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ELEMENTS OF SEQUENCE ALGEBRAS

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Abstract. Our object in this article is to consider the structure of Hadamard topological algebras. Linear forms, Hadamard homomorphisms and units are determined, as well as properties of principal, maximal and prime ideals.

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1. Introduction

Among other generalizations of fractional integro-differentiation, the processes based upon treatment of suitable sequences of complex numbers go back to the developments by A. Gel´fand & A. Leont´ev ([4]). They defined

$$\mathcal{D}^p\left\{a,f\right\}(z)=\sum_{n=p}^{\infty}\frac{a_{n-p}}{a_n}\ f_n\ z^{n-p}, \tag{1}$$

where $p\in\mathbb{N}_0,\ f(z)=\sum_{n=0}^{\infty}f_n\ z^n$ is assumed to be analytic in the unit disc and $a(z)=\sum_{n=0}^{\infty}\ a_n\ z^n$ is a suitable entire function. In particular, if $a(z)=e^z$ then (1) becomes usual derivation. More generally, let us consider the formal operator

$$\mathcal{D}\left\{a,f\right\}(z)=\sum_{n=0}^{\infty}\ a_n\ f_n\ z^n. \tag{2}$$

This expression is known as the Hadamard product composition of the functions $a(z)$ and $f(z)$. Of course (2) is a very wide generalization of the integro-differentiation notion, indeed in the case that $a_n\to\infty$. The equation (2) is invertible if all coefficients a_n are not zero and the corresponding operator is

$$\mathcal{I}\left\{a,f\right\}(z)=\sum_{n=0}^{\infty}\frac{f_n}{a_n}\quad z^n, \tag{3}$$

which may be called generalized integration under the assumption that $a_n\to\infty$. Integral representations of (2) and (3) were given by Lebedev & Smirnov ([7],

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p. 168). Therefore, appropriate choices of $a(z)$ give variants of the Riemann - Liouville, Ruscheweyh, Weyl or Dzherbashyan fractional calculus ([5], p. 429). There is a wide literature containing generalizations and development of ideas connected with this theory, including also constructions of fractional operators not necessarily supported on countable sets as can be seen in [6]. Nowadays there is a strong research in Fractional Calculus applied to classes of analytic functions. Several fractional operators have successfully been applied in obtaining characterization properties, coefficient estimates, distortion inequalities, convolution structures for various subclasses of analytic functions, etc., (e.g. [1], [2], [8], [11]). Unfortunately, algebraic structures of classes of analytic functions closed under Hadamard product compositions are in general difficult to describe. Precisely, our objective is to consider algebraic topological structures within which Hadamard product becomes continuous. In Section 2 we introduce a topological complete algebra $\mathcal{A}_0$ of sequences of complex numbers endowed with a Hadamard type product. In Section 3 we determine and characterize continuous linear forms and complex homomorphisms. Hadamard units are studied in Section 4, principal ideals in Sections 5 and 6 and prime ideals in Section 7.

2. A topological algebra of sequences

Let us consider the set $\mathcal{A}_0$ of complex sequences $a = (a_n)_{n\geq 0}$ such that the number $\rho(a) = \overline{\lim}\ |a_n|^{1/n}$ is finite. Since we are concerned with superior limits the evaluation of nth radicals causes no trouble in the case $n = 0$. Given $a = (a_n)_{n\geq 0}$, $b = (b_n)_{n\geq 0} \in \mathcal{A}_0$, $\zeta \in \mathbb{C}$ and a sequence $\varepsilon = (\varepsilon_n)_{n\geq 0}$ such that $|\varepsilon_n| = 1$ for each $n \geq 0$ we define $\zeta \cdot a + b \in \mathcal{A}_0$ and $a \odot_\varepsilon b \in \mathcal{A}_0$ as $\zeta \cdot a + b = (\zeta\ a_n + b_n)_{n\geq 0}$ and $a \odot_\varepsilon b = (\varepsilon_n\ a_n\ b_n)_{n\geq 0}$.

Proposition 1. *The set $(\mathcal{A}_0, +, \cdot, \odot_\varepsilon)$ is an Abelian - unit - complex algebra with $\varepsilon^* = (\overline{\varepsilon_n})_{n\geq 0}$ as unit.*

Remark 2. *Let $An(0)$ be the class of entire functions and for $r > 0$ let $An(r)$ be the set of analytic functions in the open circle with center zero and radius $1/r$. We´ll consider $An(r)$ with the topology of uniform convergence on compact subsets. On the other hand, let $\mathcal{A}_0(r)$ be the set of those elements $a \in \mathcal{A}_0$ such that $\rho(a) \leq r$. The map $\digamma_r : \mathcal{A}_0(r) \to An(r)$, $\digamma_r(a)(z) = \sum_{n=0}^{\infty} a_n\ z^n$ is bijective. Now, given $(a_n)_{n\geq 0} \subseteq \mathcal{A}_0(r)$, $a \in \mathcal{A}_0(r)$ we will write $a_n \to a$ if and only if $\digamma_r(a_n) \to \digamma_r(a)$ in $An(r)$, i.e. $\mathcal{A}_0(r)$ has the topology with respect to which $\digamma_r$ becomes an homeomorphism. In particular, $\mathcal{A}_0(r)$ is a complete space as can be seen by standard arguments. Moreover, if $r < s$ the topology of $An(r)$ is stronger than that induced on it by $An(s)$. In particular, the inclusion $\iota_{r,s} : An(r) \hookrightarrow An(s)$ is continuous. Of course, the same holds if we replace $An(r)$ and $An(s)$ by $\mathcal{A}_0(r)$ and $\mathcal{A}_0(s)$ respectively. Thus, $\mathcal{A}_0$ may be performed as a countable union space in the sense of Gelfand and Shilov [3]. So, a sequence $\{a_k\}_{k\geq 1}$ is said to converge to a in $\mathcal{A}_0$ if all the a_k and a belong to some particular*

$\mathcal{A}_0(r)$ and $\{a_k\}_{k\geq 1}$ converges to a in $\mathcal{A}_0(r)$. Indeed, the space $\mathcal{A}_0$ has a Fréchet structure. The same conclusion holds for the space $\mathfrak{A}_0 = An(0) \,\cup\, \cup_{r>0} An(r)$ of analytic functions at zero.

Theorem 3. *(cf. [10]) Let $a_n = (a_{nm})_{m\geq 0}$, $n = 0,\ 1,\ \ldots$ be a sequence of elements of $\mathcal{A}_0$. The following assertions are equivalent: (i) $a_n \to 0$ in $\mathcal{A}_0$. (ii) $\digamma_r(a_n) \to 0$ in $An(r)$, $r \geq 0$. (iii) There is $r > 0$ such that $\rho(a_n) \leq r$ for $n \geq 0$ and for all $\delta > 0,\ t > r$ there exist n_0 such that $|a_{nm}| \leq \delta\ t^m$ if $n \geq n_0$ and $m \in \mathbb{N}_0$.*

Corollary 4. *$(\mathcal{A}_0, +, \cdot, \odot_\varepsilon)$ is a topological algebra.*

3. Linear forms and Hadamard homomorphisms

Proposition 5. *(cf. [10]) Let $\varphi : \mathfrak{A}_0 \to \mathbb{C}$ be a linear form. Then φ is continuous if and only if for all $r \geq 0$ and all $s > r$ there is a positive constant $C_{r,s}$ such that $|\langle f, \varphi\rangle| \leq C_{r,s}\ \sup_{|z|\leq 1/s}\ |f(z)|$.*

Lemma 6. *(cf. [10]) Let $0 < r < 1$, $a = (a_n)_{n\geq 0}$, $a \in \mathcal{A}_0(r)$ such that $|a_n| < 1$ for all n. Then the element $\varepsilon^* - a$ is inversible and $(\varepsilon^* - a)^{-1} = \sum_{m=0}^{\infty}\ a^m$.*

Proposition 7. *(cf. [10]) If $r,\ s$ are positive then $\mathcal{A}_0(r)$ and $\mathcal{A}_0(s)$ are linearly homeomorphic.*

Corollary 8. *Every complex valued Hadamard homomorphism is continuous.*

Definition 9. *With the above notation, if $f \in An(r),\ g \in An(s)$ we will write*

$$f \odot_\varepsilon g = \digamma_{rs}\left[(\digamma_r)^{-1} f \odot_\varepsilon (\digamma_s)^{-1} g\right].$$

Corollary 10. *For every non-zero Hadamard homomorphism $\varkappa : \mathfrak{A}_0 \to \mathbb{C}$ there is a unique $p \in \mathbb{N}_0$ such that $\langle f, \varkappa\rangle = \varepsilon_p\ f^{(p)}(0)/p!$, $\ f \in \mathfrak{A}_0$. Moreover, $\ker(\chi)$ is a maximal closed ideal of $\mathcal{A}_0$, it is principal and $\ker(\chi) = \left\langle ((1-\delta_{pn})\ \overline{\varepsilon_n})_{n\geq 0}\right\rangle$.*

Corollary 11. *The Hadamard ring $(\mathfrak{A}_0, +, \odot_\varepsilon)$ has a trivial Jacobson radical.*

4. On Hadamard units

Theorem 12. *(cf. [10]) An element $a \in \mathcal{A}_0$ is a unit, i.e. $a \in \mathbb{U}(\mathcal{A}_0)$, if and only if $a_n \neq 0$ for all n and $\underline{\lim}\ |a_n|^{1/n} > 0$.*

Example 13. *Let $f \in \mathfrak{A}_0$ be given as $f(z) = z^{-1}\log(1-z)^{-1}$. If $|\omega| = 1$ and $\varepsilon = (\omega^n)_{n\geq 1}$ then $f \in \mathbb{U}(\mathfrak{A}_0)$, its inverse being $f^{-1}(z) = 1/(1-z/\omega^2)^2$. Here we have $f \in An(1)$ and $\digamma_1(\varepsilon^*)(z) = \omega/(w-z)$ is the Hadamard neutral product element, i.e. $g \odot_\varepsilon \digamma_1(\varepsilon^*) = g$ for all $g \in \mathfrak{A}_0$.*

5 On principal ideals

Proposition 14. *(cf. [10]) Let $a \in \mathcal{A}_0$, $M_a(b) = a \odot_\varepsilon b$, $b \in \mathcal{A}_0$. If $\rho(a) > 0$ and $s \geq 0$ then (i) $M_a : \mathcal{A}_0(s) \to \mathcal{A}_0(\rho(a)\ s)$ is injective if and only if $a_n \neq 0$ for all $n \geq 0$. (ii) It is surjective if and only if $a \in \mathbb{U}(\mathcal{A}_0)$.*

Corollary 15. *If $\rho(a) > 0$ and $s \geq 0$ then $M_a : \mathcal{A}_0(s) \to \mathcal{A}_0(\rho(a)\ s)$ is a linear homeomorphism if and only if it is surjective.*

Remark 16. *If $a \in \mathcal{A}_0$ wewrite Supp(a) for the set of those $n \in \mathbb{N}_0$ such that $a_n \neq 0$. Then*

$$\langle a \rangle = \left\{ b \in \mathcal{A}_0 : \ Supp(b) \subseteq Supp(a) \ \text{ and } \ \overline{\lim}_{n/a_n \neq 0} \left| \frac{b_n}{a_n} \right|^{1/n} < +\infty \right\}.$$

Remark 17. *Let a and b be two different generators of a principal ideal. Then $Supp(a) = Supp(b)$ and there exist constants $C \geq 1$ and $N \in \mathbb{N}_0$ such that $C^{-n} \leq |a_n/b_n| \leq C^n$ whenever a_n nor b_n are zero and $n \geq N$. Indeed, these conditions induce a partition of $\mathcal{A}_0$ into equivalent classes in such a way that any two elements belong to the same class if and only if they generate the same principal ideal.*

Proposition 18. *With the former notation, (i) If $\underline{\lim}\ |a_n|^{1/n} = 0$ and $a_n \neq 0$ for all n then $\langle a \rangle$ is not maximal. (ii) If $\underline{\lim}\ |a_n|^{1/n} > 0$ and $\langle a \rangle$ is maximal there is a unique $p \in \mathbb{N}_0$ such that $a_p = 0$.*

Proof. Given $b \in \mathcal{A}_0$ such that $\inf |b_n|^{1/n} > 0$ we get $\overline{\lim}\ |a_n/b_n|^{1/n} < +\infty$ and $\overline{\lim}\ |b_n/a_n|^{1/n} = +\infty$. Therefore $\langle a \rangle \subseteq \langle b \rangle$, $b \notin \langle a \rangle$ and (i) holds. On the other hand, let $\langle a \rangle$ be maximal and $\underline{\lim}\ |a_n|^{1/n} > 0$. Assertion (ii) follows taking into account Corollary 10 and that $a \notin \mathbb{U}(\mathcal{A}_0)$. □

Corollary 19. *A principal ideal $\langle a \rangle$ of $\mathcal{A}_0$ is maximal if and only if there is a unique index p such that $a_p = 0$ and $\underline{\lim}\ |a_n|^{1/n} > 0$.*

Remark 20. *We already know that kernels of non-zero complex Hadamard homomorphisms are closed maximal principal ideals. In particular, all maximal ideals that contain those elements $a \in \mathcal{A}_0$ such that $a_n = 0$ for all but a finite number of indexes n´s are dense in $\mathcal{A}_0$.*

Lemma 21. *(cf. [9]) If $a \in \mathcal{A}_0$ then $\langle a \rangle$ is not closed if and only if there is a sequence $\{b_n\}_{n \geq 0}$ in $\mathcal{A}_0$ such that: (i) Each sequence $(b_{np})_{n \geq 0}$ has a finite limit as $n \to +\infty$ if $p \in Supp(a)$; (ii) $\overline{\lim}_{p/a_p \neq 0}\ \lim_n\ |b_{np}|^{1/p} = +\infty$; (iii) The quantities $r_1 = \sup_n\ \overline{\lim}_p\ |a_p|^{1/p}\ |b_{np}|^{1/p}$ and $r_2 = \overline{\lim}_{p/a_p \neq 0} \left[|a_p|^{1/p}\ \lim_n\ |b_{np}|^{1/p} \right]$ are finite; (iv) If $t > \max\{r_1,\ r_2\}$ and $\delta > 0$ there is $N \in \mathbb{N}$ such that $|a_p\ (b_{np} - b_{mp})| \leq \delta\ t^p$ whenever $n \geq N$, $m \geq N$, $p \geq 0$.*

Theorem 22. *With the above notation, the following assertions are equivalent: (i) $\langle a\rangle$ is closed; (ii) $Supp(a)$ is finite or $\underline{\lim}\ |a_n|^{1/n} > 0$; (iii) $rad\ \langle a\rangle$ is principal.*

Proof. For the equivalence between (i) and (ii) see [9]. $[(ii) \Rightarrow (iii)]$ We observe that $Supp(b) \subseteq Supp(a)$ for any $b \in rad\ \langle a\rangle$. Therefore, if $\sharp(Supp(a))$ is finite or $\underline{\lim}\ |a_n|^{1/n} > 0$ then $rad\ \langle a\rangle \subseteq \langle a\rangle$ as follows by Remark 16 and (iii) holds. $[(iii) \Rightarrow (ii)]$ If $\sharp(Supp(a))$ is not finite, $\underline{\lim}\ |a_n|^{1/n} = 0$ and $p \in \mathbb{N}$ we´ll write $|a|^{1/p} = (|a_n|^{1/p})_{n\geq 0}$. Then $\rho(|a|^{1/p}) = \rho(a)^{1/p}$, i.e. $|a|^{1/p} \in \mathcal{A}_0$ and $\left\langle |a|^{1/p}\right\rangle = \{b \in \mathcal{A}_0 : \quad b^p \in \langle a\rangle\}$. Moreover, $\left\langle |a|^{1/p}\right\rangle$ is a proper subset of $\left\langle |a|^{1/(p+1)}\right\rangle$ because $\overline{\lim}\ |a_n|^{-[np(p+1)]^{-1}} = +\infty$. Since $rad\ \langle a\rangle = \cup_{p=1}^{\infty} \left\langle |a|^{1/p}\right\rangle$ it is not possible that it be principal. □

In particular, from the above proof we deduce the following

Corollary 23. *The ring $\mathcal{A}_0$ is non-Noetherian.*

6. Operations between principal ideals

Proposition 24. *The sum of principal ideals is principal.*

Proof. Given $a,\ b \in \mathcal{A}_0$ let $c = (\max\{|a_n|,\ |b_n|\})_{n\geq 0}$. Then $c \in \mathcal{A}_0$ because $\rho(c) \leq \rho(a) + \rho(b)$. It is easy to see $c \in \langle a\rangle + \langle b\rangle$. Moreover, if $\widetilde{a},\ \widetilde{b}$ belong to $\mathcal{A}_0$ the equation $c \odot_\varepsilon \widetilde{c} = a \odot_\varepsilon \widetilde{a} + b \odot_\varepsilon \widetilde{b}$ admits the solution

$$\widetilde{c} = \left(\frac{\left[a_n\ \widetilde{a}_n + b_n\ \widetilde{b}_n\right]\ \varkappa_{Supp(a)\cup Supp(b)}(n)}{\max\{|a_n|,\ |b_n|\}} \right)_{n\geq 0},$$

i.e. c is a generator of $\langle a\rangle + \langle b\rangle$. □

Corollary 25. *Every finitely generated ideal of $\mathcal{A}_0$ is principal.*

Proposition 26. *The intersection of principal ideals is a principal ideal.*

Proof. If $a,\ b \in \mathcal{A}_0$ then

$$\langle a\rangle \cap \langle b\rangle = \left\{ d \in \mathcal{A}_0 : \begin{array}{c} Supp(d) \subseteq Supp(a) \cap Supp(b), \\ \overline{\lim}_{n/a_n\neq 0}\ |d_n/a_n|^{1/n} < +\infty \quad and \\ \overline{\lim}_{n/b_n\neq 0}\ |d_n/b_n|^{1/n} < +\infty \end{array} \right\}.$$

If $c = (\min\{|a_n|,\ |b_n|\})_{n\geq 0}$ then $\rho(c) \leq \max\{\rho(a),\ \rho(b)\}$ and $c \in \mathcal{A}_0$. Since

$$Supp(c) = Supp(a) \cap Supp(b),$$

$\overline{\lim}_{n/a_n\neq 0}\ |c_n/a_n|^{1/n} \leq 1$ and $\overline{\lim}_{n/b_n\neq 0}\ |c_n/b_n|^{1/n} \leq 1$ we have $c \in \langle a\rangle \cap \langle b\rangle$. Now, if $d \in \langle a\rangle \cap \langle b\rangle$ and $d = a \odot_\varepsilon \widetilde{a} = b \odot_\varepsilon \widetilde{b}$ we seek for an element $\widetilde{c} \in \mathcal{A}_0$ such that $d = c \odot_\varepsilon \widetilde{c}$. Therefore we can write $\widetilde{c} = \left(\overline{\varepsilon_n}\ d_n/c_n\ \varkappa_{Supp(c)}(n)\right)_{n\geq 0}$. It is well defined because $\rho(\widetilde{c}) \leq \max\left\{\rho(\widetilde{a}),\ \rho(\widetilde{b})\right\}$ and c generates $\langle a\rangle \cap \langle b\rangle$. □

Proposition 27. *The ideal quotient of principal ideals is a principal ideal.*

Proof. If $a,\ b \in \mathcal{A}_0$ we write $c = (c_n)_{n\geq 0}$, with

$$c_n = \begin{cases} 0 & if \quad n \in Supp(b) - Supp(a), \\ \overline{\varepsilon_n} & if \quad n \notin Supp(b) \quad or \quad 0 < |b_n| < |a_n|\,, \\ \overline{\varepsilon_n}\ a_n\ /\ b_n & if \quad 0 < |a_n| \leq |b_n| \end{cases}$$

for each $n \in \mathbb{N}_0$. Thus we obtain $c \in \mathcal{A}_0$ and if $y \in (\langle a\rangle : \langle b\rangle)$ we show that $y = c \odot_\varepsilon z$ for some $z \in \mathcal{A}_0$. For, if $a \odot_\varepsilon x = b \odot_\varepsilon y$ with $x \in \mathcal{A}_0$ and $n \in \mathbb{N}_0$ it suffices to define

$$z_n = \begin{cases} 0 & if \quad n \in Supp(b) - Supp(a), \\ y_n & if \quad n \notin Supp(b) \quad or \quad 0 < |b_n| < |a_n|\,, \\ x_n & if \quad 0 < |a_n| \leq |b_n|\,. \end{cases}$$

Moreover, if $w \in \mathcal{A}_0$ and $t = c \odot_\varepsilon w$ there is $s \in \mathcal{A}_0$ such that $a \odot_\varepsilon s = b \odot_\varepsilon t$. In this case if $n \in \mathbb{N}_0$ we define

$$s_n = \begin{cases} 0 & if \quad n \in Supp(b) - Supp(a) \cup \mathbb{N}_0 - Supp(b), \\ w_n\ b_n\ /\ a_n & if \quad 0 < |b_n| < |a_n|\,, \\ w_n & if \quad 0 < |a_n| \leq |b_n| \end{cases}$$

and the claim holds, i.e. $(\langle a\rangle : \langle b\rangle) = \langle c\rangle$. □

7. On prime ideals

Theorem 28. *A principal ideal $I = \langle a\rangle$ is prime if and only if $\underline{\lim}\ |a_n|^{1/n} > 0$ and $\sharp(\mathbb{N}_0 - Supp(a)) = 1$.*

Proof. If $a_{n_0} = 0$ for a unique non-negative integer n_0 and $\underline{\lim}\ |a_n|^{1/n} > 0$ then $I = \left\langle (1-\delta_{nn_0})_{n\geq 0}\right\rangle$. By Corollary 10 it is a maximal ideal and hence a prime one. Moreover, if I is prime then $\sharp(\mathbb{N}_0 - Supp(a)) = 1$ and $\underline{\lim}\ |a_n|^{1/n} > 0$. To begin with, if $m_1,\ m_2 \notin Supp(a)$ we can write $a = b \odot_\varepsilon ((1-\delta_{nm_1})\ \overline{\varepsilon_n})_{n\geq 0}$, with $b = (b_n)_{n\geq 0}$ and $b_{m_1} = 1,\ b_{m_2} = 0$ and $b_n = a_n$ in other case. But $b \notin I$ since $b_{m_1} \neq 0$ and $((1-\delta_{nm_1})\ \overline{\varepsilon_n})_{n\geq 0} \notin I$ because $\overline{\varepsilon_{m_2}} \neq 0$, i.e. I becomes non-prime. If $\mathbb{N}_0 - Supp(a) = \emptyset$ then $\underline{\lim}\ |a_n|^{1/n} = 0$ and there is an infinite sequence (n_k) such that $\lim |a_{n_k}|^{1/n_k} = 0$. We consider the analytic branches of $\log(1-z)$ and log z on the respectively simply connected regions $\{z \in \mathbb{C}: \quad \mathbb{R}e(z) < 1\}$ and $\{z \in \mathbb{C}: \quad \mathbb{R}e(z) > 0\}$ so that $\log 1 = 0$. Now, if $m \in \mathbb{N}_0$ we define

$$c_m = \begin{cases} \overline{\varepsilon_m}\ a_m & if \quad m \notin (n_k), \\ \overline{\varepsilon_m}\ a_m^{\log m/\log|a_m|+1} & if \quad m = n_k,\ k - odd, \\ a_m^{-\log m/\log|a_m|} & if \quad m = n_k,\ k - even, \end{cases}$$

$$d_m = \begin{cases} 1 & if \quad m \notin (n_k), \\ a_m^{-\log m/\log|a_m|} & if \quad m = n_k,\ k - odd, \\ \overline{\varepsilon_m}\ a_m^{\log m/\log|a_m|+1} & if \quad m = n_k,\ k - even. \end{cases}$$

The inequality $\max\{\rho(c),\ \rho(d)\} \leq \max\{1,\ \rho(a)\}$ shows that $c,\ d \in \mathcal{A}_0$ and it is clear that $a = c \odot_\varepsilon d$. But $\lim_{j\to\infty} \left|c_{n_{2j}}/a_{n_{2j}}\right|^{1/n_{2j}} = \lim_{j\to\infty} \left|a_{n_{2j}}\right|^{-1/n_{2j}} = +\infty$ and $\lim_{j\to\infty} \left|d_{n_{2j+1}}/a_{n_{2j+1}}\right|^{1/n_{2j+1}} = \lim_{j\to\infty} \left|a_{n_{2j+1}}\right|^{-1/n_{2j+1}} = +\infty$ and so c nor d belong to I. Consequently, $\sharp(\mathbb{N}_0 - Supp(a)) = 1$. Now, by a similar argument we conclude that $\underline{\lim}\ |a_n|^{1/n} > 0$ and the condition is necessary. □

Corollary 29. *Every principal prime ideal I of $\mathcal{A}_0$ is maximal and closed.*

Corollary 30. *Every principal prime ideal of $\mathcal{A}_0$ is the kernel of a complex Hadamard homomorphism.*

Remark 31. *In what follows, we analyze relationships between prime ideals of $\mathcal{A}_0$ and ultrafilters of the class $\mathcal{P}(\mathbb{N}_0)$ of subsets of $\mathbb{N}_0$. Recall that if X is a non-empty set and $\mathfrak{F} \subseteq \mathcal{P}(X)$, $\mathfrak{F} \neq \emptyset$, then $\mathfrak{F}$ is a filter if the following two conditions hold: (i) If $A \in \mathfrak{F},\ B \subseteq A$ then $B \in \mathfrak{F}$. (ii) If $A,\ B \in \mathfrak{F}$ then $A \cup B \in \mathfrak{F}$. It is known that a necessary and sufficient condition in order that $\mathfrak{F}$ be a maximal filter (or ultrafilter) is (iii) For all $A \in \mathcal{P}(X)$, $A \in \mathfrak{F}$ or $A^c \in \mathfrak{F}$. We denote $\Pr(\mathcal{A}_0)$ and $Uf(\mathbb{N}_0)$ for the classes of prime ideals of $\mathcal{A}_0$ and ultrafilters of $\mathbb{N}_0$ respectively.*

Theorem 32. *(cf. [10]) The following functions are well defined*

$$\begin{cases} \Phi : \Pr(\mathcal{A}_0) \to Uf(\mathbb{N}_0) \\ \Phi(\pi) = \{A \in \mathcal{P}(\mathbb{N}_0): \quad a \in \pi \ \ if \ \ a \in \mathcal{A}_0 \ \ and \ \ Supp(a) \subseteq A\}, \end{cases}$$

$$\begin{cases} \Psi : Uf(\mathbb{N}_0) \to \Pr(\mathcal{A}_0) \\ \Psi(\mathfrak{F}) = \{a \in \mathcal{A}_0: \quad Supp(a) \in \mathfrak{F}\}. \end{cases}$$

Theorem 33. *(cf. [10]) Given $\pi \in \mathrm{Pr}(\mathcal{A}_0)$, the following assertions are equivalent:*

(i) $\Psi[\Phi(\pi)] = \pi$.
(ii) $\pi \in \mathrm{Im}(\Psi)$.
(iii) Every element of π *is supported in a proper subset of* $\mathbb{N}_0$.
(iv) π *is minimal.*

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