wNovi Sad J. Math.
wVol. 31, No. 2, 2001, 73-87

RICCI TYPE IDENTITIES FOR BASIC DIFFERENTIATION AND CURVATURE TENSORS IN OTSUKI SPACES [1]

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Abstract. In the Otsuki spaces use is made of two non-symmetric affine connection: one for contravariant and the other for covariant indices. In the present work we study the Ricci type identities for the basic differentiation and curvature tensors in these spaces.

AMS Mathematics Subject Classification (2000): 53B05

Key words and phrases: Otsuki space, basic differentiation, Ricci type identity, curvature tensors and pseudotensors.

1. Introduction

T. Otsuki has defined and investigated [6] the so-called *regular general connection* consisting of two affine connections: *contravariant* $'\Gamma$ and *covariant part* $''\Gamma$. Besides, he introduced a tensor field P of the type $(1,1)$ ($det(P^i_j) \neq 0$), with the condition ([6], (3.13))

$$P^i_{j,k} +'' \Gamma^i_{pk} P^p_j -' \Gamma^p_{jk} P^i_p = 0, \tag{1}$$

where the comma signifies usual partial derivative, i.e. $P^i_{j,k} = \partial P^i_j / \partial x^k$.

In space with this connection one defines the so-called *basic covariant derivative*, for example

$$V^i_{j;k} = V^i_{j,k} +' \Gamma^i_{pk} V^p_j -'' \Gamma^p_{jk} V^i_p, \tag{2}$$

and *non-basic covariant derivative*, for example

$$\nabla_k V^i_j = P^i_p P^q_j V^p_{q;k}, \tag{3}$$

and the corresponding differentials

$$\overline{D} V^i_j = V^i_{j;k} dx^k, \quad D V^i_j = \nabla_k V^i_j dx^k. \tag{4}$$

The relation (1) is equivalent to

$$\nabla_k Q^i_j = 0, \tag{5}$$

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where $(Q^i_j) = (P^i_j)^{-1}$, i.e.

$$P^i_s Q^s_j = P^s_j Q^i_s = \delta^i_j. \tag{6}$$

Apart from T. Otsuki the cited spaces have been investigated also by A. Moór [3], M. Prvanović [7], [8], Dj. F. Nadj [5] and others.

2. Ricci type identities for basic differentiation of the first and second kind

2.0. The *Otsuki space* O_N is defined as an N-dimensional differentiable manifold on which, with respect to local coordinates $x^i (i = 1, 2, \dots, N)$, is given a tensor field $P^i_j (det(P^i_j) \neq 0)$ and the connection coefficients $'\Gamma^i_{jk}$, $''\Gamma^i_{jk}$, which are non-symmetric in general case and the relations (1) is in force.

Since the connection coefficients $'\Gamma^i_{jk}$ and $''\Gamma^i_{jk}$ are generally non-symmetric with respect to j, k, one can define two kinds of basic covariant derivative for a tensor V of the type (u, v):

$$V^{i_1 \cdots i_u}_{j_1 \cdots j_v \underset{1}{\mid} m} = V^{i_1 \cdots i_u}_{j_1 \cdots j_v , m} + \sum_{\alpha=1}^{u} {'\Gamma}^{i_\alpha}_{pm} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {''\Gamma}^{p}_{j_\beta m} \binom{j_\beta}{p} V^{\cdots}_{\cdots}, \tag{7}$$

$$V^{i_1 \cdots i_u}_{j_1 \cdots j_v \underset{2}{\mid} m} = V^{i_1 \cdots i_u}_{j_1 \cdots j_v , m} + \sum_{\alpha=1}^{u} {'\Gamma}^{i_\alpha}_{mp} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {''\Gamma}^{p}_{m j_\beta} \binom{j_\beta}{p} V^{\cdots}_{\cdots}, \tag{8}$$

where we have used the designations

$$\binom{p}{i_\alpha} V^{\cdots}_{\cdots} = V^{i_1 \cdots i_{\alpha-1} p i_{\alpha+1} \cdots i_u}_{j_1 \cdots j_v}, \tag{9}$$

$$\binom{j_\beta}{p} V^{\cdots}_{\cdots} = V^{i_1 \cdots i_u}_{j_1 \cdots j_{\beta-1} p j_{\beta+1} \cdots j_v}. \tag{10}$$

From here, for the Kronecker symbol we have

$$\delta^i_{j \underset{1}{\mid} m} = {'\Gamma}^i_{jm} - {''\Gamma}^i_{jm} \tag{11}$$

$$\delta^i_{j \underset{2}{\mid} m} = {'\Gamma}^i_{mj} - {''\Gamma}^i_{mj}. \tag{12}$$

In order to form the Ricci type identities, we can observe the differences

$$V^{i_1 \cdots i_u}_{j_1 \cdots j_v \underset{\lambda}{\mid} m \underset{\mu}{\mid} n} - V^{i_1 \cdots i_u}_{j_1 \cdots j_v \underset{\nu}{\mid} n \underset{\omega}{\mid} m}, \tag{13}$$

having 10 different cases:

$$(\lambda,\mu;\nu,\omega)\in\{\ (1,1;1,1),(2,2;2,2),(1,2;1,2),(2,1;2,1),\ (1,1;2,2),(1,1;1,2),(1,1;2,1),(2,2;1,2),\ (2,2;2,1),(1,2;2,1)\}, \tag{14}$$

which we are to study.

2.1. In the cited works, only the first kind of covariant derivative were used and it has been proved that (see [6], eq. (7.15), or [5] eq. (0.7))

$$\begin{aligned} V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{|} m \underset{1}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{|} n \underset{1}{|} m} &= \sum_{\alpha=1}^{u} {}'\underset{1}{R}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}V^{\cdots}_{\cdots} - \\ &- \sum_{\beta=1}^{v} {}''\underset{1}{R}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}V^{\cdots}_{\cdots} - {}''\Gamma^p_{[mn]}V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{|} p}, \end{aligned} \tag{15}$$

where

$${}'\underset{1}{R}{}^{i}_{jmn} = {}'\Gamma^i_{jm,n} - {}'\Gamma^i_{jn,m} + {}'\Gamma^p_{jm}{}'\Gamma^i_{pn} - {}'\Gamma^p_{jn}{}'\Gamma^i_{pm}, \tag{16}$$

and ${}''\underset{1}{R}$ is in the same manner expressed by ${}''\Gamma$, while $[mn]$ signifies the antisymmetrisation with respect to m, n without division by 2, that is

$${}''\Gamma^p_{[mn]} = {}''\Gamma^p_{mn} - {}''\Gamma^p_{nm}. \tag{17}$$

The identity (15) we call *the first Ricci type identity for basic differentiation* in O_N, while the thensors ${}'\underset{1}{R}$, ${}''\underset{1}{R}$ are *curvature tensors of the 1^{s}t kind* in O_N, obtained by ${}'\Gamma$, respectively ${}''\Gamma$.

2.2. In the same way one proves that in O_N is in force *the second Ricci type identity* for basic differentiation

$$\begin{aligned} V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{|} m \underset{2}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{|} n \underset{2}{|} m} &= \sum_{\alpha=1}^{u} {}'\underset{2}{R}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}V^{\cdots}_{\cdots} - \\ &- \sum_{\beta=1}^{v} {}''\underset{2}{R}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}V^{\cdots}_{\cdots} + {}''\Gamma^p_{[mn]}V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{|} p}, \end{aligned} \tag{18}$$

where

$${}'\underset{2}{R}{}^{i}_{jmn} = {}'\Gamma^i_{mj,n} - {}'\Gamma^i_{nj,m} + {}'\Gamma^p_{mj}{}'\Gamma^i_{np} - {}'\Gamma^p_{nj}{}'\Gamma^i_{mp}, \tag{19}$$

and in the same manner $''\underset{2}{R}$ by $''\Gamma$. The quantities $'\underset{2}{R}$, $''\underset{2}{R}$ are *curvature tensors of the second kind* in O_N.

2.3. For the third case, by virtue of (13) and (14), we have the next theorem:

Theorem 1. *In the space O_N the third Ricci type identity for basic differentiation is valid:*

$$\begin{aligned}(20)\qquad & V^{i_1\cdots i_u}_{j_1\cdots j_v\underset{1}{|}m\underset{2}{|}n} - V^{i_1\cdots i_u}_{j_1\cdots j_v\underset{1}{|}n\underset{2}{|}m} = \\ &= \sum_{\alpha=1}^{u}{'\underset{1}{A}}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}{''\underset{2}{A}}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}V^{\cdots}_{\cdots} + \\ &+ V^{i_1\cdots i_u}_{j_1\cdots j_v<[mn]>} + V^{i_1\cdots i_u}_{j_1\cdots j_v\le[mn]\ge} + {''\Gamma}^p_{[mn]}V^{i_1\cdots i_u}_{j_1\cdots j_v\underset{1}{|}p},\end{aligned}$$

where

$$(21)\qquad {'\underset{1}{A}}{}^{i}_{jmn} = {'\Gamma}^i_{jm,n} - {'\Gamma}^i_{jn,m} + {'\Gamma}^p_{jm}{'\Gamma}^i_{np} - {'\Gamma}^p_{jn}{'\Gamma}^i_{mp},$$

$$(22)\qquad {''\underset{2}{A}}{}^{i}_{jmn} = {''\Gamma}^i_{jm,n} - {''\Gamma}^i_{jn,m} + {''\Gamma}^p_{mj}{''\Gamma}^i_{pn} - {''\Gamma}^p_{nj}{''\Gamma}^i_{pm},$$

$$(23)\qquad V^{i_1\cdots i_u}_{j_1\cdots j_v<mn>} = \sum_{\alpha=1}^{u}{'\Gamma}^{i_\alpha}_{[pm]}\binom{p}{i_\alpha}V^{\cdots}_{\cdots,n} - \sum_{\beta=1}^{v}{''\Gamma}^{p}_{[j_\beta m]}\binom{j_\beta}{p}V^{\cdots}_{\cdots,n},$$

$$\begin{aligned}(24)\qquad V^{i_1\cdots i_u}_{j_1\cdots j_v\le mn\ge} &= \sum_{\alpha=1}^{u-1}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{u}({'\Gamma}^{i_\alpha}_{pm}{'\Gamma}^{i_\beta}_{ns} - {'\Gamma}^{i_\alpha}_{mp}{'\Gamma}^{i_\beta}_{sn})\binom{p}{i_\alpha}\binom{s}{i_\beta}V^{\cdots}_{\cdots} - \\ &- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}({'\Gamma}^{i_\alpha}_{pm}{''\Gamma}^{s}_{nj_\beta} - {'\Gamma}^{i_\alpha}_{mp}{''\Gamma}^{s}_{j_\beta n})\binom{p}{i_\alpha}\binom{j_\beta}{s}V^{\cdots}_{\cdots} + \\ &+ \sum_{\alpha=1}^{v-1}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{v}({''\Gamma}^{p}_{j_\alpha m}{''\Gamma}^{s}_{nj_\beta} - {''\Gamma}^{p}_{mj_\alpha}{''\Gamma}^{s}_{j_\beta n})\binom{j_\alpha}{p}\binom{j_\beta}{s}V^{\cdots}_{\cdots}.\end{aligned}$$

Proof. We shall prove (20) for the tensor V^{hi}_{jkl}, from where one can anticipate the general formula (20). So,

$$\begin{aligned}(25)\qquad V^{hi}_{jkl\underset{1}{|}m} &= V^{hi}_{jkl,m} + {'\Gamma}^h_{pm}V^{pi}_{jkl} + {'\Gamma}^i_{pm}V^{hp}_{jkl} - \\ &- {''\Gamma}^p_{jm}V^{hi}_{pkl} - {''\Gamma}^p_{km}V^{hi}_{jpl} - {''\Gamma}^p_{lm}V^{hi}_{jkp}.\end{aligned}$$

Further, we have

$$\begin{aligned}(26)\qquad V^{hi}_{jkl\underset{1}{|}m\underset{2}{|}n} &= (V^{hi}_{jkl\underset{1}{|}m})_{\underset{2}{|}n} = (V^{hi}_{jkl\underset{1}{|}m})_{,n} + {'\Gamma}^h_{ns}V^{si}_{jkl\underset{1}{|}m} + {'\Gamma}^i_{ns}V^{hs}_{jkl\underset{1}{|}m} - \\ &- {''\Gamma}^s_{nj}V^{hi}_{skl\underset{1}{|}m} - {''\Gamma}^s_{nk}V^{hi}_{jsl\underset{1}{|}m} - {''\Gamma}^s_{nl}V^{hi}_{jks\underset{1}{|}m} - {''\Gamma}^s_{nm}V^{hi}_{jkl\underset{1}{|}s}.\end{aligned}$$

Substituting into (26) by virtue of (25), one obtains

$$
\begin{aligned}
V^{hi}_{jkl\underset{1}{|}m\underset{2}{|}n} - V^{hi}_{jkl\underset{1}{|}n\underset{2}{|}m} &= {}'\underset{1}{A}{}^{h}_{pmn}V^{pi}_{jkl} + {}'\underset{1}{A}{}^{i}_{pmn}V^{hp}_{jkl} - \\
&- {}''\underset{2}{A}{}^{p}_{jmn}V^{hi}_{pkl} - {}''\underset{2}{A}{}^{p}_{kmn}V^{hi}_{jpl} - {}''\underset{2}{A}{}^{p}_{lmn}V^{hi}_{jkp} + \\
&+ V^{hi}_{jkl<[mn]>}V^{hi}_{jkl\leq[mn]\geq} + {}''\Gamma^{p}_{[mn]}V^{hi}_{jkl\underset{1}{|}p},
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
V^{hi}_{jkl<mn>} &= {}'\Gamma^{h}_{[pm]}V^{pi}_{jkl,n} + {}'\Gamma^{i}_{[pm]}V^{hp}_{jkl,n} - \\
&- {}''\Gamma^{p}_{[jm]}V^{hi}_{pkl,n} - {}''\Gamma^{p}_{[km]}V^{hi}_{jpl,n} - {}''\Gamma^{p}_{[lm]}V^{hi}_{jkp,n},
\end{aligned}
$$

$$
\begin{aligned}
V^{hi}_{jkl\leq mn\geq} &= ({}'\Gamma^{h}_{pm}{}'\Gamma^{i}_{ns} - {}'\Gamma^{h}_{mp}{}'\Gamma^{i}_{sn})V^{ps}_{jkl} - ({}'\Gamma^{h}_{pm}{}''\Gamma^{s}_{nj} - {}'\Gamma^{h}_{mp}{}''\Gamma^{s}_{jn})V^{pi}_{skl} - \\
&- ({}'\Gamma^{h}_{pm}{}''\Gamma^{s}_{nk} - {}'\Gamma^{h}_{mp}{}''\Gamma^{s}_{kn})V^{pi}_{jsl} - ({}'\Gamma^{h}_{pm}{}''\Gamma^{s}_{nl} - {}'\Gamma^{h}_{mp}{}''\Gamma^{s}_{ln})V^{pi}_{jks} - \\
&- ({}'\Gamma^{i}_{pm}{}''\Gamma^{s}_{nj} - {}'\Gamma^{i}_{mp}{}''\Gamma^{s}_{jn})V^{hp}_{skl} - ({}'\Gamma^{i}_{pm}{}''\Gamma^{s}_{nk} - {}'\Gamma^{i}_{mp}{}''\Gamma^{s}_{kn})V^{hp}_{jsl} - \\
&- ({}'\Gamma^{i}_{pm}{}''\Gamma^{s}_{nl} - {}'\Gamma^{i}_{mp}{}''\Gamma^{s}_{ln})V^{hp}_{jks} + ({}''\Gamma^{p}_{jm}{}''\Gamma^{s}_{nk} - {}''\Gamma^{p}_{mj}{}''\Gamma^{s}_{kn})V^{hi}_{psl} + \\
&+ ({}''\Gamma^{p}_{jm}{}''\Gamma^{s}_{nl} - {}''\Gamma^{p}_{mj}{}''\Gamma^{s}_{ln})V^{hi}_{pks} + ({}''\Gamma^{p}_{km}{}''\Gamma^{s}_{nl} - {}''\Gamma^{p}_{mk}{}''\Gamma^{s}_{ln})V^{hi}_{jps}.
\end{aligned}
$$

We see that (27) is a particular case of (20). So, for the tensor V^{hi}_{jkl} the equation (20) is valid. The cases of vectors

$$
V^{i}_{\underset{1}{|}m\underset{2}{|}n} - V^{i}_{\underset{1}{|}n\underset{2}{|}m} = {}'\underset{1}{A}{}^{i}_{pmn}V^{p} + {}'\Gamma^{i}_{[pm]}V^{p}_{,n} - {}'\Gamma^{i}_{[pn]}V^{p}_{,m} + {}''\Gamma^{p}_{[mn]}V^{i}_{\underset{1}{|}p},
$$

$$
V_{j\underset{1}{|}m\underset{2}{|}n} - V_{\underset{1}{|}n\underset{2}{|}m} = -{}''\underset{2}{A}{}^{p}_{jmn}V_{p} - {}''\Gamma^{p}_{[jm]}V_{p,n} + {}''\Gamma^{p}_{[jn]}V_{p,m} + {}''\Gamma^{p}_{[mn]}V_{j\underset{1}{|}p},
$$

are included in (20), which can be verified directly. In the case of a vector, the expression (24) is zero.

Also, by direct calculation we obtain that

$$
\begin{aligned}
&V^{\ i}_{j\underset{1}{|}m\underset{2}{|}n} - V^{i}_{j\underset{1}{|}n\underset{2}{|}m} = {}'\underset{1}{A}{}^{i}_{pmn}V^{p}_{j} - {}''\underset{2}{A}{}^{p}_{jmn}V^{i}_{p} + \\
&+ {}'\Gamma^{i}_{[pm]}v^{p}_{j,n} - {}'\Gamma^{i}_{[pn]}V^{p}_{j,m} - {}''\Gamma^{p}_{[jm]}V^{i}_{p,n} + {}''\Gamma^{p}_{[jn]}V^{i}_{p,m} + \\
&+ (-{}'\Gamma^{i}_{pm}{}''\Gamma^{s}_{nj} + {}'\Gamma^{i}_{mp}{}''\Gamma^{s}_{jn} + {}'\Gamma^{i}_{pn}{}''\Gamma^{s}_{mj} - {}'\Gamma^{i}_{np}{}''\Gamma^{s}_{jm})V^{p}_{s} + {}''\Gamma^{p}_{[mn]}V^{i}_{j\underset{1}{|}p},
\end{aligned}
\tag{28}
$$

and this is obtained from (20) too.

In order to prove (20) by induction method, suppose that (20) is valid, and prove that the corresponding equation is valid for a tensor $W^{i_1\cdots i_u i_{u+1}}_{j_1\cdots j_v j_{v+1}}$.

Observe the tensor

$$V^{i_1\cdots i_u}_{j_1\cdots j_v} = W^{i_1\cdots i_u i_{u+1}}_{j_1\cdots j_v j_{v+1}} U^{j_{v+1}}_{i_{u+1}} \tag{29}$$

Applying (20) to this tensor (of the type (u, v)), we get

$$\begin{aligned} & V^{\ i_1\cdots i_u}_{\ j_1\cdots j_v \underset{1}{|} m \underset{2}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{|} n \underset{2}{|} m} = \sum_{\alpha=1}^{u}{}' \underset{1}{A}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} (W^{i_1\cdots i_u i_{u+1}}_{j_1\cdots j_v j_{v+1}} U^{j_{v+1}}_{i_{u+1}}) - \\ & - \sum_{\beta=1}^{u}{}'' \underset{2}{A}{}^{p}_{j_\beta mn} \binom{j_\beta}{p} (W^{i_1\cdots i_u i_{u+1}}_{j_1\cdots j_v j_{v+1}} U^{j_{v+1}}_{i_{u+1}}) + (W^{\cdots}_{\cdots} U^{\cdots}_{\cdots})_{<[mn]>} + \\ & + (W^{\cdots}_{\cdots} U^{\cdots}_{\cdots})_{\leq[mn]\geq} + {}''\Gamma^{p}_{[mn]} (W^{\cdots}_{\cdots \underset{1}{|} p} U^{\cdots}_{\cdots} + W^{\cdots}_{\cdots} U^{\cdots}_{\cdots \underset{1}{|} p}), \end{aligned} \tag{30}$$

where we have taken into consideration that for the basic differentiation the Leibniz rule is valid.

Based on (23,24,29), we obtain

$$\begin{aligned} (W^{\cdots}_{\cdots} U^{\cdots}_{\cdots})_{<mn>} = & \sum_{\alpha=1}^{u} {}'\Gamma^{i_\alpha}_{[pm]} \binom{p}{i_\alpha} (W^{\cdots}_{\cdots,n} U^{\cdots}_{\cdots} + W^{\cdots}_{\cdots} U^{\cdots}_{\cdots,n}) - \\ & - \sum_{\beta=1}^{v} {}''\Gamma^{p}_{[j_\beta m]} \binom{j_\beta}{p} (W^{\cdots}_{\cdots,n} U^{\cdots}_{\cdots} + W^{\cdots}_{\cdots} U^{\cdots}_{\cdots,n}), \end{aligned} \tag{31}$$

$$\begin{aligned} (W^{\cdots}_{\cdots} U^{\cdots}_{\cdots})_{\leq mn\geq} = U^{\cdots}_{\cdots} \Big[& \sum_{\alpha=1}^{u-1} \sum_{\substack{\beta=2 \\ (\alpha<\beta)}}^{u} ({}'\Gamma^{i_\alpha}_{pm} {}'\Gamma^{i_\beta}_{ns} - {}'\Gamma^{i_\alpha}_{mp} {}'\Gamma^{i_\beta}_{sn}) \binom{p}{i_\alpha} \binom{s}{i_\beta} W^{\cdots}_{\cdots} - \\ & - \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} ({}'\Gamma^{i_\alpha}_{pm} {}''\Gamma^{s}_{nj_\beta} - {}'\Gamma^{i_\alpha}_{mp} {}''\Gamma^{s}_{j_\beta n}) \binom{p}{i_\alpha} \binom{j_\beta}{s} W^{\cdots}_{\cdots} + \\ & + \sum_{\alpha=1}^{v-1} \sum_{\substack{\beta=2 \\ (\alpha<\beta)}}^{v} ({}''\Gamma^{p}_{j_\alpha m} {}''\Gamma^{s}_{nj_\beta} - {}''\Gamma^{p}_{mj_\alpha} {}''\Gamma^{s}_{j_\beta n}) \binom{j_\alpha}{p} \binom{j_\beta}{s} W^{\cdots}_{\cdots} \Big]. \end{aligned} \tag{32}$$

On the other hand, based on (29), we have

$$\begin{aligned} & V^{\cdots}_{\cdots \underset{1}{|} m \underset{2}{|} n} - V^{\cdots}_{\cdots \underset{1}{|} n \underset{2}{|} m} = \{ (W^{\cdots}_{\cdots} U^{\cdots}_{\cdots})_{\underset{1}{|} m \underset{2}{|} n} \}_{[mn]} = \\ & = (W^{\cdots}_{\cdots \underset{1}{|} m \underset{2}{|} n} U^{\cdots}_{\cdots} + W^{\cdots}_{\cdots \underset{1}{|} m} U^{\cdots}_{\cdots \underset{2}{|} n} + W^{\cdots}_{\cdots \underset{2}{|} n} U^{\cdots}_{\cdots \underset{1}{|} m} + W^{\cdots}_{\cdots} U^{\cdots}_{\cdots \underset{1}{|} m \underset{2}{|} n})_{[mn]} = \\ & = (W^{\cdots}_{\cdots \underset{1}{|} m \underset{2}{|} n} - W^{\cdots}_{\cdots \underset{1}{|} n \underset{2}{|} m}) U^{\cdots}_{\cdots} + W^{\cdots}_{\cdots} (U^{\cdots}_{\cdots \underset{1}{|} m \underset{2}{|} n} - U^{\cdots}_{\cdots \underset{1}{|} n \underset{2}{|} m}) + \\ & + (W^{\cdots}_{\cdots \underset{1}{|} m} U^{\cdots}_{\cdots \underset{2}{|} n} + W^{\cdots}_{\cdots \underset{2}{|} n} U^{\cdots}_{\cdots \underset{1}{|} m})_{[mn]}. \end{aligned} \tag{33}$$

Applying the identity (28) to the tensor $U^{j_{v+1}}_{i_{u+1}}$ at the second brackets, calculating the covariant derivatives in the third brackets, substituting the expression, (31,32) into (30) and equilizing the right sides of the equations (30) and (33), after longer arranging one obtains

$$\begin{aligned}
U^{j_{v+1}}_{i_{u+1}} &\left(W^{i_1\cdots i_{u+1}}_{j_1\cdots j_{v+1}\,\underset{1}{|}\,m\,\underset{2}{|}\,n} - W^{i_1\cdots i_{u+1}}_{j_1\cdots j_{v+1}\,\underset{1}{|}\,n\,\underset{2}{|}\,m}\right) = \\
&= U^{j_{v+1}}_{i_{u+1}}\Big\{\sum_{\alpha=1}^{u+1}{}'\underset{1}{A}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}W^{\cdots}_{\cdots} - \sum_{\beta=1}^{v+1}{}''\underset{2}{A}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}W^{\cdots}_{\cdots} + \\
&+ \Big[\sum_{\alpha=1}^{u+1}{}'\Gamma^{i_\alpha}_{[pm]}\binom{p}{i_\alpha}W^{\cdots}_{\cdots,n} - \sum_{\beta=1}^{v+1}{}''\Gamma^{p}_{[j_\beta m]}\binom{j_\beta}{p}W^{\cdots}_{\cdots,n} + \\
&+ \sum_{\alpha=1}^{u}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{u+1}({}'\Gamma^{i_\alpha}_{pm}{}'\Gamma^{i_\beta}_{ns} - {}'\Gamma^{i_\alpha}_{mp}{}'\Gamma^{i_\beta}_{sn})\binom{p}{i_\alpha}\binom{s}{i_\beta}W^{\cdots}_{\cdots} - \\
&- \sum_{\alpha=1}^{u+1}\sum_{\beta=1}^{v+1}({}'\Gamma^{i_\alpha}_{pm}{}''\Gamma^{s}_{nj_\beta} - {}'\Gamma^{i_\alpha}_{mp}{}''\Gamma^{s}_{j_\beta n})\binom{p}{i_\alpha}\binom{j_\beta}{s}W^{\cdots}_{\cdots} + \\
&+ \sum_{\alpha=1}^{v}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{v+1}({}''\Gamma^{p}j_\alpha m{}''\Gamma^{s}_{nj_\beta} - {}''\Gamma^{p}mj_\alpha{}''\Gamma^{s}_{j_\beta n})\binom{j_\alpha}{p}\binom{j_\beta}{s}W^{\cdots}_{\cdots} + \\
&+ {}''\Gamma^{p}_{mn}W^{\cdots}_{\cdots\,\underset{1}{|}\,p}\Big]_{[mn]}\Big\}.
\end{aligned}$$

Because $U^{i_{v+1}}_{j_{u+1}}$ is an arbitrary tensor of the type (1,1), the last equation, in view of (23,24), becomes

$$\begin{aligned}
W\;{}^{i_1\cdots i_{u+1}}_{j_1\cdots j_{v+1}\,\underset{1}{|}\,m\,\underset{2}{|}\,n} - W^{i_1\cdots i_{u+1}}_{j_1\cdots j_{v+1}\,\underset{1}{|}\,n\,\underset{2}{|}\,m} &= \sum_{\alpha=1}^{u+1}{}'\underset{1}{A}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}W^{\cdots}_{\cdots} - \\
- \sum_{\beta=1}^{v+1}{}''\underset{2}{A}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}W^{\cdots}_{\cdots} + W^{\cdots}_{\cdots<[mn]>} + W^{\cdots}_{\cdots\leq[mn]\geq} &+ {}''\Gamma^{p}_{[mn]}W^{\cdots}_{\cdots\,\underset{1}{|}\,p},
\end{aligned}$$

i.e. (20) is valid for the tensor W of the type $(u+1, v+1)$ too, and Theorem is proved. □

The following theorems (Th. 2 - Th. 8) are proved in a similar way.

2.4. To the fourth case from (14) is related

Theorem 2. *Applying two kinds of covariant derivative in the inversed order of that in the previous case, we obtain the fourth Ricci type identity in* O_N *for* *basic differentiation:*

$$
(34)\quad \begin{aligned}
&V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{|} m \underset{1}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{|} n \underset{1}{|} m} = \\
&= \sum_{\alpha=1}^{u} {}'\underset{3}{A}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {}''\underset{4}{A}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}V^{\cdots}_{\cdots} - \\
&- V^{i_1\cdots i_u}_{j_1\cdots j_v <[mn]>} - V^{i_1\cdots i_u}_{j_1\cdots j_v \leq [mn] \geq} - {}''\Gamma^p_{[mn]}V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{|} p},
\end{aligned}
$$

where

$$
(35)\quad {}''\underset{3}{A}{}^{i}_{jmn} = {}'\Gamma^i_{mj,n} - {}'\Gamma^i_{nj,m} + {}'\Gamma^p_{mj}{}'\Gamma^i_{pn} - {}'\Gamma^p_{nj}{}'\Gamma^i_{pm},
$$

$$
(36)\quad {}''\underset{4}{A}{}^{i}_{jmn} = {}''\Gamma^i_{mj,n} - {}''\Gamma^i_{nj,m} + {}''\Gamma^p_{jm}{}''\Gamma^i_{np} - {}''\Gamma^p_{jn}{}''\Gamma^i_{mp}.
$$

2.5. Further, we have

Theorem 3. *In O_N is valid the 5^{th} Ricci type identity for basic differentiation:*

$$
(37)\quad \begin{aligned}
&V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{|} m \underset{1}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{|} n \underset{2}{|} m} = \\
&= \sum_{\alpha=1}^{u} {}'\underset{5}{A}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {}''\underset{6}{A}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}V^{\cdots}_{\cdots} + \\
&+ V^{i_1\cdots i_u}_{j_1\cdots j_v <(mn)>} + V^{i_1\cdots i_u}_{j_1\cdots j_v \ll s(mn) \gg} - {}''\Gamma^p_{mn}(V^{\cdots}_{\cdots \underset{1}{|} p} - V^{\cdots}_{\cdots \underset{2}{|} p}),
\end{aligned}
$$

where we have designated

$$
(38)\quad {}'\underset{5}{A}{}^{i}_{jmn} = {}'\Gamma^i_{jm,n} - {}'\Gamma^i_{nj,m} + {}'\Gamma^p_{jm}{}'\Gamma^i_{pn} - {}'\Gamma^p_{nj}{}'\Gamma^i_{mp},
$$

$$
(39)\quad {}''\underset{6}{A}{}^{i}_{jmn} = {}''\Gamma^i_{jm,n} - {}''\Gamma^i_{nj,m} + {}''\Gamma^p_{mj}{}''\Gamma^i_{np} - {}''\Gamma^p_{jn}{}''\Gamma^i_{pm}.
$$

$$
(40)\quad \begin{aligned}
V^{i_1\cdots i_u}_{j_1\cdots j_v \ll smn \gg} &= \sum_{\alpha=1}^{u-1}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{v} ({}'\Gamma^{i_\alpha}_{pm}{}'\Gamma^{i_\beta}_{sn} - {}'\Gamma^{i_\alpha}_{mp}{}'\Gamma^{i_\beta}_{ns})\binom{p}{i_\alpha}\binom{s}{i_\beta}V^{\cdots}_{\cdots} - \\
&- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}({}'\Gamma^{i_\alpha}_{pm}{}''\Gamma^{s}_{j_\beta n} - {}'\Gamma^{i_\alpha}_{mp}{}'\Gamma^{s}_{nj_\beta})\binom{p}{i_\alpha}\binom{j_\beta}{s}V^{\cdots}_{\cdots} + \\
&+ \sum_{\alpha=1}^{v-1}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{v} ({}''\Gamma^{p}_{j_\alpha m}{}''\Gamma^{s}_{j_\beta n} - {}''\Gamma^{p}_{mj_\alpha}{}''\Gamma^{s}_{nj_\beta})\binom{j_\alpha}{p}\binom{j_\beta}{s}V^{\cdots}_{\cdots},
\end{aligned}
$$

while (m,n) designates the symmetrisation of the corresponding expression over m,n (without division with 2).

2.6. Theorem 4. *In O_N is in force the 6^{th} Ricci type identity for basic differentiation:*

$$
\begin{aligned}
&V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{|} m \underset{1}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{|} n \underset{2}{|} m} = \\
&\qquad = \sum_{\alpha=1}^{u} {}' \underset{7}{A}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {}'' \underset{8}{A}{}^{p}_{j_\beta mn} \binom{j_\beta}{p} V^{\cdots}_{\cdots} + \\
&\qquad + V^{i_1\cdots i_u}_{j_1\cdots j_v < mn >} + V^{i_1\cdots i_u}_{j_1\cdots j_v \leq mn \gg},
\end{aligned} \tag{41}
$$

where

$$
{}' \underset{7}{A}{}^{i}_{jmn} = {}'\Gamma^{i}_{jm,n} - {}'\Gamma^{i}_{jn,m} + {}'\Gamma^{p}_{jm} {}'\Gamma^{i}_{pn} - {}'\Gamma^{p}_{jn} {}'\Gamma^{i}_{mp}, \tag{42}
$$

$$
{}'' \underset{8}{A}{}^{i}_{jmn} = {}''\Gamma^{i}_{jm,n} - {}''\Gamma^{i}_{jn,m} + {}''\Gamma^{p}_{mj} {}''\Gamma^{i}_{pn} - {}''\Gamma^{p}_{jn} {}''\Gamma^{i}_{pm}. \tag{43}
$$

$$
\begin{aligned}
V^{i_1\cdots i_u}_{j_1\cdots j_v \leq mn \gg} &= \sum_{\alpha=1}^{u-1} \sum_{\substack{\beta=2 \\ (\alpha<\beta)}}^{u} ({}'\Gamma^{i_\alpha}_{[pm]} {}'\Gamma^{i_\beta}_{sn} + {}'\Gamma^{i_\alpha}_{pn} {}'\Gamma^{i_\beta}_{[sm]}) \binom{p}{i_\alpha} \binom{s}{i_\beta} V^{\cdots}_{\cdots} - \\
&- \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} ({}'\Gamma^{i_\alpha}_{[pm]} {}''\Gamma^{s}_{j_\beta n} + {}'\Gamma^{i_\alpha}_{pn} {}''\Gamma^{s}_{[j_\beta m]}) \binom{p}{i_\alpha} \binom{j_\beta}{s} V^{\cdots}_{\cdots} + \\
&+ \sum_{\alpha=1}^{v-1} \sum_{\substack{\beta=2 \\ (\alpha<\beta)}}^{v} ({}''\Gamma^{p}_{[j_\alpha m]} {}''\Gamma^{s}_{j_\beta n} + {}''\Gamma^{p}_{j_\alpha n} {}''\Gamma^{s}_{[j_\beta m]}) \binom{j_\alpha}{p} \binom{j_\beta}{s} V^{\cdots}_{\cdots}.
\end{aligned} \tag{44}
$$

2.7. Theorem 5. *In O_N is valid the 7^{th} Ricci type identity for basic differentiation:*

$$
\begin{aligned}
&V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{|} m \underset{1}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{|} n \underset{1}{|} m} = \\
&\qquad = \sum_{\alpha=1}^{u} {}' \underset{9}{A}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {}'' \underset{10}{A}{}^{p}_{j_\beta mn} \binom{j_\beta}{p} V^{\cdots}_{\cdots} + \\
&\qquad + V^{i_1\cdots i_u}_{j_1\cdots j_v < nm >} + V^{i_1\cdots i_u}_{j_1\cdots j_v \leq nm \gg} - ({}''\Gamma^{p}_{mn} V^{\cdots}_{\cdots \underset{1}{|} p} - {}''\Gamma^{p}_{nm} V^{\cdots}_{\cdots \underset{2}{|} p}),
\end{aligned} \tag{45}
$$

where we have designated

$$
{}' \underset{9}{A}{}^{i}_{jmn} = {}'\Gamma^{i}_{jm,n} - {}'\Gamma^{i}_{nj,m} + {}'\Gamma^{p}_{jm} {}'\Gamma^{i}_{pn} - {}'\Gamma^{p}_{nj} {}'\Gamma^{i}_{pm}, \tag{46}
$$

$$
{}'' \underset{10}{A}{}^{i}_{jmn} = {}''\Gamma^{i}_{jm,n} - {}''\Gamma^{i}_{nj,m} + {}''\Gamma^{p}_{jm} {}''\Gamma^{i}_{np} - {}''\Gamma^{p}_{jn} {}''\Gamma^{i}_{pm}. \tag{47}
$$

2.8. To the following case from (14) is related

Theorem 6. *In O_N is valid the 8^{th} Ricci type identity for basic differentiation:*

$$
\begin{aligned}
&V^{i_1\cdots i_u}_{j_1\cdots j_v\underset{2}{|}m\underset{2}{|}n} - V^{i_1\cdots i_u}_{j_1\cdots j_v\underset{1}{|}n\underset{2}{|}m} = \\
&= \sum_{\alpha=1}^{u}{}'\underset{11}{A}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}{}''\underset{12}{A}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}V^{\cdots}_{\cdots} - \\
&- V^{i_1\cdots i_u}_{j_1\cdots j_v<nm>} + V^{i_1\cdots i_u}_{j_1\cdots j_v\leqslant smn\geq} + {}''\Gamma^p_{mn}V^{\cdots}_{\cdots\underset{1}{|}p} - {}''\Gamma^p_{nm}V^{\cdots}_{\cdots\underset{2}{|}p},
\end{aligned}\tag{48}
$$

where

$$
{}'\underset{11}{A}{}^{i}_{jmn} = {}'\Gamma^i_{mj,n} - {}'\Gamma^i_{jn,m} + {}'\Gamma^p_{mj}{}'\Gamma^i_{np} - {}'\Gamma^p_{jn}{}'\Gamma^i_{mp},\tag{49}
$$

$$
{}''\underset{12}{A}{}^{i}_{jmn} = {}''\Gamma^i_{mj,n} - {}''\Gamma^i_{jn,m} + {}''\Gamma^p_{mj}{}''\Gamma^i_{pn} - {}''\Gamma^p_{nj}{}''\Gamma^i_{mp}.\tag{50}
$$

$$
\begin{aligned}
V^{i_1\cdots i_u}_{j_1\cdots j_v\leqslant smn\geq} &= \sum_{\alpha=1}^{u-1}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{v}({}'\Gamma^{i_\alpha}_{mp}{}'\Gamma^{i_\beta}_{[ns]} + {}'\Gamma^{i_\alpha}_{[np]}{}'\Gamma^{i_\beta}_{ms})\binom{p}{i_\alpha}\binom{s}{i_\beta}V^{\cdots}_{\cdots} - \\
&- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}({}'\Gamma^{i_\alpha}_{mp}{}''\Gamma^{s}_{[nj_\beta]} + {}'\Gamma^{i_\alpha}_{[np]}{}''\Gamma^{s}_{mj_\beta})\binom{p}{i_\alpha}\binom{j_\beta}{s}V^{\cdots}_{\cdots} + \\
&+ \sum_{\alpha=1}^{v-1}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{v}({}''\Gamma^{p}_{mj_\alpha}{}''\Gamma^{s}_{[nj_\beta]} + {}''\Gamma^{p}_{[nj_\alpha]}{}''\Gamma^{s}_{mj_\beta})\binom{j_\alpha}{p}\binom{j_\beta}{s}V^{\cdots}_{\cdots}.
\end{aligned}\tag{51}
$$

2.9. Also we have

Theorem 7. *In an Otsuki space O_N is in force the 9^{th} Ricci type identity for basic differentiation:*

$$
\begin{aligned}
&V^{i_1\cdots i_u}_{j_1\cdots j_v\underset{2}{|}m\underset{2}{|}n} - V^{i_1\cdots i_u}_{j_1\cdots j_v\underset{2}{|}n\underset{1}{|}m} = \\
&= \sum_{\alpha=1}^{u}{}'\underset{13}{A}{}^{i_\alpha}_{pmn}\binom{p}{i_\alpha}V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}{}''\underset{14}{A}{}^{p}_{j_\beta mn}\binom{j_\beta}{p}V^{\cdots}_{\cdots} - \\
&- V^{i_1\cdots i_u}_{j_1\cdots j_v<mn>} + V^{i_1\cdots i_u}_{j_1\cdots j_v\leqslant snm\geq},
\end{aligned}\tag{52}
$$

where

$$
{}'\underset{13}{A}{}^{i}_{jmn} = {}'\Gamma^i_{mj,n} - {}'\Gamma^i_{nj,m} + {}'\Gamma^p_{mj}{}'\Gamma^i_{np} - {}'\Gamma^p_{nj}{}'\Gamma^i_{pm},\tag{53}
$$

$$''\underset{14}{A}{}^{i}_{jmn} = {}''\Gamma^{i}_{mj,n} - {}''\Gamma^{i}_{nj,m} + {}''\Gamma^{p}_{jm}{}''\Gamma^{i}_{np} - {}''\Gamma^{p}_{nj}{}''\Gamma^{i}_{mp}. \tag{54}$$

2.10. For the last case from (14) we have

Theorem 8. *In O_N is is valid the 10^{th} Ricci type identity for basic differentiation:*

$$\begin{aligned} & V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{\mid} m \underset{2}{\mid} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{\mid} n \underset{1}{\mid} m} = \\ & = \sum_{\alpha=1}^{u} {}'\underset{15}{A}{}^{i}_{pmn}\binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {}''\underset{15}{A}{}^{p}_{j_\beta mn}\binom{j_\beta}{p} V^{\cdots}_{\cdots} - \\ & - {}''\Gamma^{p}_{nm}(V^{\cdots}_{\cdots \underset{1}{\mid} p} - V^{\cdots}_{\cdots \underset{2}{\mid} p}), \end{aligned} \tag{55}$$

where

$$^{\theta}\underset{15}{A}{}^{i}_{jmn} = {}^{\theta}\Gamma^{i}_{jm,n} - {}^{\theta}\Gamma^{i}_{nj,m} + {}^{\theta}\Gamma^{p}_{jm}{}^{\theta}\Gamma^{i}_{np} - {}^{\theta}\Gamma^{p}_{nj}{}^{\theta}\Gamma^{i}_{pm}, \quad \theta = ','' . \tag{56}$$

The equation (55) can be written in another form. Namely, counting the difference in the last addend, we obtain *another form of the 10^{th} Ricci identity* for basic differentiation in O_N:

$$\begin{aligned} & V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{1}{\mid} m \underset{2}{\mid} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{2}{\mid} n \underset{1}{\mid} m} = \\ & = \sum_{\alpha=1}^{u} {}'\underset{3}{R}{}^{i}_{pmn}\binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {}''\underset{3}{R}{}^{p}_{j_\beta mn}\binom{j_\beta}{p} V^{\cdots}_{\cdots} \end{aligned} \tag{57}$$

where

$$^{\theta}\underset{3}{R}{}^{i}_{jmn} = {}^{\theta}\underset{15}{A}{}^{i}_{jmn} + {}''\Gamma^{p}_{nm}{}^{\theta}\Gamma^{i}_{[pj]}, \quad \theta = ','' . \tag{58}$$

is *the curvature tensor of the 3^{rd} kind* in O_N, determined by the connection $^{\theta}\Gamma, \quad \theta = ','' $.

Remark. The quantities $^{\theta}\underset{t}{A}{}^{i}_{jmn} \quad (\theta = ','' ; \quad t = 1, \cdots, 15)$ are not tensors and we call them *curvature pseudotensors* of the space O_N of the 1^{st} to 15^{th} kind respectively. For example, from the particular case of (41)

$$V_{j \underset{1}{\mid} m \underset{1}{\mid} n} - V_{j \underset{1}{\mid} n \underset{2}{\mid} m} = -''\underset{8}{A}{}^{p}_{jmn} V_p + {}''\Gamma^{p}_{[mj]} V_{p,n},$$

we see that $''\underset{8}{A}$ is not a tensor, because $V_{p,n} = \partial V_p / \partial x^n$ is not a tensor.

3. The Ricci type identities for basic differentiation of the third and fourth kind

One can define in O_N *two new kinds of basic covariant derivative* (in place of (7)):

$$V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{3}{|} m} = V^{i_1\cdots i_u}_{j_1\cdots j_v ,m} + \sum_{\alpha=1}^{u} {}'\Gamma^{i_\alpha}_{pm} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {}''\Gamma^{p}_{mj_\beta} \binom{j_\beta}{p} V^{\cdots}_{\cdots}, \tag{59}$$

$$V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{4}{|} m} = V^{i_1\cdots i_u}_{j_1\cdots j_v ,m} + \sum_{\alpha=1}^{u} {}'\Gamma^{i_\alpha}_{mp} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} {}''\Gamma^{p}_{j_\beta m} \binom{j_\beta}{p} V^{\cdots}_{\cdots}. \tag{60}$$

From here, it follows that

$$\delta^i_{j \underset{3}{|} m} = {}'\Gamma^i_{jm} - {}''\Gamma^i_{mj}, \quad \delta^i_{j \underset{4}{|} m} = {}'\Gamma^i_{mj} - {}''\Gamma^i_{jm}. \tag{61}$$

Analogously to (14), we can obtain 10 new Ricci type identities in O_N. For example,

$$\begin{aligned} V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{3}{|} m \underset{3}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{3}{|} n \underset{3}{|} m} &= \sum_{\alpha=1}^{u} {}' \underset{1}{R}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \\ &- \sum_{\beta=1}^{v} {}'' \underset{2}{R}{}^{p}_{j_\beta mn} \binom{j_\beta}{p} V^{\cdots}_{\cdots} + {}''\Gamma^p_{[mn]} V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{3}{|} p}, \end{aligned} \tag{62}$$

$$\begin{aligned} V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{4}{|} m \underset{4}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{4}{|} n \underset{4}{|} m} &= \sum_{\alpha=1}^{u} {}' \underset{2}{R}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \\ &- \sum_{\beta=1}^{v} {}'' \underset{1}{R}{}^{p}_{j_\beta mn} \binom{j_\beta}{p} V^{\cdots}_{\cdots} - {}''\Gamma^p_{[mn]} V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{4}{|} p}, \end{aligned} \tag{63}$$

$$\begin{aligned} & V^{\ i_1\cdots i_u}_{j_1\cdots j_v \underset{3}{|} m \underset{4}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{3}{|} n \underset{4}{|} m} = \sum_{\alpha=1}^{u} {}' \underset{1}{A}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \\ & - \sum_{\beta=1}^{v} {}'' \underset{4}{A}{}^{p}_{j_\beta mn} \binom{j_\beta}{p} V^{\cdots}_{\cdots} + \overline{V}^{i_1\cdots i_u}_{j_1\cdots j_v <[mn]>} + \overline{V}^{i_1\cdots i_u}_{j_1\cdots j_v \leq [mn] \geq} - \\ & - {}''\Gamma^p_{[mn]} V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{3}{|} p}, \end{aligned} \tag{64}$$

where

$$\overline{V}^{i_1\cdots i_u}_{j_1\cdots j_v <mn>} = \sum_{\alpha=1}^{u} {}'\Gamma^{i_\alpha}_{[pm]} \binom{p}{i_\alpha} V^{\cdots}_{\cdots ,n} - \sum_{\beta=1}^{v} {}''\Gamma^{p}_{[mj_\beta]} \binom{j_\beta}{p} V^{\cdots}_{\cdots ,n}, \tag{65}$$

$$
\begin{aligned}
\overline{V}^{i_1\cdots i_u}_{j_1\cdots j_v \leq mn \geq} = & \sum_{\substack{\alpha=1 \\ (\alpha<\beta)}}^{u-1} \sum_{\beta=2}^{u} ({}'\Gamma^{i_\alpha}_{pm} {}'\Gamma^{i_\beta}_{ns} - {}'\Gamma^{i_\alpha}_{mp} {}'\Gamma^{i_\beta}_{sn}) \binom{p}{i_\alpha} \binom{s}{i_\beta} V^{\cdots}_{\cdots} - \\
(66) \qquad & - \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} ({}'\Gamma^{i_\alpha}_{pm} {}''\Gamma^{s}_{j_\beta n} - {}'\Gamma^{i_\alpha}_{mp} {}''\Gamma^{s}_{n j_\beta}) \binom{p}{i_\alpha} \binom{j_\beta}{s} V^{\cdots}_{\cdots} + \\
& + \sum_{\substack{\alpha=1 \\ (\alpha<\beta)}}^{v-1} \sum_{\beta=2}^{v} ({}''\Gamma^{p}_{m j_\alpha} {}''\Gamma^{s}_{j_\beta n} - {}''\Gamma^{p}_{j_\alpha m} {}''\Gamma^{s}_{n j_\beta}) \binom{j_\alpha}{p} \binom{j_\beta}{s} V^{\cdots}_{\cdots}.
\end{aligned}
$$

In these identities appear the same quantities ${}^\theta\underset{1}{R}, {}^\theta\underset{2}{R}, {}^\theta\underset{3}{R}; {}^\theta\underset{1}{A}, \cdots, {}^\theta\underset{15}{A}$, but in different distribution than in the cases 2.1-2.10. Only in the one case appears a new curvature tensor $\underset{4}{R}$:

$$
\begin{aligned}
& V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{3}{|} m \underset{4}{|} n} - V^{i_1\cdots i_u}_{j_1\cdots j_v \underset{4}{|} n \underset{3}{|} m} = \\
(67) \qquad & = \sum_{\alpha=1}^{u} {}'\underset{4}{R}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} + \sum_{\beta=1}^{v} {}''\underset{3}{R}{}^{p}_{j_\beta nm} \binom{j_\beta}{p} V^{\cdots}_{\cdots},
\end{aligned}
$$

where

$$
(68) \qquad {}'\underset{4}{R}{}^{i}_{jmn} = {}'\underset{15}{A}{}^{i}_{jmn} + {}''\Gamma^{p}_{mn} {}'\Gamma^{i}_{[pj]}.
$$

4. Derived curvature tensors. Independent curvature tensors in O_N

As we have seen, the quantities ${}^\theta\underset{t}{A}$ $(\theta = {}', {}''; \ t = 1, \cdots, 15)$ are not tensors. We proved in [1] for a non-symmetric connection $\Gamma\,(= {}'\Gamma = {}''\Gamma)$ that from the curvature pseudotensors $\underset{t}{A}$ one can obtain new, so called *"derived" curvature tensors*. We can do this in an analogous way in O_N too. For examle, adding the equations (20) and (34), we get

$$
\begin{aligned}
& V^{\cdots}_{\cdots \underset{1}{|} m \underset{2}{|} n} - V^{\cdots}_{\cdots \underset{1}{|} n \underset{2}{|} m} + V^{\cdots}_{\cdots \underset{2}{|} m \underset{1}{|} n} - V^{\cdots}_{\cdots \underset{2}{|} n \underset{1}{|} m} = \\
(69) \qquad & = \sum_{\alpha=1}^{u} 2\, {}'\underset{1}{\widetilde{R}}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} 2\, {}''\underset{1}{\widetilde{R}}{}^{p}_{j_\beta mn} \binom{j_\beta}{p} V^{\cdots}_{\cdots} + \\
& + {}''\Gamma^{p}_{[mn]} (V^{\cdots}_{\cdots \underset{1}{|} p} - V^{\cdots}_{\cdots \underset{2}{|} p}),
\end{aligned}
$$

where

$$
{}^{\theta}\underset{1}{\widetilde{R}}{}^{i}_{jmn} = \frac{1}{2}({}^{\theta}\underset{1}{A} + {}^{\theta}\underset{3}{A})^{i}_{jmn} = \frac{1}{2}({}^{\theta}\underset{2}{A} + {}^{\theta}\underset{4}{A})^{i}_{jmn}, \tag{70}
$$

is a tensor.

Adding the equations (41) and (52) we obtain

$$
\begin{aligned}
& V^{\cdots}_{\cdots \underset{1}{|} m \underset{1}{|} n} - V^{\cdots}_{\cdots \underset{1}{|} n \underset{2}{|} m} + V^{\cdots}_{\cdots \underset{2}{|} m \underset{2}{|} n} - V^{\cdots}_{\cdots \underset{2}{|} n \underset{1}{|} m} = \\
& = \sum_{\alpha=1}^{u} 2' \underset{2}{\widetilde{R}}{}^{i_\alpha}_{pmn} \binom{p}{i_\alpha} V^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} 2'' \underset{3}{\widetilde{R}}{}^{p}_{j_\beta mn} \binom{j_\beta}{p} V^{\cdots}_{\cdots} + \\
& + V^{\cdots}_{\cdots \leq mn \gg} + V^{\cdots}_{\cdots \ll snm \geq}.
\end{aligned} \tag{71}
$$

By virtue of (44), (51) we see that the quantity

$$
\begin{aligned}
& V^{\ i_1 \cdots {}_u}_{\ j_1 \cdots j_v \leq mn \gg} + V^{i_1 \cdots {}_u}_{j_1 \cdots j_v \ll smn \geq} = \\
& = \sum_{\alpha=1}^{u-1} \sum_{\substack{\beta=2 \\ (\alpha<\beta)}}^{u} ({}'\Gamma^{i_\alpha}_{[pm]} {}'\Gamma^{i_\beta}_{sn} + {}'\Gamma^{i_\alpha}_{pn} {}'\Gamma^{i_\beta}_{[sm]} + {}'\Gamma^{i_\alpha}_{np} {}'\Gamma^{i_\beta}_{[ms]} + \\
& + {}'\Gamma^{i_\alpha}_{[mp]} {}'\Gamma^{i_\beta}_{ns}) \binom{p}{i_\alpha} \binom{s}{i_\beta} V^{\cdots}_{\cdots} - \\
& - \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} ({}'\Gamma^{i_\alpha}_{[pm]} {}''\Gamma^{s}_{j_\beta n} + {}'\Gamma^{i_\alpha}_{pn} {}''\Gamma^{s}_{[j_\beta m]} + {}'\Gamma^{i_\alpha}_{np} {}''\Gamma^{s}_{[m j_\beta]} + \\
& + {}'\Gamma^{i_\alpha}_{[mp]} {}''\Gamma^{s}_{n j_\beta}) \binom{p}{i_\alpha} \binom{j_\beta}{s} V^{\cdots}_{\cdots} + \\
& + \sum_{\alpha=1}^{v-1} \sum_{\substack{\beta=2 \\ (\alpha<\beta)}}^{v} ({}''\Gamma^{p}_{[j_\alpha m]} {}''\Gamma^{s}_{j_\beta n} + {}''\Gamma^{p}_{j_\alpha n} {}''\Gamma^{s}_{[j_\beta m]} + {}''\Gamma^{p}_{n j_\alpha} {}''\Gamma^{s}_{[m j_\beta]} + \\
& + {}''\Gamma^{p}_{[m j_\alpha]} {}''\Gamma^{s}_{n j_\beta}) \binom{j_\alpha}{p} \binom{j_\beta}{s} V^{\cdots}_{\cdots} = \\
& = \sum_{\alpha=1}^{u-1} \sum_{\substack{\beta=2 \\ (\alpha<\beta)}}^{u} ({}'\Gamma^{i_\alpha}_{[pm]} {}'\Gamma^{i_\beta}_{[sn]} + {}'\Gamma^{i_\alpha}_{[pn]} {}'\Gamma^{i_\beta}_{[sm]}) \binom{p}{i_\alpha} \binom{s}{i_\beta} V^{\cdots}_{\cdots} - \\
& - \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} ({}'\Gamma^{i_\alpha}_{[pm]} {}''\Gamma^{s}_{[j_\beta n]} + {}'\Gamma^{i_\alpha}_{[pn]} {}''\Gamma^{s}_{[j_\beta m]}) \binom{p}{i_\alpha} \binom{j_\beta}{s} V^{\cdots}_{\cdots} + \\
& + \sum_{\substack{\alpha=1 \\ (\alpha<\beta)}}^{v-1} \sum_{\beta=2}^{v} ({}''\Gamma^{p}_{[j_\alpha m]} {}''\Gamma^{s}_{[j_\beta n]} + {}''\Gamma^{p}_{[j_\alpha n]} {}''\Gamma^{s}_{[j_\beta m]}) \binom{j_\alpha}{p} \binom{j_\beta}{s} V^{\cdots}_{\cdots}
\end{aligned} \tag{72}
$$

is a tensor, and in (71) the quantities

$$(73)\qquad {}^{\theta}\underset{2}{\widetilde{R}}{}^{i}_{jmn}=\frac{1}{2}({}^{\theta}\underset{7}{A}+{}^{\theta}\underset{13}{A})^{i}_{jmn},\quad {}^{\theta}\underset{3}{\widetilde{R}}{}^{i}_{jmn}=\frac{1}{2}({}^{\theta}\underset{8}{A}+{}^{\theta}\underset{14}{A})^{i}_{jmn}$$

are also tensors.

We call the quantities ${}^{\theta}\underset{1}{\widetilde{R}}$, ${}^{\theta}\underset{2}{\widetilde{R}}$, ${}^{\theta}\underset{3}{\widetilde{R}}$ $\theta = ', ''$ *derived curvature tensors* of the space O_N. In addition to those presented here, one can obtain some other such tensors too (see [1]).

By a procedure analogous to that from [2] it can be proved that from the curvature tensors ${}^{\theta}\underset{1}{R},\dots,{}^{\theta}\underset{4}{R}$, ${}^{\theta}\underset{1}{\widetilde{R}}$, ${}^{\theta}\underset{2}{\widetilde{R}}$, ${}^{\theta}\underset{3}{\widetilde{R}}$ for a fixed θ ($'$ or $''$) only five of them are independent, while the rest can be expressed as linear conbinations of these five tensors.

If $'\Gamma = {}''\Gamma = \Gamma$, where Γ is a symmetric connection, then all Ricci type identities reduce to the known Ricci identity, all cited curvature tensors and pseudotensors are reduced to the Riemann-Christoffel curvature tensor of this connection, which can be easily proved from the obtained formulas.

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Received by the editors January 11, 2001