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OPERATORS H, S AND P IN THE CLASSES OF p-SEMIGROUPS AND p-SEMIRINGS

Vjekoslav Budimirović[1], Branimir Šešelja[2]

Abstract. If $p \in N$, then a p-semigroup, introduced in [3], is a generalization of the notion of an anti-inverse semigroup [2]. A similar notion is a p-semiring. The aim of the paper was to investigate the closeness of classes of these algebras under the operators H (homomorphisms), S (subalgebras) and P (direct products). It is proved that for every $p \in N$ each of these classes is closed under H and P. Conditions under which closeness under S also hold are presented. It turns out that for p even or $p = 4k + 3$ both the class of p-semigroups and the one of p-semirings are varieties. The corresponding identities are presented.

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1. Introduction

We advance some definitions from the paper [3].

Let $(S, +)$ be a semigroup and $p \in N$. Further, let τ_p be a relation on S, introduced by:

$$x\tau_p y \iff x + py + x = y \wedge py + x + py = x.$$

If $x\tau_p y$ for $x, y \in S$, then py is called the p-**element** of the element x. A semigroup $(S, +)$ is called a p-**semigroup** if each element has its p-element. For a given p, let Π_p denote the class of all p-semigroups, i.e.,

$$S \in \Pi_p \iff (\forall x \in S)(\exists y \in S)(x\tau_p y).$$

Now we generalize the foregoing notions to the structure with two binary operations.

As is known, a **semiring** $(S, +, \cdot)$ is an algebra with two binary operations, such that $(S, +)$ and $(S, \cdot)$ are semigroups:

$x + (y + z) = (x + y) + z;\quad x \cdot (y \cdot z) = (x \cdot y) \cdot z,$

and the following distributivity laws are fulfilled:

$x \cdot (y + z) = (x \cdot y) + (x \cdot z);\quad (x + y) \cdot z = (x \cdot z) + (y \cdot z).$

[1] Higher School of Chemical Technology, Narodnih heroja 10, 15000 Šabac, Yugoslavia

[2] Institute of Mathematics, University of Novi Sad, Trg D. Obradovića 4, 21 000 Novi Sad, Yugoslavia

By some authors, the first operation is commutative, and also neutral element in the first semigroup (or in both) is required (see [4, 5]). In the present paper, semirings are not generally supposed to satisfy any of these additional properties.

Let $(S, +, \cdot)$ be a semiring and $p \in N$. Let also θ_p be a relation on the semiring S, introduced by

$$x\theta_p y \Longleftrightarrow x + py + x = y \wedge py + x + py = x \wedge 4px^2 = 4px,$$

i.e., $x\theta_p y \Longleftrightarrow x\tau_p y \wedge 4px^2 = 4px$. If $x\theta_p y$ for $x, y \in S$, then py is called the **p-element** of the element x. The semiring $(S, +, \cdot)$ is called the **p-semiring** if each element has its p-element. For a given p, let Σ_p denote the class of all p-semirings, i.e.,

$$S \in \Sigma_p \Longleftrightarrow (\forall x \in S)(\exists y \in S)(x\theta_p y).$$

An example of a p-semiring, when p is an arbitrary odd positive integer, is given by the following tables.

$+$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\cdot$	e	a	b	c
e	e	e	e	e
a	e	a	e	a
b	e	e	b	b
c	e	a	b	c

The additive semigroup of this semiring is a group and it is a p-semigroup.

We will use the following results, proved in [3].

Lemma 1. *Each element x of a p-semigroup has its own identity e_x, where $e_x = 4px$.*

Lemma 2. *Let x be an arbitrary element of p-semigroup and k the smallest positive integer such that $kx = e_x$. Then $k|4p$.*

Lemma 3. *Let x be an arbitrary element of a p-semigroup and $p = 4k + 3 (k \in N_0)$. Then $2px = e_x$.*

Lemma 4. *Let S be a semigroup and p an even positive integer, then*

$$S \in \Pi_p \Longleftrightarrow (\forall x \in S)((4p + 1)x = x).$$

Lemma 5. *If $p = 4k + 1 (k \in N_0)$, then the generalized quaternion group is a p-semigroup.*

2. On p-semigroups

Theorem 1. *A homomorphic image of a p-semigroup is a p-semigroup.*

Proof. Let f be a homomorphism which maps a p-semigroup $(S_1, +)$ onto a semigroup $(S_2, +)$ and let $x_2 \in S_2$. Then there exists $x_1 \in S_1$ such that $f(x_1) = x_2$. Since S_1 is a p-semigroup, there exists $y_1 \in S_1$ for which $x_1 \tau_p y_1$, i.e., such that $x_1 + py_1 + x_1 = y_1$ and $py_1 + x_1 + py_1 = x_1$. Therefore, for $y_2 = f(y_1)$ we have:
$x_2 + py_2 + x_2 = f(x_1) + pf(y_1) + f(x_1) = f(x_1 + py_1 + x_1) = f(y_1) = y_2$,
$py_2 + x_2 + py_2 = pf(y_1) + f(x_1) + pf(y_1) = f(py_1 + x_1 + py_1) = f(x_1) = x_2$.
So, $(S_2, +)$ is a p-semigroup. □

Theorem 2. *Let $S_i, i \in I$ be a family of semigroups and $p \in N$. Then $\prod(S_i, i \in I)$ is a p-semigroup if and only if S_i is a p-semigroup for every $i \in I$.*

Proof. Let $(S_i, +), i \in I$ be a family of p-semigroups and $x \in S$. Then, for every $i \in I$, there exists $a_i \in S_i$, such that $x(i)\tau_p a_i$. Let $a \in S$ be a function such that $a(i) = a_i$ for every $i \in I$. Then

$$(x + pa + x)(i) = x(i) + pa(i) + x(i) = x(i) + pa_i + x(i) = a_i = a(i), i \in I.$$

Hence, $x + pa + x = a$.

We also have $(pa+x+pa)(i) = pa(i)+x(i)+pa(i) = pa_i+x(i)+pa_i = x(i)$, so, $pa + x + pa = x$. Hence, $x\tau_p a$.

Conversely, let $S = \prod_{i \in I} S_i$ be a p-semigroup. Then for an arbitrary $x \in S$, there exists $a \in S$, such that $x\tau_p a$, respectively $x+pa+x = a$ and $pa+x+pa = x$. Let $x_i \in I, i \in I$, be arbitrary elements from the semigroups S_i. Let $x \in S$ be a function such that $x(i) = x_i$ for every $i \in I$. Then there exists $a \in S$ such that $x\tau_p a$. Furthermore, $(x+pa+x)(i) = a(i)$ and $(pa+x+pa)(i) = x(i)$, respectively $x(i) + pa(i) + x(i) = a(i)$, and $pa(i) + x(i) + pa(i) = x(i)$ for every $i \in I$. Since $x(i) = x_i$ for every $i \in I$, then $x_i + pa(i) + x_i = a(i)$ and $pa(i) + x_i + pa(i) = x_i$. Thus, for each $x_i \in S_i, i \in I$ there exists $a(i) \in S_i$, such that $x_i\tau_p a(i)$, so all the semigroups $S_i, i \in I$ are p-semigroups. □

Corollary 1. *The class of p-semigroups ($p \in N$) is closed under the operators H and P.*

In the following we give necessary and sufficient conditions under which a sub-semigroup of a p-semigroup is a p-semigroup too.

Theorem 3. *Let p be an odd positive integer and S a p-semigroup. Then every sub-semigroup of S is a p-semigroup if and only if $2px = e_x$ for every $x \in S$.*

Proof. Let each sub-semigroup of p-semigroup S be a p-semigroup and x an arbitrary element from an arbitrary p-sub-semigroup A of S. If k is the smallest positive integer such that $kx = e_x$, then, by Lemma 2, $k \mid 4p$. The semigroup $\langle x\rangle = \{e_x, x, 2x, \ldots, (k-1)x\}$ is a sub-semigroup of the semigroup A. Since $\langle x\rangle$ is a p-semigroup, then there exists $r \in \{0, 1, 2, \ldots, k-1\}$ such that $y = rx(0x = e_x)$ and $x\tau_p y$. Hence, $x + p(rx) + x = rx, p(rx) + x + p(rx) = x$. From the second equality we have that $r(2px) + x = x$, respectively $r(2px) = e_x$. If r is an odd positive integer, then $2px = e_x$. If $r = 0$, then from the first equality we have that $2x = e_x$, so $2px = e_x$. Let us consider the case when r is an even positive integer. If $r = 4r_0(r_0 \in N)$, then from the equality $x + p(rx) + x = rx$ we have: $rpx + 2x = rx, r_0(4px) + 2x = rx, 2x = rx$ ([2]). Since all elements of the cyclic group $\langle x\rangle$ are distinct and $r = 4r_0 \neq 2$, we conclude that r can not be of the form $4r_0$. Let $r = 4r_2 + 2(r_2 \in N_0)$. Then from the equality $x + p(rx) + x = rx$ we get: $r(px)+2x = rx, (4r_2+2)(px)+2x = (4r_2+2)x, r_2(4px)+2px+2x = 4r_2x+2x, 2px+2x = 4r_2x+2x, 2px+2x+(4p-2)x = 4r_2x+2x+(4p-2)x, 2px+4px = 4r_2x + 4px, 2px = 4r_2x, p(2px) = p(4r_2x), \frac{p-1}{2}(4px) + 2px = r_2(4px), 2px = e_x$. Hence, in any case $2px = e_x$.

Conversely, let $2px = e_x$ for every $x \in S$. Let x be an arbitrary element of any sub-semigroup A of S. Let $y = 2x$. It is clear that $y \in A$. Furthermore: $x+py+x = x+p(2x)+x = 2x+2px = 2x = y, py+x+py = p(2x)+x+p(2x) = x$, so $x\tau_p y$. Thus, the sub-semigroup A is a p-semigroup. □

Theorem 4. *Let p be an even positive integer or $p = 4k + 3(k \in N_0)$, and S a p-semigroup. Then every sub-semigroup of S is a p-semigroup.*

Proof. Let $p = 4k + 3(k \in N_0)$. By Lemma 3, $2px = e_x$ for every $x \in S$, so, by Theorem 3, each sub-semigroup of the semigroup S is a p-semigroup, too.

Let p be an even positive integer and let x be an arbitrary element of an arbitrary sub-semigroup A of S. Then $y = 2px + 2x$ is from the semigroup A, too. Since p is even, then $p(2px) = e_x$, so we have:

$$\begin{aligned} py + x + py &= p(2px + 2x) + x + p(2px + 2x) \\ &= p(2px) + 2px + x + p(2px) + 2px \\ &= 2px + x + 2px = 4px + x = x, \end{aligned}$$

$$x + py + x = x + p(2px + 2x) + x = p(2px) + 2px + 2x = 2px + 2x = y$$

Therefore, the sub-semigroup A is a p-semigroup. □

Corollary 2. *Let p be an even positive integer or $p = 4k + 3(k \in N_0)$. Then the class of p-semigroups is closed under the operator S.*

Summing up, we have the following.

Theorem 5. *If p is even or $p = 4k + 3(k = 0, 1, 2, \ldots)$ then the class of p-semigroups is a variety.*

We provide an explicit description of the above varieties.

Theorem 6. *Let $\Im$ be the variety of semigrups. The following holds:*

(a) If $p = 4k + 3 (k \in N_0)$ then Π_p is an equational class determined by the identity

$$(2p + 1)x = x;$$

(b) If p is even then Π_p is an equational class determined by the identity

$$(4p + 1)x = x.$$

Proof. (a) Let $S \in \Pi_p$. By Lemma 3, $2px = e_x$, respectively $(2p+1)x = x$ for every $x \in S$.

Conversely, let $(\forall x \in S)((2p + 1)x = x)$. Let $y = 2x$ and let us prove that $x\tau_p y$. We have:

$$\begin{aligned} x + py + x &= x + p(2x) + x = x + (2p + 1)x = x + x = y, \\ py + x + py &= p(2x) + x + p(2x) = 2px + (2p + 1)x = 2px + x = x. \end{aligned}$$

(b) The proof follows by Lemma 4 immediately. □

If $p = 4k+1 (k \in N_0)$, then the class Π_p is not a variety, since it is not closed under the operator S. Indeed, if

$$S = \{e_a, a, 2a, \ldots, (4p - 1)a, b, a + b, 2a + b, \ldots, (4p - 1)a + b\}$$

is a general quaternion group, then it has the property $2pa \neq e_a$. Hence, Theorem 3 is not satisfied. Observe that there are semigroups in the class Π_p which satisfy conditions of Theorem 3. Such is, e.g., the cyclic group $\{e_a, a\}$.

3. On p-semirings

Theorem 7 *A homomorphic image of a p-semiring is a p-semiring.*

Proof. Let f be a homomorphism which maps a p-semiring $(S_1, +, \cdot)$ onto a semiring $(S_2, +, \cdot)$ and $x_2 \in S_2$. Similarly as in Theorem 1 we prove that there exists $y_2 \in S_2$, such that $x_2\tau_p y_2$. Furthermore

$$\begin{aligned} 4px_2^2 &= 4p(f(x))^2 = 4p(f(x) \cdot f(x)) = 4pf(x \cdot x) \\ &= 4pf(x^2) = f(4px^2) = f(4px) = 4pf(x) = 4px_2, \end{aligned}$$

so $x_2\theta_p y_2$. □

Theorem 8. *Let $\{S_i, i \in I\}$ be a family of p-semirings and $p \in N$. Then, $S = \prod_{i \in I} S_i$ is a p-semiring if and only if S_i is a p-semiring for every $i \in I$.*

Proof. Let $(S_i, +, \cdot), i \in I$, be p-semirings and $x \in S$. Similarly as in Theorem 2 we prove that there exists $a \in S$ such that $x\tau_p a$. Furthermore $(4px^2)(i) = 4px^2(i) = 4px(i) = (4px)(i)$ for every $i \in I$, so $4px^2 = 4px$. Hence, $x\theta_p a$.

Conversely, let $S = \prod_{i \in I} S_i$ be a p-semiring. Similarly as in Theorem 2 we prove that there exist $a(i) \in S_i$ such that $x_i \tau_p a(i)$, for every $x_i \in S_i, i \in I$. Since $4px^2 = 4px$, then $(4px^2)(i) = (4px)(i)$, thus $4px^2(i) = 4px(i)$, for every $i \in I$. Hence, $x_i \theta_p a(i)$ for every $i \in I$, so all semirings $S_i, i \in I$, are p-semirings. □

Corollary 3. *The class of p-semiring $(p \in N)$ is closed under the operators H and P.*

Theorem 9. *Let p be even or $p = 4k + 3 (k \in N_0)$ and S a p-semiring. Then every sub-semiring of S is a p-semiring.*

Proof. If $(S, +, \cdot)$ is a p-semiring, then $(S, +)$ is a p-semigroup. If $(A, +, \cdot)$ is a sub-semiring of a p-semiring $(S, +, \cdot)$, then $(A, +)$ is a sub-semigroup of p-semigroup $(S, +)$. By Theorem 2.5., $(A, +)$ is a semigroup. Since $4px^2 = 4px$ for every $x \in S$, then $(A, +, \cdot)$ is a p-semiring. Hence, each sub-semiring of p-semiring S is a p-semiring, too. □

Corollary 4. *The class Σ_p of p-semirings, for even p or $p = 4k + 3 (k \in N_0)$ is a variety.*

Theorem 10. *If p is an even integer, then Σ_p is an equational class determined by the identities $(4p+1)x = x$ and $4px^2 = 4px$. If $p = 4k+3 (k \in N_0)$, then Σ_p is an equational class determined by the identities $(2p + 1)x = x$ and $4px^2 = 4px$.*

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