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VARIATION OF AN ELEMENT IN THE MATRIX OF THE FIRST DIFFERENCE OPERATOR AND MATRIX TRANSFORMATIONS

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Abstract. In this paper we deal with some new properties of the operator of first difference represented by the infinite matrix Δ. We study the operator represented by the perturbed matrix $\Delta'_{pq}(a')$ obtained from Δ by changing one element. Then we give necessary and sufficient conditions for a matrix A to map $s_\alpha\left(\left(\Delta'_{pq}(a')\right)^\mu\right)$ into s_β, μ being an integer.

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1. Introduction

We are interested in the study of the first difference operator. This one can be represented by the infinite matrix Δ. Many authors have given results on this last operator, see for instance Malkowsky [8],[9], Kizmaz [2], Çolak and Et [1] and more recently de Malafosse [6]. These authors gave many characterizations of the operators A mapping the space $(\Delta^\mu)^{-1}(l^\infty)$ into l^∞, that is $A \in (l^\infty(\Delta^\mu), l^\infty)$. Malkowsky [8], [9] and Malkowsky and Parashar [7] found new Schauder bases in the spaces $c_0(\Delta^\mu)$ and $c(\Delta^\mu)$. They gave many results concerning AK and BK spaces considering the Λ-strongly null and Λ-strongly convergent sequences and have studied extensions of some results given by Wilansky [12]. Note that many authors have dealt with the Cesàro operator and there is a simple relation between this operator and the operator represented by Δ. Recall that the spectrum of the Cesàro operator C_1 in certain spaces has been studied by Reade [11], Okutoyi [10] and de Malafosse [5]. Here are recalled some properties of Δ considered as an operator from the space s_α into itself. Further, as in [5], we deal with matrix perturbation and consider the new matrix $\Delta'_{pq}(a')$ obtained from Δ by changing only one element in the p-th row and in the q-th column of the infinite matrix and deduce some results on the spaces $s_\alpha\left(\left(\Delta'_{pq}(a')\right)^\mu\right)$. Then we deal with matrix transformations between matrix domains such as $s_\alpha\left(\left(\Delta'_{pq}(a')\right)^\mu\right)$ or s_β.

The paper is organized as follows. In the second section we recall some results and definitions concerning the infinite matrix theory. In the third section some properties of the spaces $s_r(\Delta^\mu)$, $s_r\left((\Delta^+)^\mu\right)$ and $s_1(\Delta^\mu)$ are given. Next, we

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assert some results concerning the operators C_1, Δ, Δ^+ and Σ in relation to s_r. In the fourth section Δ is replaced by $\Delta'_{pq}(a')$, (the matrix obtained from Δ by replacing the coefficient a_{pq} by a') and we study the new equation $\Delta'_{pq}(a')X = B$.Then, under some conditions, we characterize the matrix transformations mapping $s_\alpha\left(\Delta'_{pq}(a')\right)$ into s_β. Further, we give an upper bound of the distance $\|X_{pq}(a') - Z\|_{s_1}$, where Z is the solution of $\Delta X = B$ and $X_{pq}(a')$ the solution of equation $\Delta'_{pq}(a')X = B$, whenever it exists. In the final section, we deal with matrix transformations lying in the set $\left(s_\alpha\left(\left(\Delta'_{pq}(a')\right)^\mu\right), s_\beta\right)$, μ being a given integer.

2. Notations and preliminary results

In the following, we shall consider infinite linear systems defined by

$$\sum_{m=1}^{\infty} a_{nm}x_m = b_n \quad n = 1, 2, ...$$

Such a system can be written as a matrix equation $AX = B$, where $A = (a_{nm})_{n,m\geq 1}$ and X, B are the one column matrices defined respectively by $(x_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$. The following spaces have been defined, for instance, in [3] and [5]. For a sequence $\alpha = (\alpha_n)_{n\geq 1}$, where $\alpha_n > 0$ for every $n \geq 1$, we consider the Banach algebra

$$S_\alpha = \left\{ A = (a_{nm})_{n,m\geq 1} \ / \ \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\alpha_n}\right) < \infty \right\}, \tag{1}$$

normed by

$$\|A\|_{S_\alpha} = \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\alpha_n}\right). \tag{2}$$

S_α admits a unit element $I = (\delta_{nm})_{n,m}$ (δ_{nm} being equal to 1 if $n = m$ and equal to 0 otherwise). Denote by s the set of all sequences. We also define the Banach space s_α of one-row matrices by

$$s_\alpha = \left\{ X = (x_n)_n \in s \ / \ \sup_{n\geq 1}\left(\frac{|x_n|}{\alpha_n}\right) < \infty \right\}, \tag{3}$$

normed by

$$\|X\|_{s_\alpha} = \sup_{n\geq 1}\left(\frac{|x_n|}{\alpha_n}\right). \tag{4}$$

We shall say that the sequence $X = (x_n)_n$ belongs to Γ if

$$\overline{\lim_{n\to\infty}}\left(\left|\frac{x_{n-1}}{x_n}\right|\right) < 1.$$

For any subset E of s, we put

$$AE = \{Y \in s \;/\; \exists X \in E \quad Y = AX\}. \tag{5}$$

If F is a subset of s, we shall denote

$$F(A) = F_A = \{X \in s \;/\; Y = AX \in F\}. \tag{6}$$

We can see that $F(A) = A^{-1}F$. If A maps E into F, we write that $A \in (E, F)$. It is well-known that $A \in (s_1, s_1)$ if and only if $A \in S_1$ (see [4]).

For any sequence $\zeta = (\zeta_n)_{n \geq 1}$, we shall put $D_\zeta = (\zeta_n \delta_{nm})_{n,m \geq 1}$.

A being an infinite matrix, let us define the matrix $A\langle t\rangle = (a'_{nm})_{n,m\geq 1}$, (with $a'_{nn} \neq 0\ \forall n$) obtained from A by addition of the row $t = (t_m)_{m \geq 1}$. In the same way, set for any scalar u: ${}^tB\langle u\rangle = (u, b_1, b_2, ...)$. Then we have the following result given in [3], in which $a^* = (1/a'_{nn})_{n\geq 1}$:

Proposition 1 *If $\parallel I - D_{a^*}A\langle t\rangle \parallel_{S_\alpha} < 1$ and $D_{a^*}B\langle u\rangle \in s_\alpha$, then solutions of $AX = B$ in the space s_α are*

$$X = [D_{a^*}A\langle t\rangle]^{-1} D_{a^*}B\langle u\rangle \qquad u \in C.$$

3. Some new properties of the operator $\Delta^{(\mu)}, \mu$ being any real.

In this section we give some properties of Δ^μ and $(\Delta^+)^\mu$ in relation to the space s_r and we investigate the spectrum of each operator represented by the matrices C_1, Δ, Δ^+ and Σ in relation to the space s_r.

3.1. Properties of Δ^μ in relation to s_r

The well-known operator $\Delta^{(\mu)}: s \to s$, where μ is an integer ≥ 1, is represented by the infinite lower triangular matrix Δ^μ, where $\Delta = \begin{pmatrix} 1 & & O \\ -1 & 1 & \\ O & . & . \end{pmatrix}$.

We have for every $X = (x_n)_{n\geq 1}$, $\Delta X = (y_n)_{n\geq 1}$ with $y_1 = x_1$ and $y_n = x_n - x_{n-1}$ if $n \geq 2$. We can express the following result, in which $\Delta^+ = {}^t\Delta$ and $e = (1, 1, ...)$.

Proposition 2 *([6]) i) The operator represented by Δ is bijective from s_r into itself, for every $r > 1$ and Δ^+ is bijective from s_r into itself, for all r, $0 < r < 1$.*

ii) Δ^+ is surjective and not injective from s_r into itself, for all $r > 1$.

iii) $\forall r \neq 1$ and for every integer $\mu \geq 1$ $(\Delta^+)^\mu s_r = s_r$.

iv) We have successively

α) If μ is a real > 0 and $\mu \notin N$, then Δ^μ maps s_r into itself when $r \geq 1$ but not for $0 < r < 1$.

If $-1 < \mu < 0$, then Δ^{μ} maps s_r into itself when $r > 1$ but not for $r = 1$.

β) If $\mu > 0$ and $\mu \notin N$, then $(\Delta^{+})^{\mu}$ maps s_r into itself when $0 < r \leq 1$ but not if $r > 1$.

If $-1 < \mu < 0$, then $(\Delta^{+})^{\mu}$ maps s_r into itself for $0 < r < 1$ but not for $r = 1$.

v) For a given integer $\mu \geq 1$, we have successively

$$\begin{cases} \forall r > 1 : A \in (s_r(\Delta^{\mu}), s_r) \Leftrightarrow \sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n} \right) < \infty, \\ \forall r \in]0,1[: A \in \left(s_r\left((\Delta^{+})^{\mu} \right), s_r \right) \Leftrightarrow \sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n} \right) < \infty. \end{cases}$$

vi) For every integer $\mu \geq 1$

$$s_1 \subset s_1(\Delta^{\mu}) \subset s_{(n^{\mu})_{n \geq 1}} \subset \bigcap_{r > 1} s_r.$$

vii) If $\mu > 0$ and $\mu \notin N$ then q is the greatest integer strictly less than $(\mu + 1)$. $\forall r > 1$

$$Ker\left((\Delta^{+})^{\mu} \right) \bigcap s_r = span\,(V_1, V_2, ..., V_q),$$

where:

$$(7) \qquad \begin{cases} V_1 = {}^t e,\ V_2 = {}^t \left(A_1^1, A_2^1, ... \right),\ V_3 = {}^t \left(0, A_2^2, A_3^2, ... \right), ... \\ V_q = {}^t \left(0, 0, ..., A_{q-1}^{q-1}, A_q^{q-1}, ..., A_n^{q-1}, ... \right); \end{cases}$$

$A_i^j = \frac{i!}{(i-j)!}$, with $0 \leq j \leq i$, being the number of permutations of i things taken j at a time.

3.2. Spectrum of each operator C_1, Δ, Δ^{+} and Σ in relation to the space s_r

We give here some spectral properties of several well-known operators. Recall that $C_1 = (a_{nm})_{nm \geq 1}$ is the Cesàro operator of order 1, defined by the infinite matrix

$$\begin{cases} a_{nm} = 1/n & \text{if } m \leq n, \\ a_{nm} = 0 & \text{otherwise.} \end{cases}$$

(see [3], [5], [6], [7] and [12]). It is well-known that if Σ is the lower triangular matrix whose all entries below the main diagonal are equal to 1, we have $\Delta^{-1} = \Sigma$. There exists a relation between these operators. Indeed $D_{(n)_n} C_1 = \Sigma$ and $\Delta \left(D_{(n)_n} C_1 \right) = I$, which proves that $C_1^{-1} = \Delta D_{(n)_n}$. Here A is an operator mapping s_r into itself, r being a given real > 0. We shall denote by $\sigma(A)$ its spectrum, set of all complex numbers λ, such that $(A - \lambda I)$ as operator from s_r into itself, is not invertible. We obtain the next results.

Theorem 3. *([6]) One has*

$$\begin{cases} i)\ \sigma(C_1) = \{0\} \bigcup \left\{ \frac{1}{n}\ /\ n \geq 1 \right\}, \\ ii)\ \sigma(\Delta) = \overline{D}(1, 1/r), \\ iii)\ \sigma(\Delta^+) = \overline{D}(1, r). \end{cases}$$

Note that i) has been shown in [5]. Analogously, concerning the operator Σ one gets

Proposition 4. *([6]) Let $r > 1$. We have*

i) $\frac{1}{\lambda} \in \overline{D}(1, 1/r) \Leftrightarrow \lambda \in \sigma(\Sigma)$.

ii) For all $\lambda \notin \sigma(\Sigma)$, $\lambda I - \Sigma$ is bijective from s_r into itself and if $(\lambda I - \Sigma)^{-1} = (\tau_{nm})_{n,m \geq 1}$, then

$$\begin{cases} \tau_{nn} = \frac{1}{1-\lambda} \quad \forall n \geq 1, \\ \tau_{nm} = \frac{1}{(1-\lambda)^2} \left(\frac{-\lambda}{1-\lambda} \right)^{n-m-1} \quad \text{if } m \leq n, \\ \tau_{nm} = 0 \quad \text{otherwise}. \end{cases} \tag{8}$$

4. Variation of an element in the infinite matrix Δ

In this section we are interested in the perturbed matrix $\Delta'_{pq}(a')$ and deal with the equation $\Delta'_{pq}(a') X = B$ and matrix transformations from $s_\alpha\left(\Delta'_{pq}(a')\right)$ into s_β.

4.1. First properties of the equation $\Delta'_{pq}(a') X = B$

We study the case when only one element of Δ is changed. So, we consider a given row of index p, and a given column of index q and denote by a the term a_{pq} of the matrix Δ. B being given, we study what becomes the solution of the equation $\Delta X = B$, when a is replaced by another element a' in the matrix Δ; $\Delta'_{pq}(a')$, (or Δ' for short), will denote this new matrix.

We get the following result

Theorem 5. *Let B be any sequence.*

i) The equation $\Delta' X = B$ admits a unique solution either in the cases: $q \leq p - 1$, or $q = p$ and $a' \neq 0$, or $q > p$ and $a' \neq -1$.

ii) a- Let $p < q$. When $\sum_{k=1}^{q} b_k = 0$ the equation $\Delta'_{pq}(-1) X = B$ admits infinitely many solutions in s. If $p = 1$, these solutions are given for every *scalar u by*

$$ {}^tX = \Big(u + b_1, u + b_1 + b_2, ..., u + \sum_{k=1}^{q-1} b_k, u, u + b_{q+1}, u + b_{q+1} + b_{q+2}, ..., u + \sum_{k=q+1}^{n} b_k, ...\Big); $$

and for $p \geq 2$*, these solutions are*

$$ {}^tX = \Big(b_1, b_1 + b_2, ..., \sum_{k=1}^{p} b_k, u + \sum_{k=1}^{p+1} b_k, ..., u + \sum_{k=1}^{q-1} b_k, u, u + b_{q+1}, ..., u + \sum_{k=q+1}^{n} b_k, ...\Big). $$

When $\sum_{k=1}^{q} b_k \neq 0$ *the equation* $\Delta'_{pq}(-1) X = B$ *does not admit any solution in* s.

b- If $\sum_{k=1}^{p} b_k = 0$ *the equation* $\Delta'_{pp}(0) X = B$ *admits infinitely many solutions in* s *given for any scalar* u *by*

$$ {}^tX = \left(u, u + b_2, ..., u + \sum_{k=2}^{n} b_k, ...\right) \text{ for } p = 1, $$

and for $p \geq 2$

$$ {}^tX = \left(b_1, b_1 + b_2, ..., \sum_{k=1}^{p-1} b_k, u, u + b_{p+1}, u + b_{p+1} + b_{p+2}, ..., u + \sum_{k=p+1}^{n} b_k, ...\right). $$

When $\sum_{k=1}^{p} b_k \neq 0$ *the equation* $\Delta'_{pp}(0) X = B$ *does not admit any solution.*

Proof. Assertion i). The result is trivial in the two first cases, since a triangle whose elements on the main diagonal are all different from zero is invertible. It remains to deal with the case when $1 \leq p < q$. Consider the case $1 < p < q$. We see that the equation $\Delta' X = B$ is equivalent to the system

$$ \begin{cases} -x_{n-1} + x_n = b_n & \text{if } n = 1, 2..., p-1, p+1, ...; \\ -x_{p-1} + x_p + a' x_q = b_p, \end{cases} \tag{9} $$

where we use the convention $x_0 = 0$. We get

$$ x_n = \sum_{k=1}^{n} b_k \qquad \text{if } n = 1, 2..., p-1, \tag{10} $$

$$x_n = x_p + \sum_{k=p+1}^{n} b_k \quad \text{if } n = p+1, ..., q. \tag{11}$$

From the second equality given by (9) and (10) we obtain $x_p + a' x_q = \sum_{k=1}^{p} b_k$. Putting $n = q$ in (11) we have $-x_p + x_q = \sum_{k=p+1}^{q} b_k$. Since $a' \neq -1$ one deduces easily that

$$x_n = \begin{cases} \frac{1}{a'+1} \sum_{k=1}^{n} b_k - \frac{a'}{a'+1} \sum_{k=n+1}^{q} b_k & \text{if } n = p, ..., q-1, \\ \frac{1}{a'+1} \sum_{k=1}^{q} b_k, & \text{if } n = q, \\ -\frac{a'}{a'+1} \sum_{k=1}^{q} b_k + \sum_{k=q+1}^{n} b_k, & \text{if } n = q+1, q+2, ... \end{cases} \tag{12}$$

When $p = 1 < q$ then the unique solution of equation $\Delta' X = B$, is given by

$$x_n = \begin{cases} \sum_{k=1}^{n} b_k - \frac{a'}{a'+1} \sum_{k=1}^{q} b_k & \text{if } n \leq q-1, \\ \frac{1}{a'+1} \sum_{k=1}^{q} b_k & \text{if } n = q, \\ \sum_{k=1}^{n} b_k - \frac{a'}{a'+1} \sum_{k=1}^{q} b_k & \text{if } n = q+1, q+2, ... \end{cases} \tag{13}$$

which completes the proof of i).

Assertion ii) a. If $a' = -1$, take $p \geq 2$. We deduce from i) that the equation $\Delta'_{pq}(-1) X = B$ admits a solution if $x_p - x_q = \sum_{k=1}^{q} b_k = -\sum_{k=p+1}^{q} b_k$ that is, when $\sum_{k=1}^{n} b_k = 0$. Then we can take $x_q = u$ as an arbitrary scalar and the solutions are given by

$$x_n = \begin{cases} \sum_{k=1}^{n} b_k & \text{if } n = 1, 2..., p-1, \\ u + \sum_{k=1}^{n} b_k & \text{if } n = p, p+1, ..., q-1, \\ u + \sum_{k=q+1}^{n} b_k & \text{if } n = q+1, ... \end{cases} \tag{14}$$

The case $p = 1 < q$ can be studied in a similar way.

ii) b. If $p \geq 2$ the equation $\Delta'_{pp}(0) X = B$ is equivalent to the systems

$$(S_1) \qquad \begin{cases} -x_{n-1} + x_n = b_n & \text{if } n = 1, 2, ..., p-1, \\ -x_{p-1} = b_p; \end{cases}$$

and

$$(S_2) \qquad \{-x_{n-1}+x_n=b_n \quad \text{if } n=p+1,p+2,....$$

The second one is infinite. We get $x_n=\sum_{k=1}^{n}b_k$ if $n=1,2,...,p-1$, and if $-x_{p-1}=b_p=-\sum_{k=1}^{p-1}b_k$ the system (S_1) admits a unique solution. We conclude using the system (S_2) and setting $x_p=u$, that $x_n=u+\sum_{k=p+1}^{n}b_k$ if $n=p+1,....$ If $\sum_{k=1}^{p}b_k\neq 0$ then the system (S_1) and equation $\Delta'_{pp}(0)X=B$ do not admit any solution. We get an analogous result when $p=1$. □

Remark 1. *Consider the case when $p<q$ and let $B=(b_n)_n$ be a sequence such that $\sum_{k=1}^{q}b_k=0$. We note that equation $\Delta'_{pq}(a')X=B$, where $a'=-1$ admits infinitely many solutions, and a slight variation of a' implies that the new equation $\Delta'_{pq}(a')X=B$ does not admit a solution any more. We get a similar result when $p=q$ and a' is the neighborhood of zero.*

4.2. Operators mapping $s_\alpha\left(\Delta'_{pq}(a')\right)$ into s_β

In this subsection, under some conditions, we characterize the matrices $A\in(E,F)$, where $E=s_\alpha\left(\Delta'_{pq}(a')\right)$ and $F=s_\beta$. In order to assert the following results we need the next lemmas.

Lemma 6. *Let $A=(a_{nm})_{n,m\geq 1}$ and $P=(p_{nm})_{n,m\geq 1}$ be two infinite matrices satisfying for all $n\geq 1$*

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}|a_{nk}p_{km}|\,\alpha_m<\infty. \tag{15}$$

Then $A(PX)=(AP)X$ for all $X\in s_\alpha$.

Proof. If we set $A(PX)=(y_n)_{n\geq 1}$, then for every $n\geq 1$:

$$y_n=\sum_{k=1}^{\infty}a_{nk}\left(\sum_{m=1}^{\infty}p_{km}x_m\right).$$

The series intervening in the second member being convergent, since (15) holds and $X\in s_\alpha$. Condition (15) permits us to interchange the order of summation in the expression of y_n, which proves that $A(PX)=(AP)X$. □

Remark 2.
Note that $\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}|a_{nk}p_{km}|\,\alpha_m<\infty$ if and only if $\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}|a_{nk}p_{km}|\,\alpha_m<\infty$.

Now we shall consider the sequences $\alpha = (\alpha_n)_n$ and $\beta = (\beta_n)_n$, whose general terms are > 0. We get

Lemma 7. *$A \in (s_\alpha, s_\beta)$ if and only if*

$$\sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\beta_n}\right) < \infty. \tag{16}$$

This result comes from the fact that $A \in (s_\alpha, s_\beta)$ if and only if, for all $X \in s_1$, $D_{1/\beta} A D_\alpha X \in s_1$. As we have seen in the preliminary results, this last assertion is equivalent to $D_{1/\beta} A D_\alpha \in S_1$.

We shall denote by $S_{\alpha,\beta}$ the linear vector space

$$S_{\alpha,\beta} = \left\{ A = (a_{nm})_{n,m\geq 1} \;/\; \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\beta_n}\right) < \infty \right\}.$$

We see that $S_{\alpha,\alpha} = S_\alpha$.

In the remainder of the subsection we shall suppose that the matrix $A = (a_{nm})_{n,m\geq 1}$ satisfies the condition

$$\sum_{m=1}^{\infty}\sum_{k=m}^{\infty} |a_{nk}|\, \alpha_m < \infty, \text{ for all } n. \tag{17}$$

For every n, $m \geq 1$ denote by $\sigma_{nm}(\xi)$ the map defined for any scalar ξ by

$$\sigma_{nm}(\xi) = \sum_{k=m}^{p-1} a_{nk} + \xi \sum_{k=p}^{\infty} a_{nk},$$

and let $R_{nm} = \sum_{k=m}^{\infty} a_{nk}$. Then we can give the supplementary conditions:

For $a' \neq -1$

$$\sup_{n\geq 1}\left[\frac{1}{\beta_n}\left(\sum_{m=1}^{p-1}\left|\sigma_{nm}\left(\frac{1}{a'+1}\right)\right|\alpha_m + \sum_{m=p}^{q}\left|\frac{R_{nm} - a'\sum_{k=p}^{m-1} a_{nk}}{a'+1}\right|\alpha_m + \sum_{m=q+1}^{\infty}|R_{nm}|\,\alpha_m\right)\right] < \infty; \tag{18}$$

for $a' \neq 0$

$$\sup_{n\geq 1}\left[\frac{1}{\beta_n}\left(\sum_{m=1}^{p-1}\left|\sigma_{nm}\left(\frac{1}{a'}\right)\right|\alpha_m + \frac{1}{|a'|}|R_{np}|\,\alpha_p + \sum_{m=p+1}^{\infty}|R_{nm}|\,\alpha_m\right)\right] < \infty; \tag{19}$$

$$\sup_{n\geq 1}\left[\frac{1}{\beta_n}\left(\sum_{m=1}^{p-1}\left|\sigma_{nm}\left(-a'\right)\right|\alpha_m+\sum_{m=p}^{\infty}\left|R_{nm}\right|\alpha_m\right)\right]<\infty; \tag{20}$$

and

$$\sup_{n\geq 1}\left[\frac{1}{\beta_n}\left(\sum_{m=1}^{q}\left|\sigma_{nm}\left(1-a'\right)\right|\alpha_m+\sum_{m=q+1}^{\infty}\left|R_{nm}\right|\alpha_m\right)\right]<\infty. \tag{21}$$

We obtain the following results.

Theorem 8.
i) If $1<p<q$ and $a'\neq -1$,

$$A\in\left(s_\alpha\left(\Delta'\right),s_\beta\right) \text{ if and only if (18) holds.}$$

ii) If $p=q\geq 2$ and $a'\neq 0$,

$$A\in\left(s_\alpha\left(\Delta'\right),s_\beta\right) \text{ if and only if (19) holds.}$$

iii) If $q=p-1$,

$$A\in\left(s_\alpha\left(\Delta'\right),s_\beta\right) \text{ if and only if (20) holds.}$$

iv) If $q<p-1$,

$$A\in\left(s_\alpha\left(\Delta'\right),s_\beta\right) \text{ if and only if (21) holds.}$$

Proof. Throughout the proof we shall set $\left(\Delta'\right)^{-1}=\left(c_{nm}\right)_{n,m\geq 1}$, $A\left(\Delta'\right)^{-1}=\left(c'_{nm}\right)_{n,m\geq 1}$ and put for every n: $\chi_n=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left|a_{nk}\right|\left|c_{km}\right|\alpha_m$, when this double series exists. i) Now $A\in\left(s_\alpha\left(\Delta'\right),s_\beta\right)$ if and only if $A\left(\left(\Delta'\right)^{-1}X\right)\in s_\beta$ for all $X\in s_\alpha$. We can prove that

$$A\left(\left(\Delta'\right)^{-1}X\right)=\left(A\left(\Delta'\right)^{-1}\right)X \quad \text{for all } X\in s_\alpha. \tag{22}$$

Indeed, we deduce from (12) in Theorem 5, that

$$c_{nm}=\begin{cases}1 & \text{if } 1\leq m\leq n\leq p-1 \text{ or } q+1\leq m\leq n,\\ \frac{1}{a'+1} & \text{if } p\leq n \text{ and } m\leq n\leq q, \text{ or } 1\leq m\leq q \text{ and } q+1\leq n;\\ -\frac{a'}{a'+1} & \text{if } p\leq n<m\leq q,\\ 0 & \text{otherwise}.\end{cases}$$

Since (17) holds, we can write

$$\begin{aligned}\chi_n &= \sum_{m=1}^{p-1}\sum_{k=m}^{p-1}\left|a_{nk}\right|\alpha_m+\sum_{m=q+1}^{\infty}\sum_{k=m}^{\infty}\left|a_{nk}\right|\alpha_m+\sum_{m=p+1}^{q}\sum_{k=p}^{m}\left|\frac{a_{nk}a'}{a'+1}\right|\alpha_m+\\ &\quad \frac{1}{\left|a'+1\right|}\left(\sum_{m=1}^{p}\sum_{k=p}^{\infty}\left|a_{nk}\right|\alpha_m+\sum_{m=p+1}^{q}\sum_{k=m}^{q}\left|a_{nk}\right|\alpha_m+\sum_{m=p+1}^{q}\sum_{k=q+1}^{\infty}\left|a_{nk}\right|\alpha_m\right).\end{aligned}$$

Hence the series χ_n is convergent for every $n \geq 1$ and using Lemma 6 and Remark 2, identity (22) is proved. We see that under (17) $A \in (s_\alpha(\Delta'), s_\beta)$ if and only if $A(\Delta')^{-1} \in S_{\alpha,\beta}$. The calculation gives

$$c'_{nm} = \begin{cases} \sum_{k=m}^{p-1} a_{nk} + \frac{1}{a'+1} \sum_{k=p}^{\infty} a_{nk} & \text{if } 1 \leq m \leq p-1, \\ -\frac{a'}{a'+1} \sum_{k=p}^{m-1} a_{nk} + \frac{1}{a'+1} \sum_{k=m}^{\infty} a_{nk} & \text{if } p \leq m \leq q, \\ \sum_{k=m}^{\infty} a_{nk} & \text{if } m \geq q+1; \end{cases}$$

for every n. And the condition $A(\Delta')^{-1} \in S_{\alpha,\beta}$ is equivalent to (18), which proves i).

ii) By a simple calculation we get

$$c_{nm} = \begin{cases} 1 & \text{if } m \leq n \leq p-1 \text{ or } p+1 \leq m \leq n, \\ 1/a' & \text{if } n \geq p \text{ and } m \leq p, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

We see that for all n

$$\chi_n = \sum_{m=1}^{p-1} \sum_{k=m}^{p-1} |a_{nk}| \alpha_m + \sum_{m=p+1}^{\infty} \sum_{k=m}^{\infty} |a_{nk}| \alpha_m + \sum_{m=1}^{p} \sum_{k=p}^{\infty} \left| \frac{a_{nk}}{a'} \right| \alpha_m.$$

And since (17) holds this series is convergent for every n. Reasoning as above, we have for every n

$$c'_{nm} = \begin{cases} \sum_{k=m}^{p-1} a_{nk} + \frac{1}{a'} \sum_{k=p}^{\infty} a_{nk} & \text{if } 1 \leq m \leq p-1, \\ \frac{1}{a'} \sum_{k=p}^{\infty} a_{nk} & \text{if } m = p, \\ \sum_{k=m}^{\infty} a_{nk} & \text{if } m \geq p+1. \end{cases}$$

We conclude writing that $A(\Delta')^{-1} \in S_{\alpha,\beta}$.

iii) Doing similar calculations, we obtain

$$c_{nm} = \begin{cases} 1 & \text{if } m \leq n \leq p-1 \text{ or } p \leq m \leq n, \\ -a' & \text{if } n \geq p \text{ and } m \leq p-1, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

We see that for each n

$$\chi_n = \sum_{m=1}^{p-1} \sum_{k=m}^{p-1} |a_{nk}| \alpha_m + \sum_{m=p}^{\infty} \sum_{k=m}^{\infty} |a_{nk}| \alpha_m + \sum_{m=1}^{p-1} \sum_{k=p}^{\infty} |a_{nk} a'| \alpha_m$$

is convergent since (17) holds. Further, we get for every n

$$c'_{nm}=\begin{cases}\sum\limits_{k=m}^{p-1}a_{nk}-a'\sum\limits_{k=p}^{\infty}a_{nk} & \text{if } 1\leq m\leq p-1,\\ \sum\limits_{k=m}^{\infty}a_{nk} & \text{if } m\geq p.\end{cases}$$

Reasoning as above we obtain iii).

Assertion iv). Here the equation $\Delta' X=B$ is equivalent to

$$\begin{cases}-x_{n-1}+x_n=b_n & \text{if } n=1,2...,p-1,p+1,...\\ a'x_q-x_{p-1}+x_p=b_p.\end{cases}$$

We deduce that the solution is

$$(25)\qquad\begin{cases}x_n=\sum\limits_{k=1}^{n}b_k & \text{if } n=1,2...,p-1,\\ x_n=\sum\limits_{k=q+1}^{n}b_k+(1-a')\sum\limits_{k=1}^{q}b_k & \text{for } n\geq p.\end{cases}$$

Then

$$c_{nm}=\begin{cases}1 & \text{if } m\leq n\leq p-1 \text{ and } 1\leq m\leq q, \text{ or } q+1\leq m\leq n,\\ 1-a' & \text{if } n\geq p \text{ and } 1\leq m\leq q,\\ 0 & \text{if } m>n.\end{cases}$$

Under (17) we see that the series

$$\chi_n=\sum_{m=1}^{q}\sum_{k=m}^{p-1}|a_{nk}|\,\alpha_m+\sum_{m=q+1}^{\infty}\sum_{k=m}^{\infty}|a_{nk}|\,\alpha_m\sum_{m=1}^{q}\sum_{k=p}^{\infty}|a_{nk}\,(1-a')|\,\alpha_m$$

is convergent for every $n\geq 1$ and identity (22) is proved. We conclude, since for each n

$$c'_{nm}=\begin{cases}\sum\limits_{k=m}^{p-1}a_{nk}+(1-a')\sum\limits_{k=p}^{\infty}a_{nk} & \text{if } 1\leq m\leq q,\\ \sum\limits_{k=m}^{\infty}a_{nk} & \text{if } m\geq q+1.\end{cases}$$

Remark 3 *Note that if A is a matrix satisfying (17), we have $A\in(s_\alpha(\Delta'_{11}(a')),s_\beta)$ $(a'\neq 0)$ if and only if*

$$(26)\qquad \sup_{n\geq 1}\left[\frac{1}{\beta_n}\left(\frac{1}{|a'|}|R_{n1}|\,\alpha_1+\sum_{m=2}^{\infty}|R_{nm}|\,\alpha_m\right)\right]<\infty.$$

4.3. The distance between two solutions of an infinite linear system

Given B, let $X_{pq}(a')$ (or X') denote the solution of $\Delta'_{pq}(a')X = B$. We shall denote by $Z = \left(\sum_{k=1}^{n} b_k\right)_{n\geq 1}$ the unique solution of $\Delta X = B$, for short. Then we see that if $q \neq p$, $p-1$, $Z = X_{pq}(0)$; if $p \geq 2$, then $Z = X_{pp-1}(-1)$ and if $p = q$, $Z = X_{pp}(1)$ for all $p \geq 1$. We have the following results:

Corollary 9. *For a given matrix $B \in s_1$, and a given real a', we have:*
i) If $q > p$ and $a' \neq -1$,

$$\|X_{pq}(a') - Z\|_{s_1} = \left|\left(\frac{a'}{a'+1}\right)\sum_{k=1}^{q} b_k\right| \tag{27}$$

ii) If $a' \neq 0$ for each $p \geq 1$, we have

$$\|X_{pp}(a') - Z\|_{s_1} = \left|\left(\frac{a'-1}{a'}\right)\sum_{k=1}^{p} b_k\right|. \tag{28}$$

iii) For all $p \geq 2$:

$$\|X_{pp-1}(a') - Z\|_{s_1} = \left|(a'+1)\sum_{k=1}^{p-1} b_k\right|. \tag{29}$$

iv) If $q < p-1$ and $a' \neq -1$:

$$\|X_{pq}(a') - Z\|_{s_1} = \left|a'\sum_{k=1}^{q} b_k\right| \tag{30}$$

Proof. i) is deduced from the proof of the previous theorem, since $X' - Z = (\xi_n)_{n\geq 1}$, where

$$\xi_n = \begin{cases} 0 & \text{if } n \leq p-1, \\ -\frac{a'}{a'+1}\sum_{k=1}^{q} b_k & \text{if } n \geq p. \end{cases}$$

Hence $\|X' - Z\|_{s_1} = \sup_{n\geq p}(|\xi_n|) = \left|\left(\frac{a'}{a'+1}\right)\sum_{k=1}^{q} b_k\right|$. If $p = 1$ one can verify (27) using similar calculations. Analogously we can prove ii), iii) and iv) using (23), (24) and (25) in the proof of Theorem 8.

5. Matrix transformations mapping $s_\alpha\left(\left(\Delta'_{pq}(a')\right)^\mu\right)$ into s_β, μ being any integer

In this section we generalize results given in [1], [6] and [7] concerning matrices mapping $s_1(\Delta^\mu)$ into s_1. Malkowsky [7] introduced the sequence

$\left(R_{nm}^{(\mu)}\right)_{n,m\geq 1}$, defined in the following way: $R_{nm}^{(1)} = R_{nm} = \sum_{k=m}^{\infty} a_{nk}$, $R_{nm}^{(s)} = \sum_{k=m}^{\infty} R_{nk}^{(s-1)}$ $\forall s \geq 2$. He proved that $A \in (s_1(\Delta^{\mu}), s_1)$ if and only if

$$\begin{cases} \text{i) For every } n \text{, the series } \sum_{m=1}^{\infty} m^{\mu} a_{nm} \text{ is convergent,} \\ \text{ii)} \sup_n \left(\sum_{m=1}^{\infty} \left| R_{nm}^{(\mu)} \right| \right) < \infty. \end{cases}$$

In [1], a necessary and sufficient condition is given for $A \in (s_1(\Delta^{+\mu}), s_1)$. Let us recall the following result given in [6], in which we define for any $\mu \in C$

$$\begin{pmatrix} \mu + k - 1 \\ k \end{pmatrix} = \begin{cases} \dfrac{\mu(\mu+1)\dots(\mu+k-1)}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0 \,. \end{cases}$$

Theorem 10. *Let μ be a complex number. Assume that $A = (a_{nm})_{n,m\geq 1}$ satisfies the condition: for all $n \geq 1$ and $\lambda \neq 1$*

$$\sum_{m=1}^{\infty} \sum_{j=0}^{\infty} \begin{pmatrix} \mu + j - 1 \\ j \end{pmatrix} \frac{|a_{n,m+j}|}{|1-\lambda|^{\mu+j}} r^m < \infty. \tag{31}$$

For every $\lambda \neq 1$, we have $A \in (s_r((\Delta - \lambda I)^{\mu}), s_r)$ if and only if

$$\sup_{n\geq 1} \left[\sum_{m=1}^{\infty} \left| \sum_{j=0}^{\infty} \begin{pmatrix} \mu + j - 1 \\ j \end{pmatrix} \frac{a_{n,m+j}}{(1-\lambda)^{\mu+j}} \right| r^{m-n} \right] < \infty. \tag{32}$$

Under (31) in which $\lambda = 0$ and $r = 1$, $A \in (s_1(\Delta^{\mu}), s_1)$ if and only if

$$\sup_{n\geq 1} \left(\sum_{m=1}^{\infty} \left| \sum_{j=0}^{\infty} \begin{pmatrix} \mu + j - 1 \\ j \end{pmatrix} a_{n,m+j} \right| \right) < \infty.$$

Now, we need a result generalizing i) in Proposition 2.

Proposition 11. *i) $\alpha \in \Gamma$ if and only if there exists $\nu \geq 1$ such that*

$$\gamma_{\nu} = \sup_{n\geq \nu+1} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) < 1.$$

ii) If $\alpha \in \Gamma$, then Δ is bijective from s_{α} into itself.

iii) Let r be a real > 0. Then Δ is bijective from s_r into itself if and only if $r > 1$.

Proof. i) is obvious. Assertion ii) Denote for any integer $\nu \geq 1$ by $\Sigma^{(\nu)}$ the infinite matrix

$$\begin{pmatrix} \left[\Delta^{(\nu)}\right]^{-1} & & O \\ & 1 & \\ O & & . \end{pmatrix},$$

where $\Delta^{(\nu)}$ is the finite matrix whose elements are those of the ν first rows and of the ν first columns of Δ. We get $\Sigma^{(\nu)}\Delta = (a_{nm})_{n,m\geq 1}$, with $a_{nn} = 1$ for all n; $a_{n,n-1} = -1$ for all $n \geq \nu + 1$; and $a_{nm} = 0$ otherwise. We see that if $\alpha \in \Gamma$, there exists an integer $\nu \geq 1$ such that $\left\|I - \Sigma^{(\nu)}\Delta\right\|_{s_\alpha} < 1$. We see that $\Sigma^{(\nu)}B \in s_\alpha$ for all $B \in s_\alpha$. Then the equation $\Delta X = B$ being equivalent to

$$\left(\Sigma^{(\nu)}\Delta\right) X = \Sigma^{(\nu)}B$$

admits only one solution in s_α for all $B \in s_\alpha$. This proves that Δ is bijective from s_α into itself.

Assertion iii). The necessity is a direct consequence of ii). Conversely, assume that Δ is bijective from s_r into itself and let $B = (r^n)_{n\geq 1} \in s_r$. The equation $\Delta X = B$ admits the unique solution $X = \left(\sum_{i=1}^{n} r^i\right)_{n\geq 1} \in s_r$. Then

$$\frac{\sum_{i=1}^{n} r^i}{r^n} = \frac{r - r^{n+1}}{(1-r)\,r^n} = O(1) \quad \textit{as } n \to \infty,$$

which implies that $r > 1$.

Remark 4. *The converse of ii) in the previous proposition is false. Indeed, consider the sequence* $\alpha = (\alpha_n)_{n\geq 1}$ *defined by*

$$\alpha_n = \begin{cases} \gamma^{2j} & \textit{if } n = 2j, \\ \gamma^{2j} & \textit{if } n = 2j+1, \end{cases}$$

for a given $\gamma > 1$. *First we see that for all* $\nu \geq 1$:$\sup_{n\geq \nu+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) = 1$, *that is* $\alpha \notin \Gamma$. *Furthermore, we see that*

$$\frac{x_n - x_{n-1}}{\alpha_n} = \frac{x_n}{\alpha_n} - \frac{x_{n-1}}{\alpha_{n-1}}\frac{\alpha_{n-1}}{\alpha_n} = O(1) \;\; \textit{as } n \to \infty,$$

since $\frac{\alpha_{n-1}}{\alpha_n} = \frac{1}{\gamma^2}$ *if* n *is even, and* $\frac{\alpha_{n-1}}{\alpha_n} = 1$ *if* n *is odd. This proves that* $\Delta X \in s_\alpha$ *for all* $X \in s_\alpha$. *Therefore the equation* $\Delta X = B$, *where* $B \in s_\alpha$ *admits only one solution in* s_α, *since there exists* $M > 0$ *such that*

$$\frac{\left|\sum_{k=1}^{n} b_k\right|}{\gamma^n} \leq M \sum_{j=0}^{\infty} \frac{1}{\gamma^{2j}} \leq M \frac{\gamma^2}{\gamma^2 - 1} \qquad \textit{for all } n.$$

This proves that Δ is bijective from s_α into itself.

Here we deal with the matrix transformations mapping $s_\alpha\left(\left(\Delta'_{pq}\left(a'\right)\right)^\mu\right)$ into s_β. We have

Theorem 12. *Let μ, p, q be integers ≥ 1 and a' any scalar. If $\alpha \in \Gamma$ and $\frac{\alpha_{n+1}}{\alpha_n} = O\left(1\right)$ as $n \to \infty$, then*

$$\left(s_\alpha\left(\left(\Delta'\right)^\mu\right), s_\beta\right) = S_{\alpha,\beta}. \tag{33}$$

Proof. We only have to prove that $s_\alpha\left(\Delta'\right) = s_\alpha$ for all p, q, a'. Then, $s_\alpha\left(\left(\Delta'\right)^\mu\right) = s_\alpha$ and we deduce the theorem using Proposition 1.

First consider the case when $p = q$ and $a' = 0$. We are going to show that $s_\alpha\left(\Delta'\right) = s_\alpha$. Take $Z = \left(z_n\right)_n \in s_\alpha\left(\Delta'\right)$. Then $B = \Delta' Z \in s_\alpha$, which implies that the equation $\Delta^+ X_p = -B_p$, where ${}^t X_p = \left(x_p, x_{p+1}, ...\right)$ and ${}^t B_p = \left(b_{p+1}, b_{p+2}, ...\right)$ admits the solution ${}^t Z_p = \left(z_p, z_{p+1}, ...\right)$ in s_α. Indeed, since $\alpha \in \Gamma$ we have

$$\left\| I - \Sigma^{(\nu)} \Delta^+ \left\langle e_1 \right\rangle \right\|_{s_\alpha} = \left\| I - \Sigma^{(\nu)} \Delta \right\|_{s_\alpha} = \sup_{n \geq \nu+1} \left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1;$$

and $B \in s_\alpha$ implies

$$\frac{b_{n+p-1}}{\alpha_n} = \frac{b_{n+p-1}}{\alpha_{n+p-1}} \frac{\alpha_{n+p-1}}{\alpha_{n+p-2}} ... \frac{\alpha_{n+1}}{\alpha_n} = O\left(1\right) \quad \text{as } n \to \infty,$$

which proves that $B_p \left\langle u_o \right\rangle \in s_\alpha$. Using Proposition 1, we deduce that the solutions of the equation $\Delta^+ X_p = -B_p$ belong to s_α and can be written in the form $X_p = -\Delta^{-1} B_p \left\langle u \right\rangle$ for any scalar u. Then there exists a scalar u_o such that $Z_p = -\Delta^{-1} B_p \left\langle u_o \right\rangle \in s_\alpha$. We conclude that $Z \in s_\alpha$, since $\alpha \in \Gamma$ implies

$$\frac{z_n}{\alpha_n} = \frac{z_n}{\alpha_{n-p+1}} \frac{\alpha_{n-p+1}}{\alpha_{n-p+2}} ... \frac{\alpha_{n-1}}{\alpha_n} = O\left(1\right) \quad \text{as } n \to \infty.$$

We have proved that $s_\alpha\left(\Delta'\right) \subset s_\alpha$. Conversely, we see easily that $Z = \left(z_n\right)_n \in s_\alpha$ implies $\Delta' Z \in s_\alpha$, since $\alpha \in \Gamma$.

Now we consider the case when $q > p$ and $a' = -1$. Take $Z = \left(z_n\right)_n \in s_\alpha\left(\Delta'\right)$. Then $B = \Delta' X \in s_\alpha$, reasoning as above we see that the equation $\Delta^+ X_q = -B_q$, admits $Z_q = -\Delta^{-1} B_q \left\langle u_o \right\rangle$ as a solution for a well chosen u_0. This proves that $Z \in s_\alpha$. Conversely, if $X = \left(x_n\right)_n \in s_\alpha$, then $\Delta' X \in s_\alpha$.

Finally we consider the case when $q > p$ and $a' \neq -1$ or $p = q$ and $a' \neq 0$ or $q < p$. Take $Z = \left(z_n\right)_n \in s_\alpha\left(\Delta'\right)$. Then $B = \Delta' Z \in s_\alpha$. As we have defined $\Sigma^{(\nu)}$ from Δ in the proof of Proposition 11, we define here $\Sigma'^{(\nu)} = \left(c'_{nm}\right)_{n,m \geq 1}$ from Δ'. If we put $Z = \Sigma'^{(\nu_0)} Z'$ with $\nu_0 = \sup\left(p, q\right)$, then the equation $\Delta' Z = B$ is equivalent to $\left(\Delta' \Sigma'^{(\nu_0)}\right) Z' = B$. One sees that the solution $Z' = (z'_n)_{n\geq 1}$ of the previous equation satisfies

$$\text{(34)} \qquad \begin{cases} z'_n = b_n & \text{for } n \leq \nu_0, \\ z'_{\nu_0+1} = \sum_{m=1}^{\nu_0} c'_{\nu_0,m} b_m + b_{\nu_0+1}, & \\ z'_n - z'_{n-1} = b_n & \text{if } n \geq \nu_0 + 2. \end{cases}$$

Then $\Delta Z'_{\nu_0} = -B_{\nu_0}$, where

$${}^t Z'_{\nu_0} = \left(z'_{\nu_0+1}, z'_{\nu_0+2}, ...\right) \text{ and } {}^t B'_{\nu_0} = \left(\sum_{m=1}^{\nu_0} c'_{\nu_0,m} b_m + b_{\nu_0+1}, b_{\nu_0+2}, ..., b_n, ... \right).$$

Since $\alpha \in \Gamma$, we deduce that Z'_{ν_0} and $Z \in s_\alpha$. We have shown that $s_\alpha(\Delta') \subset s_\alpha$. The converse is trivially verified. This proves that $\Delta'_{pq}(a')$ is bijective from s_α into itself.

Remark 5. *Note that we cannot have $s_\alpha\left(\Delta'_{pq}(a')\right) = s_\alpha$ for all p, q, $\mu \geq 1$. Consider for instance the space $s_1\left(\Delta'_{2,2}(0)\right)$. It can be shown that*

$$s_1\left(\Delta'_{2,2}(0)\right) = \left\{ {}^t\left(x_1, x_2, x_3, x_3 + x_4, ..., \sum_{k=3}^{n} x_k, ...\right) / \quad (x_n)_{n\geq 1} \in s_1 \right\},$$

and we see that $X_0 = (n)_n \in s_1\left(\Delta'_{2,2}(0)\right) - s_1$.

Remark 6. *Note that in the cases when $q > p$ and $a' \neq -1$, or $p = q$ and $a' \neq 0$, or $p = q = 1$ and $a' = 0$, or $q < p$, (33) holds under the single hypothesis $\alpha \in \Gamma$.*

Remark 7. *Consider the case $p = q \geq 2$ and $a' \neq 0$ and let $A = (a_{nm})_{n,m\geq 1}$ be a matrix such that (17) holds. If $\alpha \in \Gamma$, then (19) is equivalent to (16).*

Indeed, from Theorem 8, we have $A \in \left(s_\alpha\left(\Delta'_{pq}(a')\right), s_\beta\right)$ iff (19) holds, and we conclude using Theorem 13.

Analogously, assume that $q = p - 1$ and A satisfies condition (17). If $\alpha \in \Gamma$, (20) is equivalent to (16).

From Theorem 12 we deduce

Corollary 13. *i) Let r_1 and r_2 be two reals, with $r_1 > 1$ and $r_2 > 0$ and p, q, $\mu \geq 1$. Then*

$$A \in \left(s_{r_1}\left(\left(\Delta'_{pq}(a')\right)^\mu\right), s_{r_2}\right) \text{ if and only if } \sup_{n\geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| \frac{r_1^m}{r_2^n} \right) < \infty.$$

ii) If $r_1 > 1$, we get

$$A \in \left(s_{r_1}\left(\left(\Delta'_{pq}(a')\right)^\mu\right), l^\infty\right) \text{ if and only if } \sup_{n\geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| r_1^m \right) < \infty.$$

Proof. i) We see that $\alpha = (r_1^n)_{n \geq 1} \in \Gamma$, since $\frac{\alpha_{n-1}}{\alpha_n} = \frac{1}{r_1} < 1$, moreover $r_1^{n+p-1}/r_1^n = r_1^{p-1} = O(1)$ as $n \to \infty$. ii) is obvious.

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