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INFINITELY NARROW SOLITON SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS

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Abstract. We construct an infinitely narrow N-soliton solution in an associated sense to a system of conservation laws in the frame of Colombeau's generalized functions with arbitrary initial data which are infinitely narrow solitons. With additional assumptions on the first derivatives of initial data, the solution is modified so that it also becomes the solution to the system of conservation laws in the sense of pointwise equality also.

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1. Introduction

Infinitely narrow solitons are used in quantum mechanics to describe the zero mass potential (see [12] and references therein).

Maslov, Omel'yanov and Tsupin ([7], [8]) have constructed an asymptotical solution $G_\varepsilon(x,t) = A\cosh^{-1}(c(x-vt)/\varepsilon)$, $\varepsilon \in (0,1)$ to the Inviscid Burgers' equation (in [7] it is called Hopf's equation)

$$\partial G/\partial t + G\partial G/\partial x = 0 \tag{1}$$

which is called an infinitely narrow soliton.

Biagioni and Oberguggenberger have constructed in [1] a generalized solution to the same equation in the frame of Colombeau's generalized functions represented by $G_\varepsilon(x,t) = 3c\,\mathrm{sech}^2(\sqrt{c}(x-ct)/(2\sqrt{\varepsilon}))$.

We refer to [12] for some other ideas based on a construction of an algebra of generalized functions which is different from $\mathcal{G}$.

In [1], [7] and [8], the solution is obtained by approximating (1) with the modified KdW-equation $U_t + UU_x + \mu U_{xxx} = 0$, where μ is an infinitely small number (in Colombeau's setting, μ is associated with zero).

The aim of this paper is to construct infinitely narrow N-soliton solutions to systems of conservation laws, by avoiding the approximating procedure, in the space of Colombeau's generalized functions. The results concerning scalar conservation laws are given in [10].

We use the simplified version of Colombeau's algebra.

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Let Ω be an open subset of $\mathbb{R}^n$.

$\mathcal{E}(\Omega)$ is defined to be a set of all functions $(\varepsilon, x) \mapsto F_\varepsilon(x)$, $(\varepsilon, x) \in (0,1) \times \Omega$ which are smooth on Ω for every fixed ε.

$\mathbb{R}_M$ is the set of all $A_\varepsilon : (0,1) \to \mathbb{R}$ such that there exist $N \in \mathbb{N}_0$, $C > 0$ and $\eta > 0$ such that $|A_\varepsilon| \leq C\varepsilon^{-N},\ \varepsilon < \eta$.

$\mathcal{E}_M(\Omega)$ is the set of all $G_\varepsilon \in \mathcal{E}(\Omega)$ such that for every compact set $K \subset \Omega$ and every $\beta \in \mathbb{N}_0^n$ there exist $N \in \mathbb{N}_0$, $C > 0$ and $\eta > 0$ such that $|\partial^\beta G_\varepsilon(x)| \leq C\varepsilon^{-N},\ \varepsilon < \eta,\ x \in K$.

$\mathbb{R}_0$ is the set of all $A \in \mathbb{R}_M$ such that for every $N \in \mathbb{N}$ there exist $C > 0$ and $\eta > 0$ such that $|A_\varepsilon| \leq C\varepsilon^{N},\ \varepsilon < \eta$.

$\mathcal{N}(\Omega)$ is the set of all $G \in \mathcal{E}_M$ such that for every $\beta \in \mathbb{N}_0^n$, every compact set $K \subset \Omega$ and every $N \in \mathbb{N}_0$ there exist $C > 0$ and $\eta > 0$ such that $|\partial^\beta G_\varepsilon(x)| \leq C\varepsilon^{N},\ \varepsilon < \eta,\ x \in K$.

The spaces of Colombeau's generalized complex numbers and generalized functions are defined by $\overline{\mathbb{R}} = \mathbb{R}_M / \mathbb{R}_0$ and $\mathcal{G} = \mathcal{E}_M / \mathcal{N}$, respectively. We denote by G or $[G_\varepsilon]$ the class of equivalence for G_ε.

We say that $G \in \mathcal{G}(\Omega)$ is of the bounded type if there exist $C > 0$ and $\eta > 0$ such that $\sup_{x \in \Omega} |G_\varepsilon(x)| \leq C,\ \varepsilon < \eta$.

$A \in \overline{\mathbb{R}}$ is associated to $c \in \mathbb{R}$, $A \approx c$, if $\lim_{\varepsilon \to 0} A_\varepsilon = c$. Generalized complex numbers A and B are associated if $A - B \approx 0$.

$G \in \mathcal{G}$ is associated to $H \in \mathcal{G}$, $G \approx H$ if

$$\int G(x)\psi(x)dx - \int H(x)\psi(x)dx \approx 0 \text{ for every } \psi \in C_0^\infty.$$

$G \in \mathcal{G}(\Omega)$ is null in $\Omega' \subset \Omega$ if $G_{|\Omega'} = 0$ in $\mathcal{G}(\Omega')$. The support of G is the complement of the largest open set Ω such that $G = 0$ in Ω. The subspace of compactly supported generalized functions is denoted by $\mathcal{G}_c$.

The space $\mathcal{O}_M(\mathbb{R}^n)$ consists of smooth functions whose all derivatives are polynomially bounded.

Generalized functions G and H are pointwisely equal (equal in pointwise sense) in $\mathcal{G}(\Omega)$ if $G(x) = H(x)$ in $\overline{\mathbb{R}}$ for every $x \in \Omega$.

Definition 1. *A solution $G(x,t) \in (\mathcal{G}(\mathbb{R}^2))^n$ to a system of n nonlinear equations is called an infinitely narrow N-soliton (N-INS, for short) if it can be written in the form*

$$G(x,t) = \sum_{i=1}^{N} G^i(x - v^i t - x^i) + Z(x,t), \tag{2}$$

where $Z(x,t) \in (\mathcal{G}_c(\mathbb{R}^2))^n$, $v^i \in \mathbb{R}$, $i \in \{1, \ldots, N\}$ and $G^i = (G_1^i, \ldots, G_n^i) \in (\mathcal{G}(\mathbb{R}))^n$ has the following properties

1. *G^i has a representative G_ε^i of bounded type for every $i = \{1, \ldots, N\}$.*

2. $\operatorname{supp} G^i_{j,\varepsilon} \subset [-h_\varepsilon, +h_\varepsilon]$, *$h_\varepsilon \to 0$ as $\varepsilon \to 0$ for every $i \in \{1.\ldots, N\}$, $j \in \{1, \ldots, n\}$.*

3. The generalized function represented by $G^i_{j,\varepsilon}/\varepsilon$ is associated to the delta distribution, $\delta(x)$, multiplied with some constant, $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, n\}$.

Then G^i is called an infinitely narrow soliton (INS, for short), v^i is called the speed of the soliton and $G^i(0)$ is its amplitude, $i \in \{1, \ldots, N\}$.

2. Main result

We consider the following system of conservation laws

$$\partial_t F(U) + \partial_x H(U) = 0 \tag{3}$$

with the initial data

$$\begin{aligned} U|_{t=0} &= A(x) = (A_1(x), \ldots, A_n(x)) \in (\mathcal{G}(\mathbb{R}))^n, \\ A_j(x) &= \sum_{i=1}^{N} G^i_j(x - x^i), \; j \in \{1, \ldots, n\}, \end{aligned} \tag{4}$$

where $F = (f_1, \ldots, f_n)$ and $H = (h_1, \ldots, h_n)$ are in $(\mathcal{O}_M(\mathbb{R}^n))^n$ and G^i satisfies the assumptions 1-3 in Definition 1. We shall call such data N-infinitely narrow data (N-IND, for short) and $x^i \in \mathbb{R}$ is called an initial position for the i-th soliton, $i \in \{1, \ldots, N\}$. We are looking for an infinitely narrow N-soliton solution to (3) in the associated sense.

In the sequel we will use the notation $(y_1, \ldots, y_n)$ and y for points in $\mathbb{R}^n$ and $\mathbb{R}$, respectively.

Theorem 1 *Let N-IND A be of the form (4).*
1. If there exist i and j such that

$$f_j(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n) = h_j(y_1, \ldots, y_{j-1}, 0, y_{j+1}, \ldots, y_n) = 0, \tag{5}$$

then there exists an N-INS solution to (3), (4) in the associated sense.

2. In addition to (5), assume that for every $i \in \{1, \ldots, N\}$, $j, k \in \{1, \ldots, n\}$ the following condition holds

$$\begin{aligned} v^i &:= \sum_{s=1}^{n} \partial_{y_s} h_k(G^i(0)) \frac{d}{dy} G^i_s(0) / \sum_{s=1}^{n} \partial_{y_s} f_k(G^i(0)) \frac{d}{dy} G^i_s(0) \\ &= \sum_{s=1}^{n} \partial_{y_s} h_j(G^i(0)) \frac{d}{dy} G^i_s(0) / \sum_{s=1}^{n} \partial_{y_s} f_j(G^i(0)) \frac{d}{dy} G^i_s(0) \end{aligned} \tag{6}$$

and

(7) $$(v^i - v^j)/(x^i - x^j) \neq (v^i - v^k)/(x^i - x^k).$$

Then there exists an N-INS solution to (3), (4) in the associated and pointwise sense.

Proof.
1. Let $v^1, \ldots, v^N$ be arbitrary real numbers. Then $G(x,t) = \sum_{i=1}^N G^i(x - v^i t - x^i) \in (\mathcal{G}(\mathbb{R}^2))^n$ is N-INS in the associated sense. Let us prove this. We use essentially the fact that the above system is in a conservation law. Let $A_\varepsilon = \bigcup_{i\in\{1,\ldots,N\}}\{(x,t) \in \mathbb{R}^2 |\ |x - v^i t - x^i| \leq h_\varepsilon\}$. If $\phi \in C_0^\infty(\mathbb{R}^2)$, then for every $j \in \{1, \ldots, n\}$

$$\begin{aligned}
& \int \partial_t f_j\left(\sum_{i=1}^N G^i(x - v^i t - x^i)\right)\phi(x,t)dxdt \\
+ & \int \partial_x h_j\left(\sum_{i=1}^N G^i(x - v^i t - x^i)\right)\phi(x,t)dxdt \\
= & -\int_{A_\varepsilon} f_j\left(\sum_{i=1}^N G^i(x - v^i t - x^i)\right)\partial_t\phi(x,t)dxdt \\
- & \int_{A_\varepsilon} h_j\left(\sum_{i=1}^N G^i(x - v^i t - x^i)\right)\partial_x\phi(x,t)dxdt \\
- & \int_{\mathbb{R}^2\setminus A_\varepsilon} f_j\left(\sum_{i=1}^N G^i(x - v^i t - x^i)\right)\partial_t\phi(x,t)dxdt \\
- & \int_{\mathbb{R}^2\setminus A_\varepsilon} h_j\left(\sum_{i=1}^N G^i(x - v^i t - x^i)\right)\partial_x\phi(x,t)dxdt \\
= & \ I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By (5) and by the fact that all of the supports of G^i's are contained in the set $\{(x,t),\ |x - v^i t - x^i| \leq h_\varepsilon\}$ by property 2 in Definition 1, we have $\operatorname{supp}\sum_{i=1}^N G^i \subset A_\varepsilon$. This implies $I_3 = I_4 = 0$. By the assumption 1 in Definition 1, $f_j(\sum_{i=1}^N G^i)$ and $h_j(\sum_{i=1}^N G^i)$, $j \in \{1, \ldots, n\}$ are bounded. Since $\operatorname{mes}\{(x,t),\ |x - v^i t - x^i| \leq h_\varepsilon\} \cap \operatorname{supp}\phi \to 0$ as $\varepsilon \to 0$, $I_1, I_2 \to 0$ as $\varepsilon \to 0$.

2. First, let N equals one. Then, after the substitution of $G^1(x - v^1 t - x^1)$ into (3),

$$-v^1\sum_{s=1}^n \partial_{y_s} f_j(G^1(0))\frac{d}{dy}G_s^1(0) + \sum_{s=1}^n \partial_{y_s} h_j(G^1(0))\frac{d}{dy}G_s^1(0) = 0$$

for every $j \in \{1, \ldots, n\}$ at the space point $x = v^1 t + x^1$ (at all other points we have 0=0 since the property of the support of G^1). This is possible if and only if (6) holds.

Let us now consider an arbitrary $N \in \mathbb{N}$. In this case the problem of the soliton interaction occurs. We shall solve this problem by adding some functions that have their supports contained in the regions of interaction of solitons.

Put

$$\begin{aligned} G(x,t) &= \sum_{i=1}^{N} G^i(x - v^i t - x^i) + \sum_{i,k=1}^{N} (B_1^{ik}(x,t) G_1^i(x - v^i t - x^i) G_1^k(x - v^k t - x^k), \\ &\quad \ldots, B_n^{ik}(x,t) G_n^i(x - v^i t - x^i) G_n^k(x - v^k t - x^k)) \end{aligned}$$

where $B_l^{jk} \in \mathcal{G}(\mathbb{R}^2)$, $l \in \{1, \ldots, n\}$ will be chosen later such that G solves (3) in the pointwise sense.

The support of G is contained in $\bigcup_{i=1}^{N} \operatorname{supp} G^i$, so by the same arguments as in the proof of the first part of the theorem one can see that this is a solution in the associated sense.

We have to prove the pointwise equality. There are three different cases.

a) $x \neq v^i t + x^i$ for every $i \in \{1, \ldots, N\}$. Then the left-hand side of (3) obviously equals zero due to (5). The condition means that x is not in any of the supports of solitons.

b) There exists only one $i \in \{1, \ldots, n\}$ such that $x = v^i t + x^i$. Then only $G^i(x,t) \neq 0$ and by (6) one can see that the left-hand side of (3) equals zero. The condition means that x is in a support of a soliton and there are no interactions of the solitons at the point x.

c) $x = v^i t + x^i = v^k t + x^k$. Note that condition (7) means that there are no more than two interacting solitons at the same time and in this case we have that the i-th and k-th soliton interact at the point x.

After substitution of $G(x,t)$ into the k-th equation of system (3) the j-th equation becomes

$$\begin{aligned} & \partial_t f_j \left(\sum_{m=1}^{N} G_1^m + \sum_{m,l=1}^{N} B_1^{ml} G_1^l G_1^m, \ldots, \sum_{m=1}^{N} G_n^m + \sum_{m,l=1}^{N} B_n^{ml} G_n^l G_n^m \right) \\ + \;& \partial_x g_j \left(\sum_{m=1}^{N} G_1^m + \sum_{m,l=1}^{N} B_1^{ml} G_1^l G_1^m, \ldots, \sum_{m=1}^{N} G_n^m + \sum_{m,l=1}^{N} B_n^{ml} G_n^l G_n^m \right) \\ = \;& \sum_{s=1}^{n} \left(\partial_{y_s} f_j \left(\sum_{m=1}^{N} G_1^m + \sum_{m,l=1}^{N} B_1^{ml} G_1^l G_1^m, \ldots, \sum_{m=1}^{N} G_n^m + \sum_{m,l=1}^{N} B_n^{ml} G_n^l G_n^m \right) \right) \\ & \cdot \Big(\sum_{m=1}^{N} -v^m \frac{d}{dy} G_s^m + \sum_{m,l=1}^{N} -v^m B_s^{ml} \frac{d}{dy} G_s^m G_s^l - v^l B_s^{ml} G_s^m \frac{d}{dy} G_s^l + \partial_t B_s^{ml} G_s^m G_s^l \Big) \end{aligned}$$

$$+ \quad \left(\partial_{y_s} h_j \left(\sum_{m=1}^{N} G_1^m + \sum_{m,l=1}^{N} B_1^{ml} G_1^l G_1^m, \dots, \sum_{m=1}^{N} G_n^m + \sum_{m,l=1}^{N} B_n^{ml} G_n^l G_n^m\right)\right)$$
$$\cdot \left(\sum_{m=1}^{N} \frac{d}{dy} G_s^m + \sum_{m,l=1}^{N} B_s^{ml} \frac{d}{dy} G_s^m G_s^l + B_s^{ml} G_s^m \frac{d}{dy} G_s^l + \partial_x B_s^{ml} G_s^m G_s^l\right).$$

If $x = v^i t + x^i = v^k t + x^k$ then all of $B_r^{ml}(0)$ equals zero if m, l are not equal to i or k, $r \in \{1, \dots, n\}$. We shall take $B_r^{ml} \equiv 0$ if $m \geq l$. By using the fact that $f_j(0) = h_j(0) = 0$, $j \in \{1, \dots, n\}$ we obtain that at the points $x = v^i t + x^i = v^k t + x^k$ the above sum equals (for $i < k$)

$$\sum_{s=1}^{n} \big(\partial_{y_s} f_j\big(\sum_{m=1}^{N} G_1^m(0) + \sum_{m,l=1}^{N} B_1^{ml} G_1^l(0) G_1^m(0), \dots, \sum_{m=1}^{N} G_n^m(0)$$
$$+ \quad \sum_{m,l=1}^{N} B_n^{ml} G_n^l(0) G_n^m(0)\big)\big)$$
$$\cdot \left(-v^i \frac{d}{dy} G_s^i(0) - v^k \frac{d}{dy} G_s^k(0) - v^i B_s^{ik} \frac{d}{dy} G_s^i(0) G_s^k(0) \partial_t B_s^{ml} G_s^m(0) G_s^l(0)\right)$$
$$+ \quad \big(\partial_{y_s} h_j\big(\sum_{m=1}^{N} G_1^m(0) + \sum_{m,l=1}^{N} B_1^{ml} G_1^l(0) G_1^m(0), \dots, \sum_{m=1}^{N} G_n^m(0)$$
$$+ \quad \sum_{m,l=1}^{N} B_n^{ml} G_n^l(0) G_n^m(0)\big)\big)$$
$$\cdot \left(\frac{d}{dy} G_s^i(0) + \frac{d}{dy} G_s^k(0) + B_s^{ik} \frac{d}{dy} G_s^i(0) G_s^k(0) + \partial_x B_s^{ml} G_s^m(0) G_s^l(0)\right).$$

Let us remark that the values of B_s^{ml} are evaluated at the point of intersection. Chose B such that $B_l^{ij}(v^i t + x^i, t) = -1/G_l^j(0) =: C_{1,l} \in \overline{\mathbb{R}}$, $l \in \{1, \dots, n\}$, when $x = v^i t + x^i = v^j t + x^j$. Then, for every $l \in \{1, \dots, n\}$ the following equality

$$\sum_{s=1}^{n} \partial_{y_s} f_j(G_1^k(0), \dots, G_n^k(0))(-v^k \frac{d}{dy} G_s^k(0) + \partial_t B_s^{ik} G_s^i(0) G_s^k(0)$$
$$+ \quad v^k G_s^i(0) \frac{d}{dy} G_s^k(0) / G_s^k(0))$$
$$+ \quad \sum_{s=1}^{n} \partial_{y_s} h_j(G_1^k(0), \dots, G_n^k(0))(\frac{d}{dy} G_s^k(0) + \partial_x B_s^{ik} G_s^i(0) G_s^k(0)$$
$$- \quad G_s^i(0) \frac{d}{dy} G_s^k(0) / G_s^k(0)) = 0$$

should be true. Finally, put

$$\partial_t B_s^{ik}(x,t) = -v^k \frac{d}{dy} G_s^k(0)/(G_s^k(0))^2 =: C_{2,s} \in \overline{\mathbb{R}}$$

and

$$\partial_x B_s^{ik}(x,t) = \frac{d}{dy} G_s^k(0)/(G_s^k(0))^2 =: C_{3,s} \in \overline{\mathbb{R}}.$$

Since G^k is a solution in a pointwise sense at all points, it satisfies

$$\sum_{s=1}^{n} -v^k \partial_{y_s} f_j(G_1^k(0),\dots,G_s^k(0)) \frac{d}{dy} G_s^k(0) + \partial_{y_s} h_j(G_1^k(0),\dots,G_n^k(0)) \frac{d}{dy} G_s^k(0) = 0$$

for every $j \in \{1,\dots,n\}$. Let us note that $v^k C_{3,s} + C_{2,s} = 0$ and this implies that one can take $B_s^{ik}(x,t) = C_1 - x^k C_{3,s} + C_{2,s} t + C_{3,s} x$, $s \in \{1,\dots,n\}$ for $i < k$, otherwise $B_s^{ik}(x,t) = 0$, $s \in \{1,\dots,n\}$.

This proves the theorem since

$$\begin{aligned} Z(x,t) \quad := \quad & \sum_{i,k=1}^{N} (B_1^{ik}(x,t) G_1^i(x - v^i t - x^i) G_1^k(x - v^k t - x^k), \\ & \dots, B_n^{ik}(x,t) G_n^i(x - v^i t - x^i) G_n^k(x - v^k t - x^k)) \end{aligned}$$

has a compact support (i.e. satisfies equation (2)). □

Remark 1. *If we allow $v^i \in \overline{\mathbb{R}}$, $i \in \{1,\dots,n\}$ then we have the following special cases.*

a) If v^i is associated with a real number then we can assume that the resulting function G^i represents an "approximate" soliton with the velocity converging to some real value.

b) If a representative of v^i, v^i_ε, converge to infinity as $\varepsilon \to 0$, then we obtain infinitely fast solitons mentioned in [11].

Remark 2 *Due to the boundedness condition for N-IND in Theorem 1, the most natural class for it is the class of the generalized functions G^i_j represented by $G^i_{j,\varepsilon}(y) = g^i_j(y/\varepsilon)$, where g^i_j is in C_0^∞, $0 \in \operatorname{supp} g^i_j$. The solutions of this form are in fact obtained in [1], [7] and [8].*

Remark 3 *The case when the initial positions x^i_s of G^i_s, $i, s \in \{1,\dots,n\}$, are not necessarily the same, then the above theorem holds true, and the proof is almost the same. One just has to take care of some technical details, for example, one has to put the condition*

$$(v^i - v^j)/(x^i_s - x^j_s) \neq (v^i - v^k)/(x^i_s - x^k_s) \tag{8}$$

instead of (7).

3. Comments

3.1 Multidimensional case

The first part of Theorem 1 holds in the multidimensional case $(x,t) \in \mathbb{R}^{m+1}$ if the supports of the solitons are real algebraic varieties (real zeros of the polynomials). Instead of the initial points x^i we have initial varieties S^i, $S^i = \{x \in \mathbb{R}^m,\ P^i(x) = 0\}$, where P^i is some polynomial, $i \in \{1,\dots,n\}$. Solitons are now functions given by $G^i(P^i(x) - v^i t)$, $G^i \approx 0$, G^i is bounded, $\operatorname{supp} G^i \subset \{(x,t),\ P^i(x) = v^i t\} + B_0(h_\varepsilon)$, where $B_0(h_\varepsilon)$ denotes the ball with the center at zero and its radius equals h_ε and the generalized function determined by the representative $G^i_\varepsilon(x)/\varepsilon$ is associated with the multidimensional delta distribution, $i \in \{1,\dots,n\}$. The proof that there exists the N-INS solution to (3) in the associated sense follows from Appendix in [9], by estimating the measure of the set $\operatorname{supp} G^i \cap \operatorname{supp} \phi$, $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ by $V(\varepsilon)$ which tends to zero as $\varepsilon \to 0$. The solitons may be given by the functions represented by $(g^i_1(y/\varepsilon),\dots,g^i_n(y/\varepsilon))$, $g^i_j \in C_0^\infty(\mathbb{R})$ whose supports contain the sets S^i, $i \in \{1,\dots,N\}$, $j \in \{1,\dots,n\}$. The question when the second part of the theorem holds is open.

3.2 Conservation laws in evolution form

In the case of $f_i(u) \equiv u_i$ in Section 2 (evolution case) condition (6) may be substituted by a more appropriate one. There exists N-INS solution in the associated and pointwise sense to (3) if N-IND in (4) satisfies: For every $i \in \{1,\dots,n\}$, $\frac{d}{dy}G^i(0)$ is an eigenvector to the matrix $DH(G^i(0)) = [\partial_j h_k(G^i(0))]$ and v^i is its eigenvalue.

3.3 Inviscid Burgers' Equation

Considering this simpler case (see [10]), we shall give some interesting features of infinitely narrow solitons.

As in the case of the systems, there always exists an N-INS solution in the associated sense for arbitrary N-IND. Now, let us consider the solutions in association and pointwise sense. In this case we do not need assumption on the derivatives of the given N-IND. It is only necessary that $v^i = G^i(0)$ and that (7) holds. One can say that the solitons are determined by initial positions and amplitudes in this case, contrary to the systems. Also, if all first derivatives of the initial data at the initial position x^i equals zero, $\frac{d}{dy}G^i(0) = 0$, then there exist a solution in associated and pointwise sense with arbitrary velocities v^i, $i \in \{1,\dots,n\}$.

One can see that there are no exact solutions to (1) in $\mathcal{G}$, since it would imply $\frac{d}{dy}(G^2/2 - vG) = 0$, for $G(\xi) = G(x - vt - x^0)$, which means, by [11], that $G^2/2 - vG = \text{const}$, what is impossible.

Let us note that B^{ij}, defined in the proof of Theorem 1, are the real constants, equal $-1/v^i$ or $-1/v^j$, i. e. the solution to (1) is given by

$$G(x,t) = \sum_{i=1}^{N} G^i(x - v^i t - x^i) - \sum_{i,j=1 i<j}^{N} (1/v^j) G^i(x - v^i t - x^i) G^j(x - v^j t - x^j).$$

In this case, two solitons will interact when $t = (x^i - x^j)/(v^i - v^j)$. At this moment, the value of G will be v^1 or v^2 in $x = v^1 t + x^1 = v^2 t + x^2$ and zero elsewhere. For other values of t we shall have the same situation as before the interacting. This process looks like the one soliton just goes over the another without change of the shape or velocity.

Example 1 *Let us find the infinitely narrow soliton solution to (1) when the condition ensuring that there are no more than two-soliton interactions, (7), is not satisfied. Let all initial positions be at one point, say zero. For such condition there is an infinitely narrow N-soliton solution, too, but now we have to give stronger assumptions on the initial data functions* G^i*. Let* $\frac{d}{dy} G^i(0) = v^i g_0$*, for some fixed constant* g_0*, i. e.* $\frac{d}{dy} G^i(0) / \frac{d}{dy} G^j(0) = v^i / v^j$*. Then we have an infinitely narrow N-soliton solution to (1), given by the representative*

$$G_\varepsilon(x,t) = \sum_{i=1}^{N} G^i_\varepsilon((x - v^i t)/\varepsilon) + \alpha \prod_{i=1}^{N} G^i_\varepsilon((x - v^i t)/\varepsilon),$$

where

$$\begin{aligned} \alpha &= -(v^1 + \ldots + v^N)/(2v^1 \ldots v^N) \\ &\pm ((N^2 - 4N)(v^1 + \ldots + v^N)^2 + 4N((v^1)^2 + \ldots (v^N)^2))^{1/2}/(2Nv^1 \ldots v^N). \end{aligned}$$

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