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APPLICATIONS OF DECOMPOSABLE MEASURES ON NONLINEAR DIFFERENCE EQUATIONS

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Abstract. A class of nonlinear difference equations important in the theory of the architecture of computers with parallel processing (neural nets) is examined. The integral with respect to decomposable measure is used as main tool. By a generalization of the theory of generalized functions the limit behavior of the considered difference equation is characterized.

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1. Introduction

In this paper we shall use a class of non-additive real-valued set functions generally non-additive with respect to the usual addition of real numbers, but which are "additive" (decomposable) with respect to new operation (t-conorm, pseudo-addition, etc.). Namely, the idea of generalizing the usual sum and product by triangular conorm (briefly: t-conorm and t-norm, respectively) is due to Menger. Schweizer and Sklar [19] introduced this notion in today's form defining Menger spaces as special cases of more general probabilistic spaces.
M. Sugeno [20] considered monotone set functions (fuzzy measures, capacities in the Choquet sense) and defined an integral for this set functions. These results found important applications in the fuzzy set theory.
Sugeno investigated special non-aditive set function m, with the property:

$$m(A \cup B) = m(A) + m(B) + \lambda m(a)m(B)$$

where $A \cap B = \emptyset$ and $\lambda > -1$.
Following this line, only taking a general t-conorm $\perp$ Dubois and Prade [2], and S. Weber [21] suggest the investigation of $\perp$ $-$decomposable measures:

$$m(A \cup B) = m(A) \perp m(B)$$

for $A \cap B = \emptyset$. It turns out that these important measures have many interesting properties (S. Weber [21],[21], E. Pap [9] - [15]).

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Using decomposable measures and the corresponding integral we have developed the $g-$calculus, which is very useful in the theory of nonlinear differential and integral equations (V. Maslov, S.N. Samborski [6], E. Pap [12],[14],[15] and E. Pap, N. Ralević [16],[17]).
In this paper we apply decomposable measures on a class of nonlinear difference equations important in the theory of architecture of computers with parallel processing (neural nets).

The problem of constructing computers with parallel processing (neural nets) is an optimization problem in the following sense. If we denote by K the number of elementary processors (neurons) then the complexity of the algorithm grows with the number K, in the worst case as K^2. So it is important to examine the limit case $K \to \infty$. But for most of considered problems the limit equation (obtained as limit of difference equations) has no differentiable solution, since the initial condition usually gives this. By a generalization of the theory of generalized functions the limit behavior of considered difference equation is characterized.

2. $\oplus-$decomposable measures

Let $[a,b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty,+\infty]$. We shall consider a partial order $\leq$ on $[a,b]$, which can be the usual order of the real line, but it can also be another order. All the subsequent considerations will be with respect to the order $\leq$.

The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, nondecreasing (with respect to $\leq$) associative and either a or b is a zero element, denoted by $\mathbf{0}$, i.e. for each $x \in [a,b]$

$$\mathbf{0} \oplus x = x \quad \text{holds.}$$

Let $[a,b]_+ = \{x : x \in [a,b], \ \ x \geq \mathbf{0}\}$.

The operation $\otimes$ (pseudo-multiplication) is a function $\otimes : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively nondecreasing, i.e. $x \leq y$ implies $x \otimes z \leq y \otimes z, \ z \in [a,b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a,b]$, i.e. for each $x \in [a,b]$

$$\mathbf{1} \otimes x = x.$$

We suppose, further, $\mathbf{0} \otimes x = \mathbf{0}$ and that $\otimes$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.,

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z).$$

Some of them are:

$$x \oplus y = (x^p + y^p)^{\frac{1}{p}}, \ p > 0 \quad \text{and} \quad x \otimes y = x \cdot y \ \text{ on } [0,+\infty]$$

$$\text{or} \quad x \oplus y = \max\{x,y\} \quad \text{and} \quad x \otimes y = x + y \ \text{ on } [-\infty,+\infty).$$

Pseudo-addition $\oplus$ is idempotent if for any $x \in [a,b]$

$$x \oplus x = x \quad \text{holds.}$$

Let X be a non-empty set. Let Σ be a $\sigma-$algebra of subsets of X.

A set function $m : \Sigma \to [a,b]_+$ (or semiclosed interval) is a $\bigoplus-$decomposable measure if there hold

$$m(\emptyset) = \mathbf{0} \quad \text{(if } \oplus \text{ is not idempotent)}$$

$$m(A \cup B) = m(A) \oplus m(B)$$

for $A, B \in \Sigma$ such that $A \cap B = \emptyset$.
In the case when $\oplus$ is idempotent, it is possible that m is not defined on an empty set.

A $\bigoplus-$decomposable measure m is $\sigma - \bigoplus-$decomposable if

$$m(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} m(A_i)$$

hold for any sequence (A_i) of pairwise disjoint sets from Σ.

Let m be a $\sigma - \bigoplus-$decomposable measure. A function $f : X \to [a,b]$ is measurable from below if for any $c \in [a,b]$ the sets $\{x : f(x) \leq c\}$ and $\{x : f(x) < c\}$ belong to Σ. f is measurable, if it is measurable from below and the sets $\{x : f(x) \geq c\}$ and $\{x : f(x) > c\}$ belong to Σ.

Let f and g be two functions defined on X and with values in $[a,b]$. Then, we define for any $x \in X$

$$(f \oplus g)(x) = f(x) \oplus g(x) \ ,$$

$$(f \otimes g)(x) = f(x) \otimes g(x)$$

and for any $c \in [a,b]$

$$(c \otimes f)(x) = c \otimes f(x).$$

We suppose further that $([a,b], \oplus)$ and $([a,b], \otimes)$ are complete lattice ordered semigroups. A complete lattice means that for each set $A \subset [a,b]$ bounded from above (below) there exists $\sup A$ ($\inf A$). Further, we suppose that $[a,b]$ is endowed with a metric d compatible with sup and inf and which satisfies at least one of the following conditions:

(a) $d(x \oplus y,\ x' \oplus y') \leq d(x,x') + d(y,y')$

(b) $d(x \oplus y,\ x' \oplus y') \leq \max\{d(x,x'),\ d(y,y')\}$.

Both conditions (a) and (b) imply that :

$$d(x_n, y_n) \to 0 \quad \text{implies} \quad d(x_n \oplus z,\ y_n \oplus z) \to 0.$$

Condition (b) implies

$$d(\bigoplus_{i=1}^{n} x_i, \bigoplus_{j=1}^{n} y_j) \leq \min_j \max_i d(x_i, y_j).$$

We suppose further the monotonicity of the metric d, i.e.

$$x \leq z \leq y \quad \text{implies} \quad d(x,y) \geq \max\{d(y,z), d(x,z)\}.$$

Let ε be a positive real number, and $B \subset [a,b]$. A subset $\{l_i^\varepsilon\}$ of $[a,b]$ is a $\varepsilon-$net if for each $x \in B$ there exists l_i^ε such that $d(l_i^\varepsilon, x) \leq \varepsilon$. If we have $l_i^\varepsilon \leq x$, then we shall call $\{l_i^\varepsilon\}$ a lower $\varepsilon-$net. If $l_i^\varepsilon \leq l_{i+1}^\varepsilon$ holds, then $\{l_i^\varepsilon\}$ is monotone.

We define the characteristic function

$$\chi_A(x) = \begin{cases} \mathbf{0}, & x \notin A \\ \mathbf{1}, & x \in A \end{cases}.$$

A mapping $e : X \to [a,b]$ is an elementary (measurable) function if it has the following representation

$$e = \bigoplus_{i=1}^{\infty} u_i \otimes \chi_{A_i} \quad \text{for} \quad u_i \in [a,b]$$

and $A_i \in \Sigma$ disjoint if $\oplus$ is not idempotent.

Definition 1 *The integral of a simple function $s = \bigoplus_{i=1}^{n} u_i \otimes \chi_{A_i}$ for $u_i \in [a,b]$ with disjoint $A_1, A_2, ...A_n$, if $\oplus$ is not idempotent, is defined by*

$$\int_X^{\oplus} s \otimes dm := \bigoplus_{i=1}^{n} u_i \otimes m(A_i).$$

The integral of an elementary function

$$e = \bigoplus_{i=1}^{\infty} u_i \otimes \chi_{A_i} \quad \textit{for} \quad u_i \in [a,b] \ \ (i \in \mathbb{N}) \ \ \textit{with} \ \ (A_i)$$

disjoint if $\oplus$ is not idempotent, is defined by

$$\int_X^{\oplus} e \otimes dm := \bigoplus_{i=1}^{\infty} u_i \otimes m(A_i).$$

The integral of a bounded measurable (from below for $\oplus$ idempotent) function $f : X \to \overline{[a,b]}$, for which, if $\oplus$ is not idempotent for each $\varepsilon > 0$, there exists a monotone $\varepsilon-$net in $f(X)$, is defined by

$$\int_X^{\oplus} f \otimes dm := \lim_{n\to\infty} \int_X^{\oplus} \varphi_n(x) \otimes dm,$$

(where (φ_n) is the sequence of elementary functions constructed in Theorem 1 in [15]).

Example 1. *For any function g we can define*

$$m(A) = \sup_{x\in A} g(x) \qquad A \in \mathcal{B},$$

where $\mathcal{B}$ is the Borel $\sigma-$algebra on $[-\infty, \infty)$.
Taking $\oplus = \max = \sup$, $\otimes = +$, we obtain

$$\int_{\mathbf{R}}^{\oplus} f \otimes dm = \sup_{x\in\mathbf{R}} (f(x) + g(x)),$$

f bounded above.

Example 2. *If $\oplus$ is a strict pseudo-addition with a monotone generator g, $g \circ m : \Sigma \to [0, g(c)]$ and $c \in [a,b]$ is an additive measure and we have (see [13], [15]) for the simple function*

$$\int_X^{\oplus} s \otimes dm = g^{-1}(\sum_{i=1}^{n} g(u_i) \cdot (g \circ m)(A_i))$$

and for the measurable function f

$$\int_X^{\oplus} f \otimes dm = g^{-1}(\int_X (g \circ f) \cdot dx),$$

where $dx = d(g \circ m)$ is the Lebesgue measure and $u \otimes v = g^{-1}(g(u) \cdot g(v))$.

3. $\oplus-$difference equation

We shall investigate the following difference equation

(1) $$u_{m,n}^{k+1} = u_{m-1,n}^{k} \oplus u_{m,n-1}^{k}$$

where $k = 0, 1, 2, ...;$ $m, n = 0, \pm 1, \pm 2, ...$
with the initial conditions

(2) $$u_{m,n}^{0} = \begin{cases} \mathbf{1}, & n = 0,\ m \geq 0 \\ \mathbf{1}, & m = 0,\ n \geq 0 \\ \mathbf{0}, & \text{otherwise} \end{cases}.$$

Example 3. *Let* $\oplus = \min$ *and* $\otimes = +$ *on* $[-\infty, +\infty]$. *Then we have* $\mathbf{0} = +\infty$ *and* $\mathbf{1} = 0$. *The equation* (1) *reduces on the Bellman equation*

$$u_{m,n}^{k+1} = \min\{u_{m-1,n}^k, u_{m,n-1}^k\}$$

where $k = 0, 1, 2, ...;\ \ m, n = 0, \pm 1, \pm 2, ...,$

with the initial conditions

$$u_{m,n}^0 = \begin{cases} 0, & n = 0,\ m \geq 0 \\ 0, & m = 0,\ n \geq 0 \\ +\infty, & \text{otherwise} \ . \end{cases}$$

Example 4. *Let* $\oplus = \max$ *and* $\otimes = +$ *on* $[-\infty, +\infty)$. *Then we have* $\mathbf{0} = -\infty$ *and* $\mathbf{1} = 0$. *The equation* (1) *reduces to the equation*

$$u_{m,n}^{k+1} = \max\{u_{m-1,n}^k, u_{m,n-1}^k\}$$

where $k = 0, 1, 2, ...;\ \ m, n = 0, \pm 1, \pm 2, ...,$

with the initial conditions

$$u_{m,n}^0 = \begin{cases} 0, & n = 0,\ m \geq 0 \\ 0, & m = 0,\ n \geq 0 \\ -\infty, & \text{otherwise} \ . \end{cases}$$

Example 5. *Let* $\oplus = \max$ *and* $\otimes = \min$ *on* $[0, +\infty]$. *Then we have* $\mathbf{0} = 0$ *and* $\mathbf{1} = +\infty$. *The equation* (1) *reduces to the equation*

$$u_{m,n}^{k+1} = \max\{u_{m-1,n}^k, u_{m,n-1}^k\}$$

where $k = 0, 1, 2, ...;\ \ m, n = 0, \pm 1, \pm 2, ...,$

with the initial conditions

$$u_{m,n}^0 = \begin{cases} -\infty, & n = 0,\ m \geq 0 \\ +\infty, & m = 0,\ n \geq 0 \\ 0, & \text{otherwise} \ . \end{cases}$$

We define the operator T by $T : C^2 \to C$, where

$$C = \{u_{m,n}^k : k = 0, 1, 2...; m, n = 0, \pm 1, \pm 2, ...\}$$

and T acts in the following way

$$Tu_{m,n}^k = u_{m-1,n}^k \oplus u_{m,n-1}^k.$$

This operator is linear with respect to $\oplus$ and $\otimes$. Namely, we have by the commutativity and associativity of the operation $\oplus$

$$T(u^k_{m,n} \oplus b^k_{m,n}) = (u^k_{m-1,n} \oplus b^k_{m-1,n}) \oplus (u^k_{m,n-1} \oplus b^k_{m,n-1})$$
$$= T(u^k_{m,n}) \oplus T(b^k_{m,n}),$$

and by the distributivity of the operation $\otimes$ with respect to the operation $\oplus$ we obtain

$$T(\gamma \otimes u^k_{m,n}) = (\gamma \otimes u^k_{m,n}) \oplus (\gamma \otimes u^k_{m,n-1})$$
$$= \gamma \otimes T(u^k_{m,n}).$$

4. $\oplus-$duality

Now we shall consider the corresponding continuous analog of the problem (1) $-$ (2).
We consider functions defined on $[0, M] \times [-\infty, +\infty]^2$, where $M > 0$, and with values in $[a, b]$, i.e., $u : [0, M] \times [-\infty, +\infty]^2 \to [a, b]$. Then the corresponding equation to (1) of continuous variable with mesh size $h > 0$ has the form

(3) $$u_h(z + h, x, y) = u_h(z, x - h, y) \oplus u_h(z, x, y - h),$$

taking $z = kh$, $k = 0, 1, 2, ...$ and $u(kh, mh, nh) = u^k_{m,n}$.
The corresponding initial condition to (2) has the form

(4) $$u_h(0, x, y) = u^0(x, y) = \begin{cases} \mathbf{1}, & y = 0,\ x \geq 0 \\ \mathbf{1}, & x = 0,\ y \geq 0 \\ \mathbf{0}, & \text{otherwise}\ . \end{cases}$$

Introducing the operator $T_h : C_h^2 \to C_h$, where

$$C_h = \{u_h : u_h(kh, mh, nh) = u^k_{m,n} \ \text{ for } k = 0, 1, 2, ...;\ m, n = 0, \pm 1, \pm 2, ...\},$$

such that

$$T_h u_h(z, x, y) = u_h(z, x - h, y) \oplus u_h(z, x, y - h)$$

for $z = kh$ $(k = 0, 1, 2, ...)$, we obtain that the solution of the problem (3) $-$ (4) is given by

(5) $$u_h(z, x, y) = (T_h)^k u^0(x, y).$$

We define the pseudo-scalar product for functions with values in $[a, b]$

$$(f, g)_\oplus = \int^\oplus f(x, y) \otimes g(x, y) dx dy.$$

Example 6. *For Min-Plus we have a pseudo scalar product*

$$(f, g)_\oplus = \inf_{x,y} \{f(x, y) + g(x, y)\}.$$

Example 7. *For Max-Plus we have a pseudo scalar product*

$$(f,g)_\oplus = \sup_{x,y}\{f(x,y)+g(x,y)\}.$$

We take $k \to \infty$ or $h \to 0$ (since the interval $[0,M]$ is bounded it is the same) for a point $z_0 \in [0,M]$ that $kh \to z_0$. We have for a smooth function $f(x,y)$

$$\begin{aligned}(u_h(z_0,x,y), f(x,y))_\oplus &= ((T_h)^k u^0(x,y), f(x,y))_\oplus \\ &= (u^0(x,y), (T_h^*)^k f(x,y))_\oplus,\end{aligned}$$

where T_h^* is the adjoint operator of the operator T_h with respect to the pseudo-scalar product $(.,.)_\oplus$.
The weak $-\oplus-$limit of solutions $u_h(z_0,x,y)$ of the problem $(3)-(4)$, since

$$\lim_{h\to 0}(u_h(z_0,x,y), f(x,y))_\oplus = F(f(x,y)),$$

is a pseudo-linear functional, we shall denote by $u(z_0,x,y)$, i.e., $F(f(x,y)) = (u(z_0,x,y), f(x,y))_\oplus$. On the other hand we have

$$(u(z_0,x,y), f(x,y))_\oplus = \lim_{h\to 0}(u^0(x,y), (T_h^*)^k f(x,y))_\oplus.$$

Example 8. *Take Min-Plus on the interval* $[-\infty,+\infty]$. *Then we have for the solution* (5) *of the problem* $(3)-(4)$

$$\begin{aligned}u_h(z,x,y) &= (T_h)^k u^0(x,y) \\ &= \min_{r_1+r_2=k,\ r_i\in\{0,1,2,\dots\}}\{u^0(x-r_1h, y-r_2h)\}.\end{aligned}$$

The adjoint operator $(T_h^*)^k$ *acts in the following way*

$$(T_h^*)^k f(x,y) = \min_{r_1+r_2=k,\ r_i\in\{0,1,2,\dots\}}\{u^0(x+r_1h, y+r_2h)\}.$$

Taking the weak$-\oplus-$*limit we obtain that*

$$(u(z_0,x,y), f(x,y))_\oplus = (u^0(x,y), \min_{r_1+r_2=z_0,\ r_i\in[0,+\infty)}\{f(x+r_1, y+r_2)\}.$$

The limit equation to (3) *on smooth function* u *is*

$$\frac{\partial u}{\partial z} = \min\{-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}\},$$

whose solution can be find by the Pontrjagin maximum principle.

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