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ON RANK EQUIVALENCE AND RANK PRESERVING OPERATORS

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Abstract. In the paper [6] A. Marková proved theorem on rank equivalence operators. This theorem is true, but the proof is wrong. In this paper new, correct proof of theorem of A. Marková is presented. The form of generators preserving rank is determined. Also an interesting example of generator is showed.

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1. Introduction

Recently, E. Pap [8] introduced and developed the so-called g-calculus on an real interval $[a, b] \subset [-\infty, +\infty]$ depending on a function g. There are two arithmetical operations: pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$.

The name g-calculus comes from the term "generator", used by J. Aczél in [1].

In [7], R. Mesiar and J. Rybařik introduced the concept of g-calculus on the interval $[0, +\infty]$ with four pseudo-operations: pseudo-addition $\oplus$, pseudo-multiplication $\otimes$, pseudo-substraction $\ominus$ and pseudo-division $\oslash$. This concept was extended to the interval $[-\infty, +\infty]$.

In [6], g-calculus in the interval $[-\infty, +\infty]$ was applied to linear algebra. A generalization of a system of linear equations is the corresponding pseudo-linear problem. As was shown in [6], g-rank of matrix plays a crucial role when seeking the for existence of solution of this g-linear problem.

In the paper [6], the main theorem on g-rank matrix is Theorem 4.1. However, the proof given in [6] is wrong. A general solution of the equation

$$f(K \cdot x) = 2f(x), x \in \mathbb{R}, \tag{1}$$

where $K > 1$ is a constant, in the class of continuous strictly increasing odd bijections depends on an arbitrary function (see [3], §2, case (ii), pp. 47-50 and

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[2], [4], [5]). Some additional assumptions are needed to guarantee that the general solution of (1) is given by

$$f(x) = \begin{cases} d \cdot x^p & \text{for } x \in [0, +\infty), \\ -d \cdot (-x)^p & \text{for } x \in (-\infty, 0), \end{cases} \tag{2}$$

where d and p are positive real constants.

We construct a strictly increasing odd bijection which is a solution of (1) and is not represented by formula (2).

Example 1.1. There exist a solution of equation (1) which cannot be represented in the form (2). Let us construct such a solution.

Fix an arbitrary $x_0 > 0$ and consider the points $K^m \cdot x_0$, where $m \in \mathbb{Z}$ and $K > 1$ is the constant from (1). Let us note that

$$\lim_{m \to +\infty} K^m \cdot x_0 = +\infty \quad \text{and} \quad \lim_{m \to -\infty} K^m \cdot x_0 = 0.$$

Let us observe also that

$$\bigcup_{m=-\infty}^{+\infty} [K^m x_0, K^{m+1} x_0] = (0, +\infty). \tag{3}$$

We define a function f_0 on the interval $[x_0, Kx_0]$ in the following way. Choose a $y_0 > 0$ arbitrarily and put $f_0(x_0) = y_0$ and $f_0(Kx_0) = 2y_0$. In (x_0, Kx_0) f_0 may be arbitrary provided it is continuous and increasing in $[x_0, Kx_0]$. Using (1) we can uniquely extend f_0 onto adjacent intervals $[K^{-1}x_0, x_0]$ and $[Kx_0, K^2x_0]$ so that the extension satisfies (1). The procedure can be iterated, and so f_0 can be extended onto the intervals adjacent to the one on which the solution has been already defined. In any of the intervals $[K^m x_0, K^{m+1} x_0]$, $m \in \mathbb{Z}$ the extension will be continuous and increasing. Taking into account (3) we can extend f_0 onto the whole interval $(0, +\infty)$, and this extension, which we denote by f_0, will also be a continuous and increasing solution of (1). Since

$$\lim_{m \to +\infty} f_0(K^m \cdot x_0) = \lim_{m \to +\infty} 2^m \cdot y_0 = +\infty \;\text{ and }\; \lim_{m \to -\infty} f_0(K^m \cdot x_0) = \lim_{m \to -\infty} 2^m \cdot y_0 = 0$$

we see that f_0 maps bijectively $(0, +\infty)$ onto itself.

If, for instance, f_0 is given on $[x_0, Kx_0]$ by

$$f_0(x) = \begin{cases} y_0 + \frac{1}{2(K-1)} \cdot \frac{y_0}{x_0}(x - x_0) & \text{for } x \in [x_0, \frac{K+1}{2}x_0], \\ \frac{5}{4}y_0 + \frac{3}{2(K-1)} \cdot \frac{y_0}{x_0}(x - \frac{K+1}{2}x_0) & \text{for } x \in [\frac{K+1}{2}x_0, Kx_0], \end{cases}$$

then f_0 is increasing, $f_0(x_0) = y_0$, and f_0 is not differentiable at $x = \frac{K+1}{2}x_0$. Thus f_0 is not of the form (2).

In this paper we give a new, correct proof of A. Marková's theorem (cf. Theorem 3.1). In Theorem 3.2 we determine functions preserving rank of matrices. There are only linear functions $g(x) = c \cdot x$, $x \in \mathbb{R}$, with a positive constant c. In Section 4 an interesting example of a generator is showed.

2. Preliminaries

First off all let us introduce some definitions

Definition 2.1. *A function g is a generator on $[-\infty, +\infty]$ if and only if it is odd and strictly increasing bijection on interval $(-\infty, +\infty)$.*

Remark 2.1. If g, h are generators, then g^{-1} and $g \circ h^{-1}$ are also generators.

Definition 2.2. *(see [6], p. 126). Let $A = [a_{i,j}]$, $i, j = 1, 2, \dots, n$ be a given matrix and g a generator on $[-\infty, +\infty]$. Denote by $g(A)$ the matrix $[g(a_{i,j})]$, $i = 1, 2, \dots, n$, $\quad j = 1, 2, \dots, m$. A generator g is rank preserving operator if and only if $r(A) = r(g(A))$ for every matrix A. Two generators g and h are rank-equivalence operators if and only if $r(g(A)) = r(h(A))$ for every matrix A.*

Remark 2.2. If g, h are two rank preserving operators, then $g \circ h$ and g^{-1} are rank preserving operators.

Let us also note following

Lemma 2.1. *(see [4], theorem 6, p. 311). Let the functional equation*

$$G(a \cdot b) = G(a) \cdot G(b) \tag{4}$$

be satisfied for all $a, b \in (0, +\infty)$. A function $G : (0, +\infty) \longrightarrow \mathbb{R}$ is a continuous solution of (3) on $(0, +\infty)$ if and only if $G = 0$, or $G = 1$, or it has the following form

$$G(x) = x^p \quad \text{for all } x \in (0, +\infty), \tag{5}$$

where p is a real constant.

The following lemma on a functional equation will be used in proof of theorem 3.1.

Lemma 2.2. *Let the functional equation*

$$c \cdot F(a \cdot b) = F(a) \cdot F(b) \tag{6}$$

be satisfied for all $a, b \in (0, +\infty)$, where $c \neq 0$. A continuous bijection $F : (0, +\infty) \longrightarrow \mathbb{R}$ is a solution of (5) on $(0, +\infty)$ if and only if F has the following form

$$F(x) = c \cdot x^p \quad \text{for all } x \in (0, +\infty), \tag{7}$$

where $p \neq 0$.

Proof. Denote $G(x) := F(x)/c$. From (5) the function G satisfies equation (3). Using Lemma 2.1 we obtain (4), which proves (6). Since F is a bijection then $p \neq 0$. $\square$

Remark 2.3. If functional equation (5) is satisfied for $a, b \in \mathbb{R}$, then any solution of (5) in the class of generators on $[-\infty, +\infty]$ is given by

$$F(x) = \begin{cases} c \cdot x^p & \text{for } x \in [0, +\infty), \\ -c \cdot (-x)^p & \text{for } x \in (-\infty, 0), \end{cases}$$

Proof. From Lemma 2.2, the function F on $[0, +\infty)$ is defined by formula (6) with $p > 0$ because $F(0) = 0$. The function F is odd, then on the interval $(-\infty, 0)$ it must be defined by formula

$$F(x) = -c \cdot (-x)^p \quad \text{for } x \in (-\infty, 0).$$

It is easy to observe that F defined by the above formula on the whole $\mathbb{R}$ satisfies the equation (5).

3. Main results

Theorem 3.1. *The generators g and h are rank-equivalence operators on $[-\infty, +\infty]$ if and only if there exists a positive constant c so that $g = c \cdot h$.*

Proof. Let h and g be two generators on $[-\infty, +\infty]$. If $h = c \cdot g$, $c > 0$, then h and g are rank-equivalence operators on the same set.

Suppose that g and h are rank-equivalence operators on $[-\infty, +\infty]$. Let $a, b > 0$. Consider the matrix

$$A = \begin{bmatrix} h^{-1}(1) & h^{-1}(b) \\ h^{-1}(a) & h^{-1}(a \cdot b) \end{bmatrix}.$$

Then

$$h(A) = \begin{bmatrix} 1 & b \\ a & a \cdot b \end{bmatrix} \quad \text{and} \quad g(A) = \begin{bmatrix} g(h^{-1}(1)) & g(h^{-1}(b)) \\ g(h^{-1}(a)) & g(h^{-1}(a \cdot b)) \end{bmatrix}.$$

The rank $r(h(A)) = 1$. Because g, h are rank-equivalence operators, then also $r(g(A)) = 1$. From Remark 2.1 the function $g \circ h^{-1}$ is a generator, then $g(h^{-1}(1)) \neq 0$. Because $r(g(A)) = 1$, we have

$$g(h^{-1}(1)) \cdot g(h^{-1}(a \cdot b)) - g(h^{-1}(a)) \cdot g(h^{-1}(b)) = 0. \tag{8}$$

Denote $f := g \circ h^{-1}$. Then (8) can be written as

$$f(1) \cdot f(a \cdot b) = f(a) \cdot f(b)$$

for arbitrary $a, b > 0$, where f is a generator. From Lemma 2.2 we obtain that

$$f(x) = c \cdot x^p \quad \text{for } x > 0, \tag{9}$$

where $c = f(1)$ is a positive constant.

We will show that p in formula (9) has to be equal 1. Put

$$B = \begin{bmatrix} 0 & h^{-1}(1) & h^{-1}(1) \\ h^{-1}(1) & 0 & h^{-1}(1) \\ h^{-1}(1) & h^{-1}(1) & h^{-1}(2) \end{bmatrix} \quad \text{and} \quad h(B) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Then

$$g(B) = \begin{bmatrix} 0 & g(h^{-1}(1)) & g(h^{-1}(1)) \\ g(h^{-1}(1)) & 0 & g(h^{-1}(1)) \\ g(h^{-1}(1)) & g(h^{-1}(1)) & g(h^{-1}(2)) \end{bmatrix} = \begin{bmatrix} 0 & f(1) & f(1) \\ f(1) & 0 & f(1) \\ f(1) & f(1) & f(2) \end{bmatrix}.$$

From formula (9) we obtain

$$g(B) = \begin{bmatrix} 0 & c & c \\ c & 0 & c \\ c & c & c \cdot 2^p \end{bmatrix}.$$

The rank $r(h(B)) = 2$. The generators g and h are two rank-equivalence operators, thus also $r(g(B)) = 2$. In the matrix $g(B)$ the last row is a sum of first and second rows. In particular we obtain $c + c = c \cdot 2^p$, where c is a positive constant. It follows that $p = 1$. In view of the definition of f from (9) we obtain

$$g(h^{-1}(x)) = c \cdot x \quad \text{for } x > 0$$

which is equivalent to

$$h^{-1}(x) = g^{-1}(c \cdot x) \quad \text{for } x > 0.$$

Denote for $x > 0$

$$y = h^{-1}(x) = g^{-1}(c \cdot x)$$

where $c > 0$. Then we have

$$x = h(y) \quad \text{and} \quad c \cdot x = g(y)$$

with $c > 0$. In other words

$$g(y) = c \cdot h(y) \quad \text{for all } y > 0.$$

Since the generators g and h are odd functions, we have

$$g(y) = -g(-y), \quad \text{and} \quad h(y) = -h(-y) \quad \text{for all } y < 0.$$

Hence for negative y the equality $g(y) = c \cdot h(y)$ holds as well. Also, for $y = 0$ we have $g(0) = c \cdot h(0)$. Thus for all $y \in \mathbb{R}$ is true $g(y) = c \cdot h(y)$, i.e. $g = c \cdot h$ on $\mathbb{R}$ with a positive constant c. □

If h is the identity function, then from Theorem 3.1 we directly get

Theorem 3.2. *The generator g is rank-preserving operator on the interval $[-\infty, +\infty]$ if and only if g is a linear function given by $g(x) = c \cdot x$ with a positive constant c.*

4. An example

Example 4.1. Let $C \in \mathbb{R}^{n\times n}$ be a matrix of Vandermonde with elements defined by the formulae

$$c_{i,j} = c_i^j \quad \text{for } j = 1, 2, \ldots, n,$$

where $1 < c_1 < \cdots < c_1^n < c_2 < \cdots < c_2^n < c_3 < \cdots < c_{n-1}^n < c_n$.

From properties of Vandermonde matrices it follows that $\det C \neq 0$ and rank $r(C) = n$.

Sequence of elements of the matrix C arranged row by row is strictly increasing:

$$\begin{gathered} c_{1,1} < c_{1,2} < \cdots < c_{1,n} < c_{2,1} < c_{2,2} < \cdots \\ < c_{2,n} < \cdots < c_{n-1,n} < c_{n,1} < c_{n,2} < \cdots < c_{n,n}. \end{gathered}$$

Define other matrix $D \in \mathbb{R}^{n\times n}$. Elements of the first row are defined by the formulae

$$d_{1,j} = j + 1 \quad \text{for } j = 1, 2, \ldots, n.$$

The other rows for $i = 2, 3, \ldots, n$ we obtain by multiplying first row by the positive numbers $D_2, D_3, \ldots, D_n$, respectively, i.e.

$$d_{i,j} = D_i \cdot d_{1,j} \quad \text{for } i = 2, 3, \ldots, n \text{ and } j = 1, 2, \ldots, n.$$

Then, from the properties of determinants the rank $r(D) = 1$.

The numbers D_j, $j = 2, 3, \ldots, n$, we choose such that

$$d_{j-1,n} < D_j \cdot d_{j,1} \quad \text{for } j = 2, 3, \ldots, n.$$

Then the sequence of elements of the matrix D arranged row by row

$$\begin{gathered} d_{1,1} < d_{1,2} < \cdots < d_{1,n} < d_{2,1} < d_{2,2} < \cdots \\ < d_{2,n} < \cdots < d_{n-1,n} < d_{n,1} < d_{n,2} < \cdots < d_{n,n} \end{gathered}$$

is also strictly increasing.

Now, we define a generator g. Let

$$g(c_{i,j}) = d_{i,j} \quad \text{for } i, j = 1, 2, \ldots, n$$

and $g(0) = 0$, $g(1) = 1$. Next we define the function g on the intervals $[0, 1]$, $[1, c_{1,1}]$, $[c_{1,1}, c_{1,2}]$, $\ldots$, $[c_{n,n-1}, c_{n,n}]$, $[c_{n,n}, +\infty)$ so that it is continuous and increasing. There are infinitely many functions g with these properties. Hence we obtained a function on $[0, +\infty]$ which can be uniquely extended odd to a generator on $[-\infty, +\infty]$.

This generator has the following property: for the matrix $C \in \mathbb{R}^{n\times n}$

$$r(C) = n \quad \text{and} \quad r(g(C)) = 1.$$

Remark 4.1. For the unit matrix $E \in \mathbb{R}^{n\times n}$ and the generator g from Example 4.1

$$r(E) = n \quad \text{and} \quad r(g(E)) = n.$$

Remark 4.2. From the construction of the generator g in Example 4.1 it follows that for the generator $h = g^{-1}$ and the matrix $D \in \mathbb{R}^{n\times n}$

$$r(D) = 1 \quad \text{and} \quad r(h(D)) = n.$$

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References

[1] Aczél, J., Vorlesungen über Funktionalgleichungen und ihre Anwendungen, Berlin: VEB Deutche Verlag der Wissenschaften 1961.

[2] Aczél, J., Kuczma, M., Generalisations of a "folk-theorem" on simple functional equations in a single variable, Results Math. 19 (1991), 5-21.

[3] Kuczma, M., Functional equations in a single variable, Monografie Mat. 46, Warszawa: Polish Sci. Publ. (PWN) 1968.

[4] Kuczma, M., An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality, Katowice: Uniwersytet lski; Warsaw: Polish Sci. Publ. 1985.

[5] Kuczma, M., Choczewski, B., Ger, R., Iterative functional equations, Encyclopedia of Mathematics and Its Applications 32, Cambridge: Cambridge University Press, 1990.

[6] Marková, A., Some remarks on the pseudo-linear algebra, Tatra Mountains Math. Publ. 6 (1995), 123-130.

[7] Mesiar, R., Rybaŕik, J., Pseudo-arithmetical operations, Tatra Mountains Math. Publ. 2(1993), 185-192.

[8] Pap, E., g-calculus, Zb. Prir.-Mat. Fak., Univ. Novom Sadu, Ser. Mat. 23 (1993), 145-156.

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