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UNIFORM METHODS FOR SEMILINEAR PROBLEMS WITH AN ATTRACTIVE BOUNDARY TURNING POINT

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Abstract. Two upwind finite difference schemes are considered for the numerical solution of a class of semilinear convection-diffusion problems with a small perturbation parameter ε and an attractive boundary turning point. We show that for both schemes the maximum nodal error is bounded by a special weighted ℓ_1-type norm of the truncation error. These results are used to establish ε-uniform pointwise convergence on Shishkin meshes.

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1. Introduction

In 1995, Andreev and Savin [2] introduced a new type of stability inequality for a finite difference scheme discretizing a linear singularly perturbed boundary value problem with a small positive perturbation parameter ε. Their stability inequality uses two different norms, because of which we refer to this kind of stability inequalities as the hybrid ones. The result of this is that the maximum pointwise (ℓ_∞) error of the numerical solution is bounded by a weighted ℓ_1-type norm of the truncation error. This approach can be applied to other types of singular perturbation problems for which only ℓ_1 ε-uniform convergence results were possible to prove previously. For instance, quasilinear problems are typically analyzed in an ℓ_1 norm, see [1] and [11]. ε–uniform convergence in an ℓ_1 norm was proved in [11] for a quasilinear convection-diffusion problem without turning points. This result was recently improved in [4] to the ℓ_∞ ε-uniform convergence due to the use of a hybrid-type stability inequality. Another class of problems that have so far been treated in an ℓ_1 norm is a class of attractive turning point problems, [10] and [13]. The main purpose of the present paper is to show that the hybrid stability inequality approach can be applied also to some problems of this type and that ε-uniform convergence can be proved in the ℓ_∞ norm.

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We consider the singularly perturbed semilinear convection-diffusion problem

(1) $\mathcal{T}u := -\varepsilon u'' - p(x)b(x)u' + c(x,u) = 0$ for $x \in (0,1)$, $u(0) = \gamma_0$, $u(1) = \gamma_1$,

where $0 < \varepsilon \ll 1$,

(2) $$\begin{aligned} &p(x) > 0 \text{ on } (0,1) \text{ and is monotonically increasing, while} \\ &b(x) \geq \beta > 0 \text{ and } c_u(x,u) \geq 0 \text{ for } (x,u) \in (0,1) \times \mathbb{R}. \end{aligned}$$

In sections 2 and 3 we analyze two upwind discretization schemes for problem (1), one first-order and the other a second-order scheme. For those schemes we derive appropriate hybrid stability inequalities. This is not a trivial generalization of the Andreev and Savin [2] result, since it requires a precise problem-adapted estimate of the discrete Green's function. We apply those results in section 4 to some special cases of problem (1). By using a Shishkin-type discretization mesh we are able to prove ε-uniform pointwise accuracy of order one or two (both up to logarithmic factors), depending on the scheme and the conditions on the problem. This is illustrated by numerical experiments.

The special case analyzed in section 4 belongs to the class of single attractive boundary turning point problems. This class includes the problem

(3) $$-\varepsilon u'' - xu' + xu = 0, \quad \text{for } x \in (0,1), \quad u(0) = \gamma_0,\ u(1) = \gamma_1,$$

which models heat flow and mass transport near oceanic rises, [3]. Multiple boundary turning points ($p(0) = p'(0) = 0$) are also covered by (1) and they too arise in applications, see [8].

Our technique does not apply to interior turning point problems, such as the differential equation in (3) considered on the interval $(-1,1)$. The aforementioned paper [10] deals with single interior turning point problems but its result is also true for the single boundary turning point case. This case is what we improve on in the present paper. An additional improvement is in the simplicity of the discretization mesh as compared to the complicated mesh of Bakhvalov type used in [10]. A single interior turning point problem is considered in [13] as well, but in a more general quasilinear case, that we cannot extend our technique to.

It should be mentioned that Liseikin [5] proves first-order ε-uniform convergence in the ℓ_∞ norm for single boundary turning point problems like (3). His numerical method, however, is too complicated: the differential equation is transformed using an appropriate substitution for x and then the transformed problem is discretized on an equidistant mesh. Our method is much simpler, since we discretize the problem directly on a Shishkin-type mesh.

We would like to point out that problems with a cusp layer (for those, see [9] and the references therein) usually satisfy $c_u(x,u) \geq c_* > 0$. These problems are stable in the ℓ_∞ norm and their numerical analysis requires no hybrid stability inequality. Nevertheless, our results in sections 2 and 3 apply to them as well.

On numerical methods for singular perturbation problems in general, see [6] and [7] for instance.

2 A first-order upwind scheme

Let N, our discretization parameter, be a positive integer. Let $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary mesh and let $h_i = x_i - x_{i-1}$, $i = 1, \ldots, N$. We discretize using the following simple upwind scheme,

$$\left[T_\kappa^N u^N\right]_i = 0 \quad \text{for} \quad i = 1, \ldots, N-1, \quad u_0^N = \gamma_0, \; u_N^N = \gamma_1, \tag{4}$$

where

$$\left[T_\kappa^N v\right]_i := -\frac{\varepsilon}{\chi_i}\left(\frac{v_{i+1} - v_i}{h_{i+1}} - \frac{v_i - v_{i-1}}{h_i}\right) - p_i b_i \frac{v_{i+1} - v_i}{\chi_i} + c(x_i, v_i)$$

with $\chi_i = \kappa h_i + (1-\kappa) h_{i+1}$ and $\kappa \in [0,1]$ fixed.

The following Theorem states stability results for the difference operator T_κ^N. The proof uses a linearization technique and a barrier function argument for the discrete Green's function of the linear operator obtained. We introduce the discrete maximum norm

$$\|v\|_\infty := \max_{i=0,\ldots,N} |v_i|.$$

Theorem 1. *Assume (2) and let v and w be two arbitrary mesh functions with $v_0 = w_0$ and $v_N = w_N$. Then*

$$\|v - w\|_\infty \le \frac{1}{\beta} \sum_{j=1}^{N-1} \frac{\chi_j}{p_j} \left| \left[T_\kappa^N v - T_\kappa^N w\right]_j \right|. \tag{5}$$

Proof. Let v and w be the two mesh functions for which we want to prove (5). Following the usual practice, we define the discrete linear operator

$$\left[L_\kappa^N y\right]_i := -\frac{\varepsilon}{\chi_i}\left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) - p_i b_i \frac{y_{i+1} - y_i}{\chi_i} + \bar{c}_i y_i, \quad y_0 = y_N = 0,$$

where

$$\bar{c}_i = \int_0^1 c_u\big(x_i, w_i + s(v_i - w_i)\big)\, ds \ge 0.$$

The operators L_κ^N and T_κ^N are related by

$$L_\kappa^N v - L_\kappa^N w = L_\kappa^N \big(v - w\big) = T_\kappa^N v - T_\kappa^N w. \tag{6}$$

For the linear operator L_κ^N and arbitrary mesh functions y with $y_0 = y_N = 0$ we have

$$y_i = \sum_{j=1}^{N-1} \chi_j G_i^j \left[L_\kappa^N y\right]_j \quad \text{for} \quad i = 1, \ldots, N-1, \tag{7}$$

where G is the discrete Green's function associated with L_κ^N. For arbitrary fixed j, G satisfies

$$\left[L_\kappa^N G^j\right]_i = \delta_{ij}^N \chi_i^{-1} \quad \text{for} \quad i = 1, \ldots, N-1, \quad G_0^j = G_N^j = 0,$$

with

$$\delta_{ij}^N = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

The operator L_κ^N satisfies a discrete comparison principle since the matrix associated with L_κ^N is an M-matrix (an inverse–monotone L-matrix). This is easily verified using the M-matrix criterion with the test function $z_i = 1 - x_i$.

We construct a barrier function for G now. Let $\beta_i = \beta p_i$,

$$R_i^j := \begin{cases} 1 & \text{for} \quad i = j+1, \\ \displaystyle\prod_{\mu=j+1}^{i-1} \left(1 + \frac{\beta_\mu h_{\mu+1}}{\varepsilon}\right)^{-1} & \text{for} \quad i = j+2, \ldots, N, \end{cases}$$

$$Q_i^j := \begin{cases} 0 & \text{for} \quad i = 0, \ldots, j, \\ \displaystyle\frac{1}{\varepsilon + \beta_j h_{j+1}} \sum_{\nu=j+1}^{i} h_\nu R_\nu^j & \text{for} \quad i = j+1, \ldots, N, \end{cases}$$

and

$$B_i^j := \begin{cases} Q_N^j & \text{for } i = 0, \ldots, j, \\ Q_N^j - Q_i^j & \text{for } i = j+1, \ldots, N. \end{cases}$$

Clearly, B_i^j satisfies

(8) $$0 \leq B_i^j \leq Q_N^j \quad \text{for} \quad i = 0, \ldots, N,$$

since Q_i^j monotonically increases with i. Now we shall show that

(9) $$\left[L_\kappa^N B^j\right]_i \geq \delta_{ij}^N \chi_i^{-1} \quad \text{for} \quad i = 1, \ldots, N-1.$$

We have

$$\frac{B_i^j - B_{i-1}^j}{h_i} = \begin{cases} 0 & \text{for} \quad i = 1, \ldots, j, \\ -\displaystyle\frac{R_i^j}{\varepsilon + \beta_j h_{j+1}} & \text{for} \quad i = j+1, \ldots, N. \end{cases}$$

Thus

$$\begin{aligned} \left[L_\kappa^N B^j\right]_i &= \bar{c}_i B_i^j \geq 0 \quad \text{for } i = 1, \ldots, j-1, \\ \left[L_\kappa^N B^j\right]_j &= -\frac{\varepsilon + b_j p_j h_{j+1}}{\chi_j} \frac{B_{j+1}^j - B_j^j}{h_{j+1}} + \bar{c}_j B_j^j \geq \frac{1}{\chi_j}, \end{aligned}$$

and

$$\begin{aligned}\left[L_\kappa^N B^j\right]_i &= -\frac{\varepsilon + b_i p_i h_{i+1}}{\chi_i}\frac{B_{i+1}^j - B_i^j}{h_{i+1}} + \frac{\varepsilon}{\chi_i}\frac{B_i^j - B_{i-1}^j}{h_i} + \bar{c}_i B_i^j \\ &\geq \frac{\left(\varepsilon + b_i p_i h_{i+1}\right) R_{i+1}^j - \varepsilon R_i^j}{\chi_i\left(\varepsilon + \beta_j h_{j+1}\right)} \geq 0 \quad \text{for } i = j+1, \dots, N-1,\end{aligned}$$

because $\varepsilon R_i^j = \left(\varepsilon + \beta_i h_{i+1}\right) R_{i+1}^j$. This completes the proof of (9).

Since L_κ^N satisfies the discrete comparison principle, from (8) and (9), we get

$$0 \leq G_i^j \leq B_i^j \leq Q_N^j \quad \text{for} \quad i, j = 1, \dots, N-1. \tag{10}$$

Next we show that

$$Q_N^j \leq \frac{1}{\beta_j} \quad \text{for} \quad j = 1, \dots, N-1. \tag{11}$$

From the definition of Q we have

$$Q_N^N = 0 \quad \text{and} \quad Q_N^{j-1} = \frac{1}{\beta_{j-1}} + \frac{\varepsilon}{\beta_{j-1}}\frac{\beta_{j-1} Q_N^j - 1}{\varepsilon + \beta_{j-1} h_j}.$$

Induction for $j = N, N-1, \dots, 2$ yields (11) because of the monotonicity of p.

Finally, combine (10) and (11) with (7) and (6) with $y = v - w$. □

Remark 1 *For $p \equiv 1$ we recover the stability results from [2] for linear problems and from [4, Lemma 2] for quasilinear problems in conservative form.*

An immediate consequence of Theorem 1 for the simple upwind scheme is

$$\left\|u - u^N\right\|_\infty \leq \frac{1}{\beta}\sum_{j=1}^{N-1}\frac{\chi_j}{p_j}\left|\left[T_\kappa^N u\right]_j\right|.$$

Thus the error of the numerical solution in the maximum norm is bounded by an ℓ_1-type norm of the truncation error weighted with the inverse of the coefficient of the convection term.

3 A second-order upwind scheme

In this section we consider a second-order upwind scheme for the discretization of (1). In addition to (2) we shall assume that there exists a positive constant α such that

$$p(x)b(x) \geq \alpha c_u(x, u) \quad \text{for all} \quad (x, u) \in (0, 1) \times \mathbb{R}. \tag{12}$$

This is a technical condition required by our method of proof. Nevertheless, the equation (3) satisfies it with $\alpha = 1$.

Our scheme is a combination of the standard central difference scheme

$$\left[T_c^N v\right]_i := -\frac{\varepsilon}{\hbar_i}\left(\frac{v_{i+1}-v_i}{h_{i+1}} - \frac{v_i - v_{i-1}}{h_i}\right) - p_i b_i \frac{v_{i+1}-v_{i-1}}{2\hbar_i} + c(x_i, v_i)$$

and the midpoint-upwind scheme

$$\begin{aligned}\left[T_{mp}^N v\right]_i := -\frac{\varepsilon}{h_{i+1}}\left(\frac{v_{i+1}-v_i}{h_{i+1}} - \frac{v_i - v_{i-1}}{h_i}\right)\\ -(pb)_{i+1/2}\frac{v_{i+1}-v_i}{h_{i+1}} + c\Big(x_{i+1/2}, \frac{v_{i+1}+v_i}{2}\Big),\end{aligned}$$

where $x_{i+1/2} = x_i + h_{i+1}/2$. Let $\mathcal{I}$ denote the set of indices for which the central difference discretization is stable, i.e., $\mathcal{I} = \{i : h_i p_i b_i \le 2\varepsilon\}$. Let $I \subseteq \mathcal{I}$ be arbitrary. The discrete problem reads: find $u^N \in I\!R^{N+1}$ such that

$$\left[T_\sigma^N u^N\right]_i = 0 \text{ for } i = 1, \dots, N-1, \quad u_0^N = \gamma_0, \ u_N^N = \gamma_1, \tag{13}$$

where

$$\left[T_\sigma^N v\right]_i := \begin{cases} \left[T_c^N v\right]_i & \text{if } i \in I, \\ \left[T_{mp}^N v\right]_i & \text{otherwise.} \end{cases}$$

Before stating our stability result for T_σ^N we have to introduce some more notation. Let

$$\sigma_i = \begin{cases} \hbar_i & \text{if } i \in I, \\ h_{i+1} & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta_i = \begin{cases} 2\beta p_i & \text{if } i \in I, \\ \beta p_{i+1/2} & \text{otherwise.} \end{cases}$$

Theorem 2. *Assume (2), (12), and that $h_i \le 2\alpha$ for $i = 1, \dots, N$. Let v and w be two arbitrary mesh functions with $v_0 = w_0$ and $v_N = w_N$. Then*

$$\|v - w\|_\infty \le \sum_{j=1}^{N-1} \frac{\sigma_j}{\beta_j}\left|\left[T_\sigma^N v - T_\sigma^N w\right]_j\right|. \tag{14}$$

Proof. The stability analysis for T_σ^N is similar to that for T_κ^N. We start by linearizing T_σ^N. Let

$$\begin{aligned}\left[L_c^N y\right]_i &:= -\frac{\varepsilon}{\hbar_i}\left(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) - p_i b_i \frac{y_{i+1}-y_{i-1}}{2\hbar_i} + c_i^0 y_i,\\ \left[L_{mp}^N y\right]_i &:= -\frac{\varepsilon}{h_{i+1}}\left(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) - p_{i+1/2} b_{i+1/2}\frac{y_{i+1}-y_i}{h_{i+1}}\\ &\quad + \frac{c_i^+ y_{i+1} + c_i^- y_i}{2},\end{aligned}$$

and

$$\left[L_\sigma^N v\right]_i := \begin{cases} \left[L_c^N v\right]_i & \text{if } \ i \in I, \\ \left[L_{mp}^N v\right]_i & \text{otherwise,} \end{cases}$$

where

$$c_i^0 = \int_0^1 c_u\big(x_i, w_i + s(v_i - w_i)\big)\, ds, \quad c_i^- = \int_0^1 c_u\big(x_{i+1/2}, w_i + s(v_i - w_i)\big)\, ds$$

and

$$c_i^+ = \int_0^1 c_u\big(x_{i+1/2}, w_{i+1} + s(v_{i+1} - w_{i+1})\big)\, ds.$$

This construction ensures

$$L_\sigma^N\big(v - w\big) = T_\sigma^N v - T_\sigma^N w. \tag{15}$$

The combination of central differencing with the midpoint upwind scheme and $h_i \le 2\alpha$ guarantee that L_σ^N is an L-matrix. Using the test function $z_i = 1 - x_i$ on can easily verify L_σ^N is also an M-matrix.

The discrete Green's function associated with L_σ^N satisfies

$$\left[L_\sigma^N G^j\right]_i = \delta_{ij}^N \sigma_i^{-1} \ \text{ for } \ i = 1, \dots, N-1, \ \ G_0^j = G_N^j = 0.$$

The construction of the barrier function of G is only slightly different from that for the simple upwind operator. Let

$$R_i^j := \begin{cases} 1 & \text{for} \quad i = j+1, \\ \displaystyle\prod_{\mu=j+1}^{i-1} \left(1 + \frac{\beta_\mu h_{\mu+1}}{\varepsilon}\right)^{-1} & \text{for} \quad i = j+2, \dots, N, \end{cases}$$

$$Q_i^j := \begin{cases} 0 & \text{for} \quad i = 0, \dots, j, \\ \displaystyle\frac{1}{\varepsilon + \beta_j h_{j+1}} \sum_{\nu=j+1}^{i} h_\nu R_\nu^j & \text{for} \quad i = j+1, \dots, N, \end{cases}$$

and

$$B_i^j := \begin{cases} Q_N^j & \text{for } i = 0, \dots, j, \\ Q_N^j - Q_i^j & \text{for } i = j+1, \dots, N. \end{cases}$$

This B_i^j is a barrier function for G_i^j. □

4 Application to a special problem

We now consider the special case $p(x) = x$, $c(x,u) = xg(x,u)$, thus we are interested in the problem

(16) $\mathcal{T}u := -\varepsilon u'' - xb(x)u' + xg(x,u) = 0$ for $x \in (0,1)$, $u(0) = \gamma_0$, $u(1) = \gamma_1$.

We shall derive a uniform maximum-norm error estimates for the first-order scheme on Shishkin meshes using Theorem 1. We need first some results on the solution of (16).

Throughout this section we assume the following minimum smoothness conditions

$$b \in C^2[0,1] \quad \text{and} \quad g \in C^2([0,1] \times W),$$

where $W \subset I\!R$ is described below. Also, analogously to (2), let

(17) $$b(x) \geq \beta > 0 \quad \text{and} \quad g_u(x,u) \geq 0 \quad \text{for} \quad (x,u) \in (0,1) \times W.$$

Then we can construct an upper solution $\bar{u}$ of (16),

(18) $$\bar{u}(x) = |\gamma_0| + |\gamma_1| + \frac{G}{\beta}(2 - x), \quad G = \max_{0 \leq x \leq 1} |g(x,0)|,$$

whereas $-\bar{u}$ is a lower solution. This construction can be found in [5]. Since the operator $\mathcal{T}$ is inverse monotone, this means that problem (16) has a unique solution, $u \in C^4[0,1]$, and moreover,

$$u(x) \in W := \left[-\bar{u}(0), \bar{u}(0)\right] \quad \text{for} \quad x \in [0,1].$$

Let $u_0 \in C^3[0,1]$ be the unique solution to the reduced problem

$$-b(x)u' + g(x,u) = 0, \quad \text{for} \quad x \in (0,1), \quad u(1) = \gamma_1.$$

Let also $\mu = \sqrt{\varepsilon}$. By C, sometimes subscripted, we denote throughout the paper a generic positive constant which is independent of ε and N, the number of steps in the mesh ω.

Lemma 1. *Let (17) hold true. Then the solution u of (16) satisfies*

(19) $$\left|(u - u_0)(x)\right| \leq C\left(\mu + e^{-mx/\mu}\right),$$

(20) $$\left|u^{(i)}(x)\right| \leq C\left(\mu^{\min\{0,2-i\}} + \mu^{-i} e^{-mx/\mu}\right),$$

where $x \in [0,1]$, $i = 0,1,2,3$, and $m > 0$ is an arbitrary constant independent of ε.

Proof. See in [10]. Note that the estimates in [5] are less sharp because u_0 was not used in proving them. □

4.1 The discretization mesh

We use a slightly generalized Shishkin mesh which we denote by $S(L)$, where $L = L(N)$ stands for any quantity satisfying $L \le \ln N$ and

$$e^{-L} \le \frac{L}{N}. \tag{21}$$

Let $\tau = a\mu L$ with an arbitrary positive number a. Also, let $J = qN$ be a positive integer such that $q < 1$ and $q^{-1} \le C$. We assume that $a\mu \ln N \le q$, since N is unreasonably large otherwise. Therefore, $\tau \le q$. Then we form the mesh $S(L)$ by dividing the interval $[0, \tau]$ into J equidistant subintervals and the interval $[\tau, 1]$ into $N - J$ equidistant subintervals. Note that $x_J = \tau$. The standard Shishkin mesh uses $L = \ln N$, typically with $q = \frac{1}{2}$. The use of L instead of $\ln N$ is for practical and not theoretical reasons, since any L behaves like $\ln N$ when $N \to \infty$, see [12]. Still, as L may be less than $\ln N$ in practice, with such an L we get a mesh which is denser in the layer. This is very likely to improve the numerical results.

4.2 Analysis of the first-order upwind scheme

Let us now discretize the problem (16) on the $S(L)$ mesh by using the first-order upwind scheme (4). It is easy to see that the discrete problem has a unique solution u^N. Its uniqueness follows from (5). To show that (4) has a solution, we construct its upper and lower solutions in the same way as for the continuous problem. Indeed, using $\bar{u}$ as defined in (18), we get

$$\left[T_\kappa^N \bar{u}\right]_i = x_i \left[\frac{b_i}{\beta} \cdot \frac{h_{i+1}}{\chi_i} G + g(x_i, \bar{u}_i)\right] \ge x_i [G + g(x_i, 0)] \ge 0,$$

where we have used (17) and the fact that $S(L)$ satisfies $h_{i+1} \ge \chi_i$, since $h_{i+1} \ge h_i$ being equivalent to $\tau \le q$. Similarly, $-\bar{u}$ is a lower solution of (4). The solution u^N therefore exists, and moreover, $u_i^N \in W$ analogously to the continuous solution.

We are now ready to prove the almost first-order ε-uniform convergence result. For the technique of proof cf. [4], [11], and [12].

Theorem 3. *Let u be the solution of problem (16) satisfying (17). Then the following ε-uniform convergence result holds true for the solution u^N of the discrete problem (4) on the $S(L)$ mesh*

$$\left\|u - u^N\right\|_\infty \le C \frac{L^2}{N}.$$

Proof. Let $1 \le i \le i^* \le N - 1$ for some integers i and i^* and let

$$\Sigma_i^{i^*} = \sum_{j=i}^{i^*} \frac{\chi_j}{x_j} r_j, \quad r_j = \left|[T_\kappa^N u]_j\right|.$$

Because of Remark 1 on (5), it suffices to prove that

$$\Sigma_i^{i^*} \leq C\frac{L^2}{N} \tag{22}$$

for $i = 1$ and $i^* = N-1$. We divide this proof into several steps.

Let us first consider the fine part of $S(L)$ on the interval $(0, \tau)$ and the corresponding Σ_1^{J-1}. Note that here $\chi_j = h$ and $x_j = jh$ where h is the fine mesh step-size,

$$h = \frac{\tau}{J} \leq C\frac{\mu L}{N}.$$

Expanding the consistency error $[T_\kappa^N u]_j$ and using (20), we get

$$r_j \leq C\left[\varepsilon h\left(\frac{1}{\mu} + \frac{1}{\mu^3}e^{-mx_{j-1}/\mu}\right) + x_j h\left(1 + \frac{1}{\mu^2}e^{-mx_j/\mu}\right)\right]$$

and

$$\frac{\chi_j}{x_j} r_j \leq Ch^2\left[1 + \frac{\mu}{x_j} + \left(\frac{1}{\mu x_j} + \frac{1}{\mu^2}\right)e^{-mx_{j-1}/\mu}\right]. \tag{23}$$

From here it follows that

$$\frac{\chi_j}{x_j} r_j \leq C\left(h^2 + \frac{\mu h}{j} + \frac{h}{\mu j} + \frac{h^2}{\mu^2}\right)$$

and

$$\Sigma_1^{J-1} \leq C\left(\frac{L^2}{N} + \frac{L}{N}\sum_{j=1}^{J-1}\frac{1}{j}\right) \leq C\frac{L^2}{N},$$

since

$$\sum_{j=1}^{J-1}\frac{1}{j} \leq C\int_1^J \frac{ds}{s} \leq C\ln J \leq CL$$

(the last inequality is satisfied because, as we have mentioned, L behaves like $\ln N$ as $N \to \infty$). Thus, (22) holds true for $i = 1$ and $i^* = J-1$.

Let us now consider the coarse part of $S(L)$ on the interval $(\tau + H, 1)$, where $H \leq C/N$ is the coarse mesh step-size. The corresponding part of the sum Σ_1^{N-1} is Σ_{J+2}^{N-1}. We use (23) again but with H instead of h. This time we can estimate the exponential expression much better,

$$e^{-mx_{j-1}/\mu} \leq e^{-m(\tau+H)/\mu} \leq \left(\frac{L}{N}\right)^{am} e^{-mH/\mu},$$

where we have used (21). From here and $x_j > \tau$, we get

$$\frac{\chi_j}{x_j} r_j \leq C\left[H^2 + \left(\frac{L}{N}\right)^{am}\left(\frac{H}{\mu}\right)^2 e^{-mH/\mu}\right] \leq C\left[\frac{1}{N^2} + \left(\frac{L}{N}\right)^{am}\right].$$

As m is an arbitrary constant, we can set above that $m = 2/a$. Then (22) follows in this case, i.e. for $i = J + 2$ and $i^* = N - 1$.

To finish the proof of (22) for $i = 1$ and $i^* = N - 1$, we just have to estimate the two remaining terms,

$$\frac{\chi_j}{x_j} r_j \quad \text{for} \quad j = J, J+1.$$

When $\mu \geq 1/N$, we proceed like in the previous case, but using

$$e^{-mx_{j-1}/\mu} \leq e^{-m(\tau-h)/\mu} \leq C\left(\frac{L}{N}\right)^{am}. \tag{24}$$

Since $am = 2$, we get

$$\frac{\chi_j}{x_j} r_j \leq C\left[H^2 + \left(\frac{L}{N}\right)^2 \left(\frac{1}{\mu N}\right)^2\right] \leq C\left(\frac{L}{N}\right)^2 \leq C\frac{L^2}{N}.$$

On the other hand, if $\mu \leq 1/N$, we use a different estimate,

$$\frac{\chi_j}{x_j} r_j \leq \frac{2\varepsilon}{x_j} \max_{[x_{j-1}, x_{j+1}]} \left|u'(x)\right| + b_j \left|u_{j+1} - u_j\right| + \chi_j \left|g(x_j, u_j)\right|. \tag{25}$$

We now make use of (19) as follows,

$$\left|u_{j+1} - u_j\right| \leq \left|u_{j+1} - u_{0,j+1}\right| + \left|u_j - u_{0,j}\right| + CH \leq C\left(\mu + e^{-mx_{j-1}/\mu} + H\right).$$

Using (20), (24), $x_j \geq \tau$, and the above estimate in (25), we obtain

$$\frac{\chi_j}{x_j} r_j \leq C\left(\mu + e^{-mx_{j-1}/\mu} + H\right) \leq \frac{C}{N} \leq C\frac{L^2}{N},$$

which completes the proof of the theorem. □

4.3 Analysis of the second-order upwind scheme

We now consider the second-order scheme T_σ^N on the $S(L)$ mesh. For a special case of problem (16), we prove below an almost second-order ε-uniform accuracy,

$$\|u - u^N\|_\infty \leq C\frac{L^3}{N^2}. \tag{26}$$

We assume in this subsection that $b \in C^3[0, 1]$ and $g \in C^3([0, 1] \times W)$. Then it is possible to prove (20) for $i = 4$, using the same technique as in [10]. However, that is not enough to prove (26) for the general problem (16). Already the estimate of $|u^{(3)}(x)|$ makes it difficult to prove (26). The separate $1/\mu$-term spoils the proof on the coarse mesh when estimating the truncation error of the scheme for $xb(x)u'$. Also, the technique used in (25) cannot give in general more than first-order accuracy.

Note that we still may treat T_σ^N as a first-order scheme and prove the result of Theorem 3 for it. This means that T_σ^N cannot perform asymptotically worse than the first-order upwind scheme, but it is reasonable to expect better results even when a rigorous proof is missing.

We show below that our technique can be applied to a special case of problem (16) and that we can still prove (26) for that case. In addition to (17), let

$$g(x,\gamma_1)=0, \tag{27}$$

so that the reduced solution is $u_0 \equiv \gamma_1$.

Lemma 2. *Let 17 and 27 hold true. Then the solution u of 16 satisfies*

$$|[u(x)-\gamma_1]^{(i)}| \le C\mu^{-i}e^{-mx/\mu},$$

where $x \in [0,1]$, $i=0,\ldots,4$, and $m>0$ is an arbitrary constant independent of ε.

Proof. We prove the following sharper estimates,

$$|[u(x)-\gamma_1]^{(i)}| \le C\left(e^{-m/\mu}+\mu^{-i}e^{-B(x)/\varepsilon}\right), \tag{28}$$

where $B(x)=\int_0^x sb(s)ds$. To prove (28) for $i=0$, we linearize the operator $\mathcal{T}$,

$$\mathcal{L}v := -\varepsilon v'' - xb(x)v' + xf(x)v,$$

with

$$f(x)=\int_0^1 g_u(x, su(x)+(1-s)\gamma_1)ds,$$

so that

$$\mathcal{L}(u-\gamma_1)=\mathcal{T}u-\mathcal{T}\gamma_1=0.$$

Since $f(x)\ge 0$, $\mathcal{L}$ is an inverse monotone operator. We construct the barrier function

$$z(x)=C_1(2-x)e^{-\eta/\varepsilon}+\gamma_0 e^{-B(x)/\varepsilon}$$

with some positive η independent of ε. We get $z(0)\ge|\gamma_0|$, $z(1)\ge 0$, and

$$\mathcal{L}z(x)\ge xb(x)C_1e^{-\eta/\varepsilon}+\gamma_0[xb(x)]'e^{-B(x)/\varepsilon}.$$

Now there exists a positive constant δ independent of ε, such that $[xb(x)]'\ge 0$ for $x\in[0,\delta]$. Therefore, $\mathcal{L}z(x)\ge 0$ for $x\in[0,\delta]$. On the other hand, even if $[xb(x)]'<0$ for some $x\in[\delta,1]$, the term $\exp(-B(x)/\varepsilon)$ is exponentially small on that interval, and thus we can choose C_1 and η so that $\mathcal{L}z(x) \geq 0$ on $[\delta, 1]$ as well. Inverse monotonicity implies now that

$$|u(x) - \gamma_1| \leq z(x) \quad \text{for} \quad x \in [0,1],$$

which proves (28) for $i = 0$.

The remaining estimates for $i = 1, \ldots, 4$ can be proved by using the technique from [10]. □

We can now prove (26) for this special type of problem.

Theorem 4. *Let u be the solution of problem (16) satisfying (12), (17), and (27). Let also u^N be the solution of the discrete problem (13) with $\{1, \ldots, J-1\} \subseteq I \subseteq \mathcal{I}$, on the $S(L)$ mesh. Then (26) holds true provided N is sufficiently large but independent of ε.*

Proof. For N sufficiently large independently of ε we have $h_i x_i b_i \leq 2\varepsilon$, $i = 1, \ldots, J-1$. Thus $\{1, \ldots, J-1\} \subseteq I \subseteq \mathcal{I}$ and the central scheme T_c^N is used on the fine mesh. Furthermore if N is sufficiently large then $h_i \leq 2\alpha$, $i = 1, \ldots, N$. Therefore, Theorem 2 can be applied.

Using the technique of proof of Theorem 3 it is easy to show that

$$\sum_{j=1}^{J-1} \frac{\sigma_j}{\beta_j} |[T_c^N u]_j| = \frac{1}{2\beta} \sum_{j=1}^{J-1} \frac{1}{j} |[T_c^N u]_j| \leq C \frac{L^3}{N^2}. \tag{29}$$

Let us now consider $x_j \geq \tau$. Regardless of whether T_c^N or T_{mp}^N is used at x_j, we can apply the same approach as in the estimate (25) to get

$$\frac{\sigma_j}{\beta_j} |[T_\sigma^N u]_j| \leq C \left[\frac{\varepsilon}{x_j} \max_{[x_{j-1}, x_{j+1}]} |u'(x)| + R_j \right],$$

where

$$R_j = |u_{j+1} - \tilde{u}_j| + N^{-1} \tilde{g}_j,$$

and where $\tilde{u}_j$ stands for either u_{j-1} or u_j and $\tilde{g}_j$ is either $g(x_j, u_j)$ or $g(x_{j+1/2}, (u_{j+1} + u_j)/2)$, depending on what scheme is used at x_j. Because of (27), $u_0 \equiv \gamma_1$, and Lemma 2, we get

$$R_j \leq C \left(1 + N^{-1}\right) e^{-m x_{j-1}/\mu}.$$

This implies

$$\sum_{j=J}^{N-1} \frac{\sigma_j}{\beta_j} |[T_\sigma^N u]_j| \leq C \sum_{j=J}^{N-1} \left(\frac{\varepsilon}{\tau\mu} + 1 + \frac{1}{N} \right) e^{-m x_{j-1}/\mu} \leq C \frac{L^3}{N^2}, \tag{30}$$

where in the last step we used (24) with $m = a/3$. The assertion now follows from (29), (30), and (14). □

4.4 Numerical results

In this section we verify experimentally our convergence result for the first-order scheme. Our test problem is

$$-\varepsilon u'' - x(2-x)u' + xe^u = 0 \quad \text{for} \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{31}$$

This problem satisfies (17) with $\beta = 1$. The exact solution of this problem is not available. We therefore estimate the accuracy of the numerical solution by comparing it to the numerical solution on a finer mesh. For our tests we take $\tau = \sqrt{\varepsilon}L(N)$ and $q = 1/2$.

Indicating by u_ε^N that the numerical approximation of (31) depends on both N and ε, we estimate the uniform error by

$$\eta^N := \max_{\varepsilon=1,10^{-1},\ldots,10^{-12}} \left\| u_\varepsilon^N - \tilde{u}_\varepsilon^{8N} \right\|_\infty,$$

where $\tilde{u}_\varepsilon^{8N}$ is the approximate solution of the first-order scheme on a mesh obtained by bisecting the original mesh three times, i. e., a mesh that is 8 times finer. The rates of convergence are computed using the standard formula $r^N = \ln\left(\eta^N/\eta^{2N}\right)/\ln 2$.

	Shishkin mesh		$S_E(L)$ mesh	
N	error	rate	error	rate
64	2.437e-2	0.85	2.152e-2	0.88
128	1.352e-2	0.87	1.173e-2	0.89
256	7.408e-3	0.88	6.350e-3	0.89
512	4.015e-3	0.90	3.418e-3	0.90
1024	2.158e-3	0.90	1.830e-3	0.91
2048	1.153e-3	0.91	9.760e-4	0.91
4096	6.126e-4	0.92	5.187e-4	0.92
8192	3.242e-4	—	2.748e-4	—

Table 1: First-order upwind scheme, $\kappa = 1.00$

	Shishkin mesh		$S_E(L)$ mesh	
N	error	rate	error	rate
64	4.165e-3	1.55	2.233e-3	1.54
128	1.419e-3	1.62	7.701e-4	1.59
256	4.625e-4	1.66	2.561e-4	1.63
512	1.462e-4	1.70	8.273e-5	1.67
1024	4.512e-5	1.72	2.608e-5	1.69
2048	1.365e-5	1.75	8.055e-6	1.72
4096	4.061e-6	1.77	2.445e-6	1.74
8192	1.192e-6	—	7.308e-7	—

Table 2: Second-order upwind scheme

The results of our test computations are given in Table 1. They are clear illustrations of the almost first-order convergence proved in Theorem 3. We also see that the $S_E(L)$ mesh (the mesh with an $L(N)$ that satisfies (21) with equality) performs slightly better than the standard Shishkin mesh.

In Table 2 we present numerical results for the second-order scheme with $I = \{1, \ldots, J-1\}$ when applied to our test problem. We observe almost second-order convergence although Theorem 4 does not apply since (27) is not satisfied by our test problem.

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