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A NOTE ON SQUARE EXTENSIONS OF BANDS

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Abstract. The construction of (associative) square extension of an idempotent groupoid (semigroup) was recently introduced by A. W. Marczak and J. Płonka. They proved that all square extensions of a variety of idempotent groupoids also form a variety. In this note we explicitely describe the free objects in semigroup varieties obtained from band varieties by means of associative square extensions, which is done by solving the corresponding word problems. As a consequence, we calculate the free spectra and the p_n-sequences of the considered varieties.

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In their recent paper [6], Marczak and Płonka defined a new algebraic construction on idempotent groupoids, called *square extension*, in order to study clone-theoretical properties of certain groupoids. Namely, let $(I, *)$ be an idempotent groupoid. Assume we have given a family $\{A_i : \ i \in I\}$ of disjoint sets, and for each $i, j \in I$, a mapping $h_{ij} : A_i \to A_{i*j}$ such that the following conditions hold:

(1) $i \in A_i$ for all $i \in I$,

(2) $h_{ij}(i) = i * j$ for all $i, j \in I$,

(3) $h_{ii}(a) = i$, whenever $a \in A_i$.

Then we define the square extension of I to be a groupoid $(A, \cdot)$ such that $A = \bigcup_{i\in I} A_i$, with multiplication defined for all $a \in A_i$ and $b \in A_j$ by

$$ab = h_{ij}(a).$$

We immediately note that for all $b \in A_j$ we have $ab = aj$. Also, for all $i, j \in I$ we have $ij = i * j$, so that I is actually a subgroupoid of A.

However, if I is an idempotent semgiroup (a band), its arbitrary square extension need not to be associative. Therefore, the authors of [6] define the *associative square extension* of a band, by requiring that the square extension preserves associativity, which turned out to be equivalent with the condition

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(4) $h_{i*j,k}(h_{i,j}(x)) = h_{i,j*k}(x)$ for all $i, j, k \in I$ and $x \in A_i$.

Since we deal exclusively with semigroups in this note, we shall omit the 'associative' adjective in the sequel, and consider the above condition as a part of the definition of the square extension.

Further, it was proved in Theorem 4.2 of [6] that the square extension is in fact a varietal construction. Namely, if $\mathcal{V}$ is a variety of bands, then all square extensions of members of $\mathcal{V}$ form a variety, which we denote by $\mathcal{V}^{\mathrm{sq}}$. Moreover, as shown in [6], there is a strong connection between the equational base for $\mathcal{V}$ and $\mathcal{V}^{\mathrm{sq}}$. After some simplification, we have the following result.

Theorem 1. ([6]) *The variety $\mathcal{V}^{\mathrm{sq}}$ is defined (within the class of all semigroups) by the identities $(xy)^2 = x^2y$, $xy^2 = xy$, and $u^2 = v^2$, for all identities $u = v$ from a set $E_{\mathcal{V}}$ defining $\mathcal{V}$ within the variety $\mathcal{B}$ of all bands.*

It is known that $E_{\mathcal{V}}$ can be chosen to be a singleton, cf. [2].

Our main objective here is to solve the (free) word problem for varieties of the form $\mathcal{V}^{\mathrm{sq}}$, i.e. to give a full (algorithmic) description of their equational theories, which enables us to analyze the structure of their free algebras. As a consequence, we shall derive a nice formula for the free spectrum of such varieties (the sequence of cardinalities of finitely generated free algebras). To this end, we consider first the square extensions of rectanguar bands, and then varieties $\mathcal{V}^{\mathrm{sq}}$ such that $\mathcal{V}$ contains $\mathcal{SL}$, the variety of semilattices.

§1. Before considering square extensions of rectangular bands, we are going to take a look at the probably simplest examples of square extensions, the inflations of bands. In general an *inflation* of a semigroup S is a retractive ideal extension of S by a null semigroup N. If V is such an extension, this means that we have a mapping $\xi : V \to S$ such that $\xi^2 = \xi$ (i.e. ξ is a retraction) and ξ is an ideal homomorphism, so that for all $a, b \in V$ we have

$$\xi(ab) = a\xi(b) = \xi(a)b = \xi(a)\xi(b).$$

If we want to construct an inflation from the given components, we need to consider the set $N^* = N \setminus \{0\}$ and an arbitrary mapping $\varphi : N^* \to S$. The multiplication in $V = N^* \cup S$ is defined so that for all $x \in N$, $a \in S$ we have $xa = \varphi(x)a$ and $ax = a\varphi(x)$, cf. [9].

Now, in a square extension A of a band I, it is easy to see that the mapping $\xi : A \to I$ given by $\xi(a) = a^2$ is a retraction, but its image I is not always an ideal of A, only a right ideal, since we have $\xi(a)b = a^2b = (ab)^2 = \xi(ab)$. A characterization of inflations within square extensions is given in the following result.

Proposition 2. *Let A be a square extension of a band I. Then A is an inflation of I if and only if all mappings h_{ij} are constant, $h_{ij}(x) = i * j$. Moreover, every* *inflation V of I can be obtained as a square extension of I with the mappings h_{ij} as above and $A_i = \xi^{-1}(i)$, where $\xi : V \to I$ is the corresponding retractive ideal homomorphism.*

Proof. Assume A is an inflation of I with the corresponding inflation homomorphism $\xi : A \to I$. Then for all $a \in A_i$ we have

$$\xi(a) = (\xi(a))^2 = \xi(a^2) = \xi(i) = i = a^2.$$

It follows that $h_{ij}(a) = aj = \xi(a)j = ij = i * j$. Conversely, if A is a square extension of I with $h_{ij}(a) = i * j$ for all $a \in A_i$, then $\xi(a) = a^2$ is easily seen to be a retracitve ideal homomorphism. Finally, for the last statement of the proposition, it is straightforward to see that the given parameters h_{ij} and A_i indeed define a square extension of I, provided A is an inflation of the band I. □

Therefore, not every square extension is an inflation, as the following example shows.

Example 3. Let Σ_3 be the three-element semilattice consisting of elements $0, 1, 2$ such that $0 < 1$ and $0 < 2$, while 1 and 2 are incomparable. Now add a fourth element 3 and define $A_0 = \{0, 3\}$, $A_1 = \{1\}$ and $A_2 = \{2\}$. Further, let $h_{0,1} : A_0 \to A_0$ be the identity mapping on A_0 and $h_{0,2} : A_0 \to A_0$ be the constant zero mapping (all the rest of the funcions h_{ij}, $i, j \in \{0, 1, 2\}$, are defined in accordance with the square extension rules). Then it is not difficult to see that we obtained a square extension of Σ_3 which is not an inflation of the latter semilattice.

Let us now turn to square extensions of rectangular bands. The result below shows that examples of the above kind are actually impossible for such extensions.

Theorem 4. *Every square extension A of a rectangular band I is an inflation of I.*

Proof. Let $\mathcal{RB}$ denote the variety of rectangular bands. By Theorem 1, the variety $\mathcal{RB}^{\mathrm{sq}}$ is defined by $(xy)^2 = x^2y$, $xy^2 = xy$ and $(xyx)^2 = x^2$. From the second identity we have $x^3 = x^2$. Thus, we obtain

$$xy = xy^2 = x(yxy)^2 = xyxy^2xy = xyxyxy = (xy)^3 = (xy)^2,$$

implying $xy = x^2y$, which suffices to see that the retraction $\xi(x) = x^2$ is an ideal homomorphism in every member of $\mathcal{RB}^{\mathrm{sq}}$, as required. □

It is known (e.g. from [3, 7]) that all inflations of semigroups belonging to a variety $\mathcal{V}$ also form a variety, which is just equal to $\mathcal{V} \vee \mathcal{N}$, where $\mathcal{N}$ stands for the variety of null semigroups. Since $\mathcal{RB}$ has four subvarieties (the proper ones are $\mathcal{LZ}$, left zero bands, and $\mathcal{RZ}$, right zero bands), from the cited papers it also follows that $\mathcal{RB} \vee \mathcal{N}$ has eight subvarieties, and their inclusion diagram is the three-dimensional cube. Petrich in [8] gives equational bases for $\mathcal{RB} \vee \mathcal{N}$, $\mathcal{LZ} \vee \mathcal{N}$ and $\mathcal{RZ} \vee \mathcal{N}$, so that it is not difficult to conclude that in all of these three varieties we have $f_1 = 2$, and for all $n \geq 2$, $f_n(\mathcal{RB}^{\mathrm{sq}}) = n^2$ and $f_n(\mathcal{LZ}^{\mathrm{sq}}) = f_n(\mathcal{RZ}^{\mathrm{sq}}) = n$ (where $f_n(\mathcal{V})$ refers to the free spectrum of $\mathcal{V}$, the size of n-generated $\mathcal{V}$-free semigroup).

§2. In the rest of the paper we consider varieties of the form $\mathcal{V}^{\mathrm{sq}}$, where $\mathcal{V}$ is a variety of bands containing $\mathcal{SL}$, the variety of semilattices. Example 3 above shows that in such a case, $\mathcal{V}^{\mathrm{sq}}$ does not consist entirely of inflations. Also, every identity $u = v$ which may hold in $\mathcal{V}^{\mathrm{sq}}$ (and thus in $\mathcal{V}$) is necessarily *homotypical*, meaning that the same variables occur both in u and v.

Let w be any (semigroup) word and $\mathcal{V}$ a semigroup variety. We say that w is *idempotent in* $\mathcal{V}$ if $\mathcal{V}$ satisfies the identity $w^2 = w$. Further, let $\ell(w)$ denote the *head* of w, the leftmost variable of w.

Lemma 5. *Let $\mathcal{V}$ be a variety of bands, $\mathcal{SL} \subseteq \mathcal{V}$. A word w is idempotent in $\mathcal{V}^{\mathrm{sq}}$ if and only if $\ell(w)$ occurs in w at least twice.*

Proof. ($\Leftarrow$) Denote $a = \ell(w)$ for short. Then $w = aw_1aw_2$, where the words w_1, w_2 may be empty. Using the square extension identities, we have

$$w^2 = (aw_1aw_2)^2 = (aw_1)^2aw_2 = a(w_1a)^2w_2 = aw_1aw_2 = w.$$

($\Rightarrow$) Let w be a word in which $a = \ell(w)$ occurs only once, so that we have $w = au$, with a not occurring in u, and suppose w is idempotent in $\mathcal{V}^{\mathrm{sq}}$. Then $a^2u = (au)^2 = au$. Since $\mathcal{SL} \subseteq \mathcal{V}$, the four-element square extension A of Σ_3 from Example 3 belongs to $\mathcal{V}^{\mathrm{sq}}$. Consider a valuation in A such that a is set to 3, while all the other variables (from which the word u is constructed) have value 1. Then u evaluates to 1, while a^2 is 0. Consequently, a^2u also evaluates to zero, while au becomes $3 \cdot 1 = h_{01}(3) = 3$. A contradiction. □

We turn to the solution of the word problem of a variety of the form $\mathcal{V}^{\mathrm{sq}}$.

Lemma 6. *Let $\mathcal{V}$ be a variety of bands, $\mathcal{SL} \subseteq \mathcal{V}$, and let u, v be two idempotent words in $\mathcal{V}^{\mathrm{sq}}$. The identity $u = v$ holds in $\mathcal{V}^{\mathrm{sq}}$ if and only if it holds in $\mathcal{V}$.*

Proof. Assume that $u = v$ holds in $\mathcal{V}$. Then $u^2 = v^2$ is satisfied in $\mathcal{V}^{\mathrm{sq}}$, because for each semigroup word $w(x_1, \ldots, x_n)$, as a consequence of $(xy)^2 = x^2y = x^2y^2$, we have $w^2 = w(x_1^2, \ldots, x_n^2)$. However, by the given conditions, $\mathcal{V}^{\mathrm{sq}}$ satisfies $u^2 = u$ and $v^2 = v$. Hence, $u = v$ is true in $\mathcal{V}^{\mathrm{sq}}$. The converse implication is obvious, as $\mathcal{V}$ is a subvariety of $\mathcal{V}^{\mathrm{sq}}$. □

We now introduce a construction which will allow us to prove our main theorem. Let X be any set, and let $F_{\mathcal{V}}(X)$ denote the $\mathcal{V}$-free band freely generated by X. For a word w, let $\overline{w}$ denote the element of $F_{\mathcal{V}}(X)$ represented by w, i.e. the class of all words (over X) $\mathcal{V}$-equivalent to w. We define the following sets and functions:

(1) $A_{\overline{u}} = \{\overline{u}\} \cup \{(a, \overline{w}) : \overline{w} \in F \cup \{\lambda\},\ \overline{aw} = \overline{u},\ a \text{ does not occur in } w\}$, where λ denotes the empty word,

(2) for each $\overline{u}, \overline{v} \in F$, we define $h_{\overline{u},\overline{v}} : A_{\overline{u}} \to A_{\overline{uv}}$ given by

$$h_{\overline{u},\overline{v}}(x) = \begin{cases} \overline{uv}, & x = \overline{u}, \\ \overline{awv}, & x = (a, \overline{w}),\ a \text{ occurs in } v, \\ (a, \overline{wv}), & x = (a, \overline{w}),\ a \text{ does not occur in } v. \end{cases}$$

It is indeed a routine to check that the above parameters determine a square extension of the $\mathcal{V}$-free band on X. This extension we denote by $A(\mathcal{V}, X)$.

Lemma 7. *Let $\mathcal{V}$ be a variety of bands, $\mathcal{SL} \subseteq \mathcal{V}$, let u, v be two arbitrary words, and let a be a variable not occurring in u and v. Then $\mathcal{V}^{\mathrm{sq}}$ satisfies $au = av$ if and only if $\mathcal{V}$ satisfies $u = v$.*

Proof. ($\Leftarrow$) As in the previous lemma, if $u = v$ holds in $\mathcal{V}$, then $u^2 = v^2$ holds in $\mathcal{V}^{\mathrm{sq}}$. Therefore, $\mathcal{V}^{\mathrm{sq}}$ satisfies $au = au^2 = av^2 = av$, as desired.

($\Rightarrow$) Assume that $au = av$ holds in $\mathcal{V}^{\mathrm{sq}}$, and let X be the set of all variables occurring in au (or av, which is the same, since $\mathcal{SL} \subseteq \mathcal{V}$). Then the identity $au = av$ holds in $A(\mathcal{V}, X) \in \mathcal{V}^{\mathrm{sq}}$. Choose a valuation of variables from X in the semigroup $A(\mathcal{V}, X)$ so that a becomes (a, λ), while every variable $x \in X \setminus \{a\}$ is replaced by $\overline{x}$ (in fact, by saying that $F_{\mathcal{V}}(X)$ is freely generated by X, we identify x with $\overline{x}$ for all $x \in X$). Then au evalutes, by the above definition, to $(a, \overline{u})$, while av becomes $(a, \overline{v})$, since a does not occur in u and v. It follows that $\overline{u} = \overline{v}$, and this can happen only if the identity $u = v$ holds in $\mathcal{V}$. □

Lemma 8. *If a, b are different variables, u a word not containing a, and v a word not containing b, then the identity $au = bv$ cannot be true in a variety of the form $\mathcal{V}^{\mathrm{sq}}$, where $\mathcal{SL} \subseteq \mathcal{V}$.*

Proof. Assume to the contrary. Since $\mathcal{SL} \subseteq \mathcal{V}$, the identity $au = bv$ must be homotypical, so that it actually has the form $au_1bu_2 = bv_2av_2$, where some of the words u_1, u_2, v_1, v_2 can be empty. Replace $a \to xz$, $b \to yz$, and all the other variables by z. We obtain $xzyz = yzxz$, implying

$$xyx = x(yx)^2 = (xy)^2x = x^2yx = (yx)x^2 = yx^3 = yx,$$

which, in turn, gives $xyz = yxz$. It follows that $xyz = yxz$ holds in $\mathcal{V}^{\mathrm{sq}}$, and thus in $\mathcal{SL}^{\mathrm{sq}}$. The latter variety also satisfies $xyz = x(yz)^2 = x(zy)^2 = xzy$, yielding

$$xy = yxy = y^2x = (yx)^2 = (xy)^2 = x^2y = xyx = yx,$$

which leads to a contradiction, because Example 3 provides a noncommutative square extension of a semilattice ($3 \cdot 1 = h_{01}(3) = 3$, while $1 \cdot 3 = h_{10}(1) = 0$). □

Now we are ready to prove the main result of the paper, which describes the structure of $\mathcal{V}^{\mathrm{sq}}$-free semigroups in terms of $\mathcal{V}$-free bands and square extensions.

Theorem 9. *$A(\mathcal{V}, X)$ is isomorphic to $F_{\mathcal{V}^{\mathrm{sq}}}(X)$, the $\mathcal{V}^{\mathrm{sq}}$-free semigroup on the set X.*

Proof. We can represent $F_{\mathcal{V}^{\mathrm{sq}}}(X)$ so that it consists of elements $\widetilde{w}$, where w is a word over X, with $\widetilde{w}$ denoting the class of all words over X which are $\mathcal{V}^{\mathrm{sq}}$-equivalent to w. Consider the mapping $\psi : F_{\mathcal{V}^{\mathrm{sq}}}(X) \to A(\mathcal{V}, X)$ defined by

$$\psi(\widetilde{w}) = \begin{cases} \overline{w}, & \ell(w) \text{ occurs in } w \text{ at least twice,} \\ (a, \overline{u}), & w = au, \ a \text{ does not occur in } u. \end{cases}$$

Note that the content of the previous four lemmata exactly amounts to saying that ψ is well-defined and injective. Since ψ is obviously surjective, it remains to show that it is a homomorphism.

So, let w_1, w_2 be any words over X. If $\ell(w_1)$ occurs in w_1 at least twice, or if it occurs in w_2, then

$$\psi(\widetilde{w}_1\widetilde{w}_2) = \psi(\widetilde{w_1w_2}) = \overline{w_1w_2}.$$

On the other hand, if $\ell(w_1)$ occurs in w_1 at least twice, then $\psi(\widetilde{w}_1) = \overline{w}_1$, so that

$$\psi(\widetilde{w}_1)\psi(\widetilde{w}_2) = \overline{w}_1\psi(\widetilde{w}_2) = \overline{w}_1(\psi(\widetilde{w}_2))^2 = \overline{w}_1\overline{w}_2 = \overline{w_1w_2},$$

since by the definition of ψ, $(\psi(\widetilde{w}))^2 = \overline{w}$. If, however, $w_1 = au$ such that a occurs in w_2, but not in u, then, using some of the above arguments, we have

$$\psi(\widetilde{w}_1)\psi(\widetilde{w}_2) = (a, \overline{u})\psi(\widetilde{w}_2) = (a, \overline{u})(\psi(\widetilde{w}_2))^2 =$$

$$= (a, \overline{u})\overline{w}_2 = h_{\overline{w}_1, \overline{w}_2}(a, \overline{u}) = \overline{auw_2} = \overline{w_1w_2}.$$

The other case is when $w_1 = au$ and a does not occur in u and w_2. Then $\psi(\widetilde{w}_2\widetilde{w}_2) = (a, \overline{uw_2})$, while

$$\psi(\widetilde{w}_1)\psi(\widetilde{w}_2) = (a, \overline{u})\psi(\widetilde{w}_2) = (a, \overline{u})(\psi(\widetilde{w}_2))^2 =$$

$$= (a, \overline{u})\overline{w}_2 = h_{\overline{w}_1, \overline{w}_2}(a, \overline{u}) = (a, \overline{uw_2}),$$

establishing that ψ is indeed a homomorphic mapping. □

Immediately from the above theorem and the item (1) of the definition of $A(\mathcal{V}, X)$, we have the following conclusion.

Corollary 10. *Let $\mathcal{V}$ be a variety of bands such that $\mathcal{SL} \subseteq \mathcal{V}$. Denote $f'_n = |F_{\mathcal{V}^{\mathrm{sq}}}(n)|$ and $f_n = |F_{\mathcal{V}}(n)|$. Then $f'_0 = 0$ and for all $n \geq 1$,*

$$f'_n = f_n + n(f_{n-1} + 1).$$

Therefore, all varieties of the form $\mathcal{V}^{\mathrm{sq}}$ are locally finite.

For example, if $\mathcal{V} = \mathcal{SL}$ ($\mathcal{SL}^{\mathrm{sq}}$ is just the variety $\mathcal{K}_1$ from [6]), then $f_n = 2^n - 1$, and so $f'_n = (n+2)2^{n-1} - 1$.

Finally, recall that for an algebra A, $p_n(A)$ denotes the number of all n-ary term operations of A which depend on all of its variables for $n \geq 1$ (while $p_0(A)$ is the number of constant unary term operations of A). Furthermore, the p_n-sequence of a variety $\mathcal{V}$ is just the p_n-sequence of any algebra generating $\mathcal{V}$, e.g. the $\mathcal{V}$-free algebra over a countable set of free generators. For some background concerning the theory of p_n-sequences we refer to [4]. Now, since we are working in this section with semigroup varieties that contain the variety $\mathcal{SL}$ of semilattices, the determination of the p_n-sequence of such a variety $\mathcal{V}$ comes down to counting elements of $F_{\mathcal{V}}(n)$ represented by words which include occurences of all the generators. Hence, from our Theorem 9 we obtain

Corollary 11. *Let $\mathcal{V}$ be a variety of bands such that $\mathcal{SL} \subseteq \mathcal{V}$. Denote $p'_n = p_n(\mathcal{V}^{\mathrm{sq}})$ and $p_n = p_n(\mathcal{V})$. Then $p'_0 = 0$, $p'_1 = 2$ and for all $n \geq 2$,*

$$p'_n = p_n + np_{n-1}.$$

For example, we have $p_n(\mathcal{SL}^{\mathrm{sq}}) = n + 1$ for all $n \geq 1$, as indicated in [5, 6]. On the other hand, for $\mathcal{V} = \mathcal{NB}$, the variety of normal bands, we have $p_n = n^2$ (moreover, $\mathcal{NB}$ is uniquely determined by the sequence of square numbers, see [1]), so that

$$p_n(\mathcal{NB}^{\mathrm{sq}}) = n^2 + n(n-1)^2 = n(n^2 - n + 1),$$

for all $n \geq 2$. Finally, if $\mathcal{V}$ is the variety $\mathcal{LNB}$ of left normal bands (or dually, the variety $\mathcal{RNB}$ of right normal bands), we have $p_n = n$, and so

$$p_n(\mathcal{LNB}^{\mathrm{sq}}) = n + n(n-1) = n^2,$$

for all $n \geq 2$. Therefore, the p_n-sequences of $\mathcal{LNB}^{\mathrm{sq}}$ and $\mathcal{NB}$ differ *only* in the first term ($p_1(\mathcal{LNB}^{\mathrm{sq}}) = 2$, while $p_1(\mathcal{NB}) = 1$).

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