mathcal{A}_o$, is a total member of $\mathcal{M}(\mathcal{A})$.

Proof. Namely, if $D, E \in \mathcal{A}_o$, then by Theorem 1.3 we have $F_A(D) \wedge E = (A \wedge D) \wedge E = (A \wedge E) \wedge D = F_A(E) \wedge D$. □

Remark 3.4 In addition to the above proposition, we can also at once state that F_A is nondecreasing, $F_A(D) \leq D$ and $F_A(D) \leq A$ for all $D \in \mathcal{A}_o$, and $F_A(D \wedge E) = F_A(D) \wedge E$ for all $D, E \in \mathcal{A}_o$.

Moreover, it is also worth noticing that if the poset $\mathcal{A}$ has a least (resp. greatest) element O (resp. X), then F_O (resp. F_X) is just the zero (resp. identity) function of $\mathcal{A}_o$.

In this respect, it is also worth proving the following

Proposition 3.5 *If $F \in \mathcal{M}(\mathcal{A})$ such that the range $F(\mathcal{D}_F)$ of F has an upper bound E in $\mathcal{A}$ such that $E \in \mathcal{D}_F$, then $F(D) = F_{F(E)}(D)$ for all $D \in \mathcal{D}_F$.*

Proof. Namely, $F(D) = F(D) \wedge E = F(E) \wedge D = F_{F(E)}(D)$ for all $D \in \mathcal{D}_F$. □

Hence, by Proposition 2.2, it is clear that in particular we have

Corollary 3.6 *If $F \in \mathcal{M}(\mathcal{A})$ is total and the poset $\mathcal{A}$ has a greatest element X, then $F = F_{F(X)}$.*

Remark 3.7 A total multiplier $F \in \mathcal{M}(\mathcal{A})$ is called inner if there exists $A \in \mathcal{A}$ such that $F = F_A$.

Corollary 3.6 shows that each total multiplier on a poset with a greatest element is inner. Moreover, as a partial converse to this statement, we can also at once state

Proposition 3.8 *If each total member of $\mathcal{M}(\mathcal{A})$ is inner, then the centre $\mathcal{A}_o$ of $\mathcal{A}$ is bounded from above.*

Proof. Since $\Delta_{\mathcal{A}_o}$ is a total member of $\mathcal{M}(\mathcal{A})$, there exists $A \in \mathcal{A}$ such that $\Delta_{\mathcal{A}_o} = F_A$. Hence, $D = A \wedge D$, and thus $D \leq A$ for all $D \in \mathcal{A}_o$. □

Hence, by Proposition 2.4, it is clear that in particular we have

Corollary 3.9 *If $\mathcal{A}$ is a semilattice such that each total member of $\mathcal{M}(\mathcal{A})$ is inner, then $\mathcal{A}$ has a greatest element.*

Definition 3.10 *If $F, G \in \mathcal{M}(\mathcal{A})$ such that $\mathcal{D}_F \subset \mathcal{D}_G$ and $F(D) \leq G(D)$ for all $D \in \mathcal{D}_F$, then we write $F \leq G$.*

Concerning the above inequality, we can easily establish the following theorem.

Theorem 3.11 *If $\mathcal{A}$ is a poset such that $\mathcal{A}_o \neq \emptyset$, then $\mathcal{M}(\mathcal{A})$, together with the inequality introduced in Definition 3.10, is a poset such that the mapping $A \mapsto F_A$ of $\mathcal{A}$ into $\mathcal{M}(\mathcal{A})$ is nondecreasing.*

The importance of effective and supereffective posets is apparent from the following two obvious theorems.

Theorem 3.12 *If $\mathcal{A}$ is a poset such that $\mathcal{A}_o \neq \emptyset$, then the following assertions are equivalent:*

(1) $\mathcal{A}$ is effective;

(2) the mapping $A \mapsto F_A$ of $\mathcal{A}$ into $\mathcal{M}(\mathcal{A})$ is injective.

Theorem 3.13 *If $\mathcal{A}$ is a poset such that $\mathcal{A}_o \neq \emptyset$, then the following assertions are equivalent:*

(1) $\mathcal{A}$ is supereffective;

(2) the inverse of the mapping $A \mapsto F_A$ of $\mathcal{A}$ into $\mathcal{M}(\mathcal{A})$ is nondecreasing.

Remark 3.14 Therefore, if $\mathcal{A}$ is a supereffective poset, then by identifying each $A \in \mathcal{A}$ with F_A the poset $\mathcal{M}(\mathcal{A})$ becomes a natural extension of $\mathcal{A}$.

Finally, to provide a genuine example for nontotal multipliers, we state

Example 3.15 *Let $\mathcal{A}$ be a distributive lattice [2, p. 12] with a smallest element O and a greatest element X such that $X \neq O$. Choose $A \in \mathcal{A}$ such that $A \neq O$, and define*

$$\mathcal{D} = \big\{ D \in \mathcal{A} : \quad A \wedge D = O \big\} \qquad \text{and} \qquad F(D) = A \vee D \qquad \big(D \in \mathcal{D} \big).$$

Then $\mathcal{D}$ is a noneffective ideal of $\mathcal{A}$ and F is a maximal (nonextendable) member of $\mathcal{M}(\mathcal{A})$.

Namely, if $D, E \in \mathcal{D}$, then by the distributivity of $\mathcal{A}$

$$F(D) \wedge E = (A \vee D) \wedge E = (A \wedge E) \vee (D \wedge E) = O \vee (D \wedge E) = D \wedge E.$$

Hence, it is clear that $F(D) \wedge E = F(E) \wedge D$, and thus $F \in \mathcal{M}(\mathcal{A})$.

Moreover, if $G \in \mathcal{M}(\mathcal{A})$ such that $F \subset G$, then

$$A \wedge D = (A \vee O) \wedge D = F(O) \wedge D = G(O) \wedge D = G(D) \wedge O = O,$$

for all $D \in \mathcal{D}_G$. Therefore, $\mathcal{D}_G \subset \mathcal{D}$, and thus $G = F$.

Remark 3.16 In connection with the above example, it is also worth noticing that F is nondecreasing, but $D < F(D)$, and hence $F(D) \not\leq D$ for all $D \in \mathcal{D}$.

Namely, we evidently have $D \leq A \vee D = F(D)$ for all $D \in \mathcal{D}$. Moreover, if there were a $D \in \mathcal{D}$ such $D = F(D) = A \vee D$, then $A \leq D$, and thus $A = A \wedge D = O$ would also hold.

4. Nonexpansive multipliers

Definition 4.1 *A multiplier* $F \in \mathcal{M}(\mathcal{A})$ *is called nonexpansive if* $F(D) \leq D$ *for all* $D \in \mathcal{D}_F$ *. The family of all nonexpansive members of* $\mathcal{M}(\mathcal{A})$ *is denoted by* $\mathcal{M}'(\mathcal{A})$ *.*

Remark 4.2 Note that if $F \in \mathcal{M}'(\mathcal{A})$ and the poset $\mathcal{A}$ has a least element O such that $O \in \mathcal{D}_F$, then we necessarily have $F(O) = O$.

Moreover, it is also worth noticing that for each $D \in \mathcal{D}_F$ we have $F(D) \leq D$ if and only if $F(D) = F(D) \wedge D$. Therefore, we may naturally introduce

Definition 4.3 *If* $F \in \mathcal{M}(\mathcal{A})$ *, then we define*

$$F'(D) = F(D) \wedge D$$

for all $D \in \mathcal{D}_F$*.*

Namely, by the above observation, we can at once state

Proposition 4.4 $\mathcal{M}'(\mathcal{A}) = \big\{ F \in \mathcal{M}(\mathcal{A}) : \quad F' = F \big\}$.

Moreover, concerning the above notations, we can also easily prove

Proposition 4.5 $\mathcal{M}'(\mathcal{A}) = \big\{ F' : \quad F \in \mathcal{M}(\mathcal{A}) \big\}$.

Proof. If $F \in \mathcal{M}(\mathcal{A})$ and $D, E \in \mathcal{D}_F$, then we evidently have

$$\begin{aligned} F'(D) \wedge E &= \big(F(D) \wedge D \big) \wedge (E \wedge E) = \big(F(D) \wedge E \big) \wedge (D \wedge E) = \\ &= \big(F(E) \wedge D \big) \wedge (D \wedge E) = \big(F(E) \wedge E \big) \wedge (D \wedge D) = F'(E) \wedge D . \end{aligned}$$

Moreover, it is clear that $F'(D) = F(D) \wedge D \leq D$. Therefore, $F' \in \mathcal{M}'(\mathcal{A})$. Hence, by Proposition 4.4, it is clear that the required equality is also true. □

Now, by making use of the above propositions, we can also easily establish

Theorem 4.6 *The mapping* $F \mapsto F'$ *of the poset* $\mathcal{M}(\mathcal{A})$ *into itself is an interior operation on* $\mathcal{M}(\mathcal{A})$ *such that* $\mathcal{M}'(\mathcal{A})$ *is the family of all open members of* $\mathcal{M}(\mathcal{A})$ *.*

The importance of nonexpansive multipliers lies mainly in the following two theorems and their numerous consequences which are straightforward extensions of the duals of the corresponding results of Szász [23], [24] and Szász and Szendrei [26].

Theorem 4.7 *If* $F \in \mathcal{M}'(\mathcal{A})$ *, then*

$$F(D \wedge E) = F(D) \wedge E$$

for all $D \in \mathcal{D}_F$ *and* $E \in \mathcal{A}_o$ *with* $D \wedge E \in \mathcal{D}_F$*.*

Proof. If D and E are as in the theorem, then by the nonexpansitivity of F we have $F(D \wedge E) \leq D \wedge E \leq D$. Hence, by Theorem 1.1 (1), it follows that

$$F(D \wedge E) = F(D \wedge E) \wedge D.$$

Moreover, by using the multiplier property of F and Theorem 1.3 (2) and Proposition 4.4, we can easily see that

$$F(D \wedge E) \wedge D = F(D) \wedge (D \wedge E) = \big(F(D) \wedge D\big) \wedge E = F'(D) \wedge E = F(D) \wedge E.$$

Therefore, the required equality is also true. □

Now, as a useful consequence of Theorems 4.7 and 1.3 (1) and Proposition 4.4, we can also state

Corollary 4.8 *If F is a function from a subset $\mathcal{D}_F$ of the centre $\mathcal{A}_o$ of a poset $\mathcal{A}$ into $\mathcal{A}$ such that $\mathcal{D}_F$ is a semilattice in $\mathcal{A}$, then the following assertions are equivalent:*

(1) $F \in \mathcal{M}'(\mathcal{A})$;

(2) $F(D \wedge E) = F(D) \wedge E$ *for all* $D, E \in \mathcal{D}_F$.

Remark 4.9 Whenever $\mathcal{D}_F = \mathcal{A}_o$, the assertion (2) can be expressed in the more concise form, that is $F \circ F_E = F_E \circ F$ for all $E \in \mathcal{A}_o$.

Moreover, as a deeper characterization of nonexpansive multipliers, we can also prove the following

Theorem 4.10 *If F is a function from a nonvoid subset $\mathcal{D}_F$ of the centre $\mathcal{A}_o$ of a poset $\mathcal{A}$ into $\mathcal{A}$, then the following assertions are equivalent:*

(1) $F \in \mathcal{M}'(\mathcal{A})$;

(2) $F(D) \wedge E = F(D) \wedge F(E)$ *for all* $D, E \in \mathcal{D}_F$.

Hint. If the assertion (1) holds, then by Corollary 1.2, Theorem 1.3 and Proposition 4.4 it is clear that

$$\begin{aligned} F(D) \wedge E &= \big(F(D) \wedge F(D)\big) \wedge E = \\ &= F(D) \wedge \big(F(D) \wedge E\big) = F(D) \wedge \big(F(E) \wedge D\big) = F(D) \wedge \big(D \wedge F(E)\big) = \\ &= \big(F(D) \wedge D\big) \wedge F(E) = F'(D) \wedge F(E) = F(D) \wedge F(E) \end{aligned}$$

for all $D, E \in \mathcal{D}_F$. That is, the assertion (2) also holds. □

Now, as a useful consequence of Theorems 4.7 and 4.10, we can also state

Corollary 4.11 *If* $F \in \mathcal{M}'(\mathcal{A})$, *then*

(1) $F\big(F(D)\big) = F(D)$ *for all* $D \in \mathcal{D}_F$ *with* $F(D) \in \mathcal{D}_F$;

(2) $F(D \wedge E) = F(D) \wedge F(E)$ *for all* $D, E \in \mathcal{D}_F$ *with* $D \wedge E \in \mathcal{D}_F$.

Hint. If D is as in the assertion (1), then by Proposition 4.4, Theorem 4.7 and Corollary 1.2 (1) it is clear that

$$\begin{aligned} F\big(F(D)\big) &= F\big(F'(D)\big) = F\big(F(D)\wedge D\big) = \\ &= F\big(D\wedge F(D)\big) = F(D)\wedge F(D) = F(D)\,. \end{aligned} \qquad \square$$

From the first assertion of the above corollary we can easily get

Corollary 4.12 *If* $F\in\mathcal{M}'(\mathcal{A})$, *then* $\mathcal{D}_F\cap F(\mathcal{D}_F)$ *is the family of all fixed points of* F.

Hint. If $E\in F(\mathcal{D}_F)$, then there exists $D\in\mathcal{D}_F$ such that $E=F(D)$. Therefore, if $E\in\mathcal{D}_F$ also holds, then by Corollary 4.11 (1) we have $F(E)=F\big(F(D)\big)=F(D)=E$. That is, E is a fixed point of F. $\square$

Moreover, from Theorem 4.7 and the second assertion of Corollary 4.11, we can at once get

Corollary 4.13 *If* $F\in\mathcal{M}'(\mathcal{A})$, *then*

$$F(D) = F(E)\wedge D \qquad \text{and} \qquad F(D) = F(D)\wedge F(E)$$

for all $D, E\in\mathcal{D}_F$ *with* $D\leq E$.

Hence, it is clear that in particular we also have

Corollary 4.14 *If* $F\in\mathcal{M}'(\mathcal{A})$, *then* F *is nondecreasing.*

Moreover, from the first assertion of Corollary 4.13, we can easily get

Corollary 4.15 *If* $\mathcal{A}$ *is distributive lattice and* $F\in\mathcal{M}'(\mathcal{A})$, *then*

$$F(D\vee E) = F(D)\vee F(E)$$

for all $D, E\in\mathcal{D}_F$ *with* $D\vee E\in\mathcal{D}_F$.

Proof. If D and E are as above, then by the first statement of Corollary 4.13

$$\begin{aligned} F(D\vee E) &= F(D\vee E)\wedge(D\vee E) = \\ &= \big(F(D\vee E)\wedge D\big)\vee\big(F(D\vee E)\wedge E\big) = F(D)\vee F(E)\,. \end{aligned} \qquad \square$$

Remark 4.16 Following [24], we can also note that a lattice $\mathcal{A}$ is already distributive if each inner multiplier of $\mathcal{A}$ is join preserving.

5. Maximal multipliers

Definition 5.1 *A multiplier* $F \in \mathcal{M}(\mathcal{A})$ *is called maximal if* $G \in \mathcal{M}(\mathcal{A})$ *and* $F \subset G$ *imply that* $F = G$.

Moreover, if $F, G \in \mathcal{M}(\mathcal{A})$ *such that* $F \subset G$ *and* G *is maximal, then* G *is called a maximal extension of* F.

A straightforward application of the Hausdorff maximality principle gives the following

Theorem 5.2 *Each* $F \in \mathcal{M}(\mathcal{A})$ *has at least one maximal extension* G.

Hint. Define $\mathcal{F} = \big\{ \mathcal{G} \in \mathcal{M}(\mathcal{A}) : \ \mathcal{F} \subset \mathcal{G} \big\}$. Then, $\mathcal{F}$ is a poset with respect to set inclusion. Therefore, by the Hausdorff maximality principle, there exists at least one maximal totally ordered subset $\mathcal{G}$ of $\mathcal{F}$. Hence, it is easy to see that the relation $G = \bigcup \mathcal{G}$ has the required properties. □

The importance of maximal multipliers lies mainly in the following

Theorem 5.3 *If* $F \in \mathcal{M}(\mathcal{A})$ *is maximal, then* $\mathcal{D}_F$ *is an ideal of* $\mathcal{A}_o$.

Proof. Namely, if $D \in \mathcal{D}_F$ and $E \in \mathcal{A}_o$ such that $D \wedge E \notin \mathcal{D}_F$, then

$$G = F \cup \big\{ \big(D \wedge E, \ F(D) \wedge E \big) \big\}$$

is a function on $\mathcal{D}_F \cup \{ D \wedge E \}$ such that

$$\begin{aligned} G(D \wedge E) \wedge Q &= \big(F(D) \wedge E \big) \wedge Q = \big(F(D) \wedge Q \big) \wedge E = \\ &= \big(F(Q) \wedge D \big) \wedge E = F(Q) \wedge (D \wedge E) = G(Q) \wedge (D \wedge E) \end{aligned}$$

for all $Q \in \mathcal{D}_F$. Therefore, $G \in \mathcal{M}(A)$, and this contradicts the maximality of F. Consequently, $\mathcal{D}_F \wedge \mathcal{A}_o \subset \mathcal{D}_F$, and thus $\mathcal{D}_F$ is an ideal of $\mathcal{A}_o$. □

Remark 5.4 A similar application of the Hausdorff maximality principle shows that for each $F \in \mathcal{M}'(\mathcal{A})$ there exists a maximal member G of $\mathcal{M}'(\mathcal{A})$ such that $F \subset G$.

Moreover, by using the same reasoning as in the proof of Theorem 5.3 we can also prove that the domain $\mathcal{D}_F$ of a maximal member F of $\mathcal{M}'(\mathcal{A})$ is necessarily an ideal of $\mathcal{A}_o$.

However, these facts seem to be of no particular importance for us now since we can also prove the following theorems.

Theorem 5.5 *If* $F \in \mathcal{M}'(\mathcal{A})$ *and* F *has a unique maximal extension* F^- *in* $\mathcal{M}(\mathcal{A})$, *then* $F^- \in \mathcal{M}'(\mathcal{A})$.

Proof. Define $G = \left(F^-\right)'$. Then, by Proposition 4.5, $G \in \mathcal{M}'(\mathcal{A})$. Moreover, by Proposition 4.4, it is clear that

$$F(D) = F'(D) = F(D) \wedge D = F^-(D) \wedge D = (F^-)'(D) = G(D)$$

for all $D \in \mathcal{D}_F$, and hence $F \subset G$. Therefore, a maximal extension H of G is also a maximal extension of F. Hence, since F^- is the unique maximal extension of F, it follows that $H = F^-$. Now, since $G \subset H$ and $\mathcal{D}_G = \mathcal{D}_{F^-} = \mathcal{D}_H$, it is clear that $F^- = H = G$, and thus $F^- \in \mathcal{M}'(\mathcal{A})$ is also true. □

Theorem 5.6 *If* $F \in \mathcal{M}'(\mathcal{A})$, *then there exists at least one* $F^* \in \mathcal{M}'(\mathcal{A})$ *such that* $F \subset F^*$ *and the domain* $\mathcal{D}_{F^*}$ *of* F^* *is an ideal of* $\mathcal{A}_o$.

Proof. By Theorem 5.2 there exists at least one maximal extension G of F. Moreover, by Theorem 5.3, the domain $\mathcal{D}_G$ of G is an ideal of $\mathcal{A}_o$. Define $F^* = G'$. Then, by the definition of F' and Proposition 4.4, it is clear that $\mathcal{D}_{F^*} = \mathcal{D}_G$ and $F = F' \subset G' = F^*$. Moreover, by Proposition 4.5, we have $F^* \in \mathcal{M}'(\mathcal{A})$. Therefore, F^* has the required properties. □

The importance of those nonexpansive multipliers whose domains are ideals is apparent from the results of Section 4 and the corollaries of the subsequent theorems.

Theorem 5.7 *If* $F \in \mathcal{M}'(\mathcal{A})$ *and* $\mathcal{D}$ *is an ideal of* $\mathcal{A}_o$ *such that* $\mathcal{D} \subset \mathcal{D}_F$, *then* $F(\mathcal{D})$ *is semilattice in* $\mathcal{A}$ *such that* $F(\mathcal{D}) \wedge \mathcal{A}_o \subset F(\mathcal{D})$.

Proof. In this case, by Corollary 4.11 (2), $F(D) \wedge F(E) = F(D \wedge E) \in F(\mathcal{D})$ for all $D, E \in \mathcal{D}$. Therefore, $F(\mathcal{D})$ is a semilattice in $\mathcal{A}$.

Moreover, by Theorem 4.7, $F(D) \wedge E = F(D \wedge E) \in F(\mathcal{D})$ for all $D \in \mathcal{D}$ and $E \in \mathcal{A}_o$. Therefore, the required inclusion is also true. □

Corollary 5.8 *If* $F \in \mathcal{M}'(\mathcal{A})$ *such that* $\mathcal{D}_F$ *is an ideal of* $\mathcal{A}_o$, *then* $F(\mathcal{D}_F)$ *is semilattice in* $\mathcal{A}$ *such that* $F(\mathcal{D}_F) \wedge \mathcal{A}_o \subset F(\mathcal{D}_F)$.

Theorem 5.9 *If* $F \in \mathcal{M}'(\mathcal{A})$ *such that* $\mathcal{D}_F$ *is an ideal of* $\mathcal{A}_o$ *and* $\mathcal{D}$ *is an ideal of* $\mathcal{A}_o$ *such that* $F^{-1}(\mathcal{D}) \neq \emptyset$, *then* $F^{-1}(\mathcal{D})$ *is also an ideal of* $\mathcal{A}_o$ *and* $F^{-1}(\mathcal{D}) \subset F^{-1}(\mathcal{D}_F)$.

Proof. If $D \in F^{-1}(\mathcal{D})$, then $D \in \mathcal{D}_F$ such that $F(D) \in \mathcal{D}$. Therefore, by Theorem 4.7, $F(D \wedge E) = F(D) \wedge E \in \mathcal{D}$, and hence $D \wedge E \in F^{-1}(\mathcal{D})$ for all $E \in \mathcal{A}_o$. Therefore, $F^{-1}(\mathcal{D})$ is also an ideal of $\mathcal{A}_o$.

Moreover, by Proposition 4.4, $F(D) = F'(D) = F(D) \wedge D \in \mathcal{D}_F$, and hence $D \in F^{-1}(\mathcal{D}_F)$. Therefore, the required inclusion is also true. □

Corollary 5.10 *If* $F \in \mathcal{M}'(\mathcal{A})$ *such that* $\mathcal{D}_F$ *is an ideal of* $\mathcal{A}_o$ *and* $F^{-1}(\mathcal{A}_o) \neq \emptyset$, *then* $F^{-1}(\mathcal{A}_o)$ *is an ideal of* $\mathcal{A}_o$ *and* $F^{-1}(\mathcal{A}_o) = F^{-1}(\mathcal{D}_F)$.

Proof. Since $\mathcal{A}_o$ is an ideal of $\mathcal{A}_o$, by Theorem 5.9 $F^{-1}(\mathcal{A}_o) \subset F^{-1}(\mathcal{D}_F)$. Moreover, since $\mathcal{D}_F \subset \mathcal{A}_o$, the converse inclusion is also true. □

Theorem 5.11 *If $\mathcal{A}$ is a semilattice and $F \in \mathcal{M}'(\mathcal{A})$ such that $\mathcal{D}_F$ is an ideal of $\mathcal{A}$, then $F(\mathcal{D}_F)$ is an ideal of $\mathcal{A}$ such that $F(\mathcal{D}_F) \subset \mathcal{D}_F$.*

Proof. In this case, by Proposition 2.4, $\mathcal{A}_o = \mathcal{A}$. Therefore, by Corollary 5.8, $F(\mathcal{D}_F)$ is an ideal of $\mathcal{A}$. Moreover, by Corollary 5.10, we have $\mathcal{D}_F = F^{-1}(\mathcal{A}) = F^{-1}(\mathcal{A}_o) = F^{-1}(\mathcal{D}_F)$. Therefore, the inclusion $F(\mathcal{D}_F) = F\big(F^{-1}(\mathcal{D}_F)\big) \subset \mathcal{D}_F$ is also true. □

Now, combining Theorem 5.11 with Corollary 4.11 (1), we can also state

Corollary 5.12 *If $\mathcal{A}$ is a semilattice and $F \in \mathcal{M}'(\mathcal{A})$ such that $\mathcal{D}_F$ is an ideal of $\mathcal{A}$, then $F = F \circ F$.*

Theorem 5.13 *If $\mathcal{A}$ is a semilattice and $F, G \in \mathcal{M}'(\mathcal{A})$ such that $\mathcal{D}_F$ and $\mathcal{D}_G$ are ideals of $\mathcal{A}$ and $F(\mathcal{D}_F) = G(\mathcal{D}_G)$, then $F(D) = G(D)$ for all $D \in \mathcal{D}_F \cap \mathcal{D}_G$.*

Proof. If $D \in \mathcal{D}_F$, then by the above condition and Theorem 5.11 it is clear that

$$F(D) \in F(\mathcal{D}_F) = G(\mathcal{D}_G) = \mathcal{D}_G \cap G(\mathcal{D}_G).$$

Therefore, by Corollary 4.12, $F(D)$ is a fixed point of G. Now, if $D \in \mathcal{D}_G$ also holds, then by Proposition 4.4 and the multiplier property of G it is clear that

$$F(D) = F'(D) = F(D) \wedge D = G\big(F(D)\big) \wedge D = G(D) \wedge F(D).$$

Moreover, quite similarly we can also see that $G(D) = F(D) \wedge G(D)$. Therefore, the required equality is true. □

From the above theorem, by Remark 5.4, it is clear that in particular we also have

Corollary 5.14 *If $\mathcal{A}$ is a semilattice and F and G are maximal members of $\mathcal{M}'(\mathcal{A})$, then $F = G$ if and only if $F(\mathcal{D}_F) = G(\mathcal{D}_G)$.*

6. Effective multipliers

Definition 6.1 *A multiplier $F \in \mathcal{M}(A)$ is called effective (supereffective) if its domain $\mathcal{D}_F$ is effective (supereffective).*

By using the corresponding definitions, we can easily prove

Theorem 6.2 *If $F \in \mathcal{M}(\mathcal{A})$ is effective, then $F \in \mathcal{M}'(\mathcal{A})$.*

Proof. If $D \in \mathcal{D}_F$, then we have

$$F'(D) \wedge E = \big(F(D) \wedge D \big) \wedge E = \big(F(D) \wedge E \big) \wedge D = \\ = \big(F(E) \wedge D \big) \wedge D = F(E) \wedge (D \wedge D) = F(E) \wedge D = F(D) \wedge E$$

for all $E \in \mathcal{D}_F$. Hence, since $\mathcal{D}_F$ is effective, it follows that $F'(D) = F(D)$. Therefore, $F' = F$, and thus by Proposition 4.4 $F \in \mathcal{M}'(\mathcal{A})$. □

The importance of effective multipliers lies mainly in the following

Theorem 6.3 *If* $F \in \mathcal{M}(\mathcal{A})$ *is effective, then*

$$F^- = \Big\{ (D, A) \in \mathcal{A}_o \times \mathcal{A} : \quad \forall\ Q \in \mathcal{D}_F : \quad A \wedge Q = F(Q) \wedge D \Big\}$$

is the unique maximal extension of F.

Proof. If $(D, A), (D, B) \in F^-$, then by the definition of F^- we have

$$A \wedge Q = F(Q) \wedge D = B \wedge Q$$

for all $Q \in \mathcal{D}_F$. Hence, since D_F is effective, it follows that $A = B$. Therefore, F^- is a function.

Now, if $D \in \mathcal{D}_{F^-}$ and $E \in \mathcal{A}_o$, then by the definition of the function F^- it is clear that

$$\big(F^-(D) \wedge E \big) \wedge Q = \big(F^-(D) \wedge Q \big) \wedge E = \big(F(Q) \wedge D \big) \wedge E = F(Q) \wedge (D \wedge E)$$

for all $Q \in \mathcal{D}_F$. Hence, by the definition of the function F^-, it follows that

$$F^-(D \wedge E) = F^-(D) \wedge E .$$

Therefore, in particular, $F^- \in \mathcal{M}(\mathcal{A})$ is also true.

Finally, if $G \in \mathcal{M}(\mathcal{A})$ such that $F \subset G$, then we evidently have

$$G(D) \wedge Q = G(Q) \wedge D = F(Q) \wedge D$$

for all $D \in \mathcal{D}_G$ and $Q \in \mathcal{D}_F$. Hence, by the definition of F^-, it follows that $G \subset F^-$. Therefore, the function F^- is the unique maximal extension of F. □

Now, by using Theorems 6.3 and 3.12, we can also easily establish the following

Theorem 6.4 *If* $\mathcal{A}$ *is an effective poset and* $\mathcal{D}$ *is a nonvoid subset of* $\mathcal{A}_o$, *then the following assertions are equivalent:*

(1) $\mathcal{D}$ *is effective;*

(2) each $F \in \mathcal{M}(A)$, *with domain* $\mathcal{D}$, *has a unique maximal extension.*

Proof. Since the implication (1)$\Rightarrow$(2) follows immediately from Theorem 6.3, we need only prove the converse implication.

For this, note that if $A, B \in \mathcal{A}$ such that $A \wedge D = B \wedge D$ for all $D \in \mathcal{D}$, then both F_A and F_B are maximal extensions of the restriction of F_A to $\mathcal{D}$. Therefore, if the assertion (2) holds, then we necessarily have $F_A = F_B$. Hence, by Theorem 3.12, it follows that $A = B$ and thus the assertion (1) also holds. □

Definition 6.5 *If $\mathcal{A}$ is an effective (supereffective) poset, then the family of all maximal and effective (resp. supereffective) members of $\mathcal{M}(A)$ is denoted by $\mathfrak{M}(\mathcal{A})$ $\big($resp. $\mathfrak{M}^{\star}(\mathcal{A})\big)$.*

Remark 6.6 Note that by Remark 2.6 and Theorem 6.2 we always have $\mathfrak{M}^{\star}(\mathcal{A}) \subset \mathfrak{M}(\mathcal{A}) \subset \mathcal{M}'(\mathcal{A})$.

Moreover, if $\mathcal{A}$ is in particular a semilattice, then by Theorem 2.8 we also have $\mathfrak{M}^{\star}(\mathcal{A}) = \mathfrak{M}(\mathcal{A})$.

Concerning the equality in $\mathfrak{M}(\mathcal{A})$ we can easily establish the next useful

Theorem 6.7 *If $F, G \in \mathfrak{M}(\mathcal{A})$, then the following assertions are equivalent:*

(1) $F = G$;

(2) $F(D) = G(D)$ *for all* $D \in \mathcal{D}_F \cap \mathcal{D}_G$;

(3) there exists an effective subset $\mathcal{D}$ of $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$ and $F(D) = G(D)$ for all $D \in \mathcal{D}$.

Proof. By Theorem 5.3 and Corollary 2.16, $\mathcal{D}_F \cap \mathcal{D}_G$ is effective. Therefore, the implication (2)$\Rightarrow$(3) is true.

On the other hand, if the assertion (3) holds, then by Theorem 6.4 it is clear that the assertion (1) also holds. □

7. Inequalities for effective multipliers

Because of Theorem 6.7, we may naturally introduce the following weakening of Definition 3.10.

Definition 7.1 *If $F, G \in \mathfrak{M}(\mathcal{A})$ such that there exists an effective subset $\mathcal{D}$ of $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$ and $F(D) \leq G(D)$ for all $D \in \mathcal{D}$, then we can write $F \leq G$.*

However, in contrast to Theorem 6.7, now we can only prove the following

Proposition 7.2 *If $F, G \in \mathfrak{M}(\mathcal{A})$, then the following assertions are equivalent:*

(1) $F \leq G$;

(2) there exists an effective subset $\mathcal{Q}$ of $\mathcal{A}$ such that $\mathcal{Q}$ is an ideal of $\mathcal{A}_o$, $\mathcal{Q} \subset \mathcal{D}_F \cap \mathcal{D}_G$ and $F(Q) \leq G(Q)$ for all $Q \in \mathcal{Q}$.

Proof. If the assertion (1) holds, then there exists an effective subset $\mathcal{D}$ of $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$ and $F(D) \leq G(D)$ for all $D \in \mathcal{D}$. Define $\mathcal{Q} = \mathcal{D} \wedge \mathcal{D}_G$. Then, by Theorem 2.15, it is clear that $\mathcal{Q}$ is an effective subset of $\mathcal{A}$. Moreover, by Theorem 5.3, it is clear that $\mathcal{Q}$ is an ideal of $\mathcal{A}_o$ such that $\mathcal{Q} \subset \mathcal{D}_F \cap \mathcal{D}_G$.

On the other hand, by Theorems 5.3 and 4.7, it is clear that

$$F(D \wedge E) = F(D) \wedge E \leq G(D) \wedge E = G(D \wedge E)$$

for all $D \in \mathcal{D}$ and $E \in \mathcal{D}_G$. Therefore, we have $F(Q) \leq G(Q)$ for all $Q \in \mathcal{Q}$, and thus the assertion (2) also holds. □

Now, by using the above proposition, we can also easily prove the following

Theorem 7.3 *If $\mathcal{A}$ is an effective poset, then $\mathfrak{M}(\mathcal{A})$, together with the inequality introduced in Definition 7.1, is a supereffective poset, with the largest element $\Delta_{\mathcal{A}_o}$, such that the mapping $A \mapsto F_A$ is a nondecreasing injection of $\mathcal{A}$ into $\mathfrak{M}(\mathcal{A})$.*

Proof. It is clear that the inequality introduced in Definition 7.1 is reflexive on $\mathfrak{M}(\mathcal{A})$.

Moreover, if $F, G \in \mathfrak{M}(\mathcal{A})$ such that $F \leq G$ and $G \leq F$, then by Proposition 7.2 there exist effective subsets $\mathcal{D}$ and $\mathcal{E}$ of $\mathcal{A}$ such that $\mathcal{D}$ and $\mathcal{E}$ are ideals of $\mathcal{A}_o$, $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$ and $\mathcal{E} \subset \mathcal{D}_G \cap \mathcal{D}_F$, and moreover $F(D) \leq G(D)$ for all $D \in \mathcal{D}$ and $G(E) \leq F(E)$ for all $E \in \mathcal{E}$. Define $\mathcal{Q} = \mathcal{D} \cap \mathcal{E}$. Then, by Corollary 2.16, it is clear that $\mathcal{Q}$ is an effective subset of $\mathcal{A}$. Moreover, it is clear that $\mathcal{Q} \subset \mathcal{D}_F \cap \mathcal{D}_G$ and $F(Q) = G(Q)$ for all $Q \in \mathcal{Q}$. Therefore, by Theorem 6.7, $F = G$ is also true.

Quite similarly, if $F, G, H \in \mathfrak{M}(\mathcal{A})$ such that $F \leq G$ and $G \leq H$, then there exist effective subsets $\mathcal{D}$ and $\mathcal{E}$ of $\mathcal{A}$ such that $\mathcal{D}$ and $\mathcal{E}$ are ideals of $\mathcal{A}_o$, $\mathcal{D} \subset \mathcal{D}_F \cap \mathcal{D}_G$ and $\mathcal{E} \subset \mathcal{D}_G \cap \mathcal{D}_H$, and moreover $F(D) \leq G(D)$ for all $D \in \mathcal{D}$ and $G(E) \leq H(E)$ for all $E \in \mathcal{E}$. Hence, by defining $\mathcal{Q} = \mathcal{D} \cap \mathcal{E}$, we can at once see that $\mathcal{Q}$ is an effective subset of $\mathcal{A}$ such that $\mathcal{Q} \subset \mathcal{D}_F \cap \mathcal{D}_H$ and $F(Q) \leq H(Q)$ for all $Q \in \mathcal{Q}$. Therefore, $F \leq H$ is also true.

Since $\mathcal{A}$ is effective, by Proposition 3.3 it is clear that $F_A \in \mathfrak{M}(\mathcal{A})$ for all $A \in \mathcal{A}$. Therefore, by Theorems 3.11 and 3.12, the mapping $A \mapsto F_A$ is a nondecreasing injection of $\mathcal{A}$ into $\mathfrak{M}(\mathcal{A})$.

Moreover, we can also at once see that the identity function $\Delta_{\mathcal{A}_o}$ of $\mathcal{A}_o$ is also a member of $\mathfrak{M}(\mathcal{A})$. Furthermore, if $F \in \mathfrak{M}(\mathcal{A})$, then by Theorem 6.2 we have $F(D) \leq \Delta_{\mathcal{A}_o}(D)$ for all $D \in \mathcal{D}_F$. Therefore, $F \leq \Delta_{\mathcal{A}_o}$. Thus, $\Delta_{\mathcal{A}_o}$ is the greatest element of $\mathfrak{M}(\mathcal{A})$. Hence, by Corollary 2.14, it is clear that the poset $\mathfrak{M}(\mathcal{A})$ is supereffective. □

For supereffective multipliers, we may naturally introduce a stronger inequality.

Definition 7.4 *If* F, $G \in \mathfrak{M}^\star(\mathcal{A})$ *such that there exists a supereffective subset* $\mathcal{D}$ *of* $\mathcal{A}$ *such that* $D \subset \mathcal{D}_F \cap \mathcal{D}_G$ *and* $F(D) \leq G(D)$ *for all* $D \in \mathcal{D}$, *then we can write* $F \leq G$.

Remark 7.5 Note if in particular $\mathcal{A}$ is a semilattice, then by Theorem 2.8 $\mathfrak{M}^\star(\mathcal{A}) = \mathfrak{M}(\mathcal{A})$ and the corresponding inequalities coincide.

Therefore, in the case of a semilattice $\mathcal{A}$, the above definition is not needed. However, the following observations are not, even in this case, superfluous.

Proposition 7.6 *If* F, $G \in \mathfrak{M}^\star(\mathcal{A})$, *then the following assertions are equivalent:*

(1) $F \leq G$;

(2) $F(D) \leq G(D)$ *for all* $D \in \mathcal{D}_F \cap \mathcal{D}_G$;

Proof. If the assertion (1) holds, then there exists a supereffective subset $\mathcal{E}$ of $\mathcal{A}$ such that $\mathcal{E} \subset \mathcal{D}_F \cap \mathcal{D}_G$ and $F(E) \leq G(E)$ for all $E \in \mathcal{E}$. Therefore, if $D \in \mathcal{D}_F \cap \mathcal{D}_G$, then we have

$$F(D) \wedge E = F(E) \wedge D \leq G(E) \wedge D = G(D) \wedge E$$

for all $E \in \mathcal{E}$. Hence, since $\mathcal{E}$ is supereffective, it follows that $F(D) \leq G(D)$. Therefore, the assertion (2) also holds.

Moreover, by Theorem 5.3 and Corollary 2.16, it is clear that $\mathcal{D}_F \cap \mathcal{D}_G$ is a supereffective subset of $\mathcal{A}$. Therefore, the implication (2)$\Rightarrow$(1) is also true. □

Now, analogously to Theorem 7.3, we can also easily establish the following

Theorem 7.7 *If* $\mathcal{A}$ *is a supereffective poset, then* $\mathfrak{M}^\star(\mathcal{A})$, *together with the inequality introduced in Definition 7.4, is a supereffective poset, with the largest element* $\Delta_{\mathcal{A}_o}$, *such that the mapping* $A \mapsto F_A$ *is an order-isomorphism of* $\mathcal{A}$ *into* $\mathfrak{M}^\star(\mathcal{A})$.

Remark 7.8 Therefore, if $\mathcal{A}$ is a supereffective poset, then by identifying each $A \in \mathcal{A}$ with F_A the poset $\mathfrak{M}^\star(\mathcal{A})$ becomes a natural extension of $\mathcal{A}$.

However, it seems not to be an easy task to determine the meet operation and the centre of the poset $\mathfrak{M}^\star(\mathcal{A})$. Some partial results will be establised elsewhere.

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